On Pre-Proper Functions

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Abstract

In this paper we introduce new type of functions called pre-proper functions as generalization of proper functions. Also, we study the characterizations and basic properties of pre-proper and pre-closed functions. Moreover we study the relation between the pre-proper functions and each of proper functions, pre-closed functions and closed functions respectively and we give an example when the converse may not be true.

1.Introduction

The concept of a proper functions was first introduced by N. Bourbaki.[1], by using closed functions. The purpose of this paper is to generalize the concept of proper functions [1] to the concept of pre-proper functions. We give the definition by depending on the definition of pre-closed functions which is introduced by S.N. El-Deeb and etal.,[2] which itself depends on the concept of pre-closed set which is introduced by A.S. Mashhour and etal.,[3]. Also, we study the characterizations and basic properties of pre-proper and pre-closed functions. Moreover we study the relation between the pre-proper functions and each of proper functions, pre-closed functions and closed functions resp. and we give an example when the converse may not be true in general. Recall that a subset A of a topological space X is called pre-open if $A \subseteq int(cl(A))$ where $cl(A)$ and $int(cl(A))$ denotes the closure of A and interior of $cl(A)$ resp.[3]. The complement of a pre-open set is called a pre-closed set [3] or equivalently a subset A of a topological space X is pre-closed iff $cl(int(A)) \subseteq A$ [4] .The family of all pre-open sets of a topological space X is denoted by $PO(X)$ [5] .The intersection of all pre-closed sets containing a set A is pre-closed which is called the pre-closure of A and is denoted by $pcl(A)$ [5]. A subset A of a topological space X is pre-closed iff $A = pcl(A)$ [6].A function $f : X \to Y$ is called proper if (i) f is continuous (ii) $f \times I_Z : X \times Z \to Y \times Z$ is closed, for every topological space Z, where I_Z denotes the identity function on Z [1]. Also, a function $f : X \to Y$ is called pre-irresolute if the inverse image of every pre-open subset of Y is an pre-open set in X [7].If A is a subset of $Y \subseteq X$,the closure of A and the interior of A with respect to Y is denoted by $cl_y(A)$ and $int_y(A)$ resp..

2. Preliminaries

<u>2.1. Definition [8]:</u>

Let $(x_d)_{d\in D}$ be a net in a topological space X. Then $(x_d)_{d\in D}$ is said to have $x \in X$ as an pre-cluster point (written $x_d \overset{p}{\propto} x$) iff for each pre-neighborhood U of x and for each $d \in D$,there is some $d_0 \geq d$ such that $x_{d_0} \in U$. This is sometimes said $(x_d)_{d\in D}$ has x as an pre-cluster point iff $(x_d)_{d\in D}$ is frequently in every pre-neighborhood of x.

***<u>2.2. Theorem</u>* <u>[8]:</u>**

Let X be a topological space and $A \subseteq X$, $x \in X$ .Then $x \in pcl(A)$ if and only if there exists a net $(x_d)_{d\in D}$ in A such that $x_d \overset{p}{\propto} x$.

<u>2.3. Proposition:</u>

If X is topological space, Y is an open subset of X and A is pre-closed in X, then $A \cap Y$ is pre-closed in Y.

<u>Proof:</u>

To prove that $cl_y(int_y(A \cap Y)) \subset A \cap Y$.

$\because cl_y(int_y(A \cap Y)) = cl(int_y(A \cap Y)) \cap Y$.

And since Y is an open subset of X, then $cl(int_y(A \cap Y)) \cap Y \subseteq cl(int(A) \cap cl(Y) \cap Y$.

$\subseteq cl(int(A)) \cap Y \subseteq A \cap Y$, since A is pre-closed in X. Thus $A \cap Y$ is pre-closed in Y.

3. Pre-closed functions

3.1. Definition [2],[9]:

Let X and Y be two topological spaces. A function $f: X \to Y$ is called an pre-closed (pre-open) function if the image of every closed (open) subset of X is an pre-closed (pre-open) set in Y.

3.2.Examples:

1) Let $f: (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by :
$f(x) = 0, \forall\, x \in \Re$. Then f is an pre-closed function.

2) An inclusion function $i: F \to X$ is pre-closed iff F is an pre-closed set in X.

Since every closed set is an pre-closed set, then we have the following theorem.

3.3.Theorem:

Every closed function is an pre-closed function.

3.4.Remark:

The converse of (3.3) may not be true in general.

Example:

If F is an pre-closed set in X, then the inclusion function $i: F \to X$ is pre-closed, but not closed function.

3.5. Theorem:

Let X and Y be two topological spaces. A function $f: X \to Y$ is pre-irresolute iff the inverse image of every pre-closed subset of Y is an pre-closed set in X.

Proof:

The proof of 3.5 is obvious.

3.6. Theorem:

Let X, Y, Z be three topological spaces, and $f: X \to Y, g: Y \to Z$ be two functions. Then:-

1) If f is closed and g is pre-closed, then $g \circ f$ is pre-closed.

2) If $g \circ f$ is pre-closed and f is continuous and onto, then g is pre-closed.

3) If $g \circ f$ is pre-closed and g is one-to-one and pre-irresolute, then f is pre-closed.

Proof:

1) To prove that $g \circ f: X \to Z$ is pre-closed. Let F be a closed subset of X .Since f is closed, then $f(F)$ is a closed set in Y. But g is an pre-closed function, then $g(f(F))$ is an pre-closed set in Z, hence $(g \circ f)(F)$ is an pre-closed set in Z.Thus $g \circ f: X \to Z$ is an pre-closed function.

2) To prove that $g: Y \to Z$ is pre-closed. Let A be a closed subset of Y, since f is continuous, then $f^{-1}(A)$ is a closed set in X, since $g \circ f$ is pre-closed, then $(g \circ f)(f^{-1}(A)) = g(f \circ f^{-1}(A))$ is pre-closed in Z. Since f is onto, then $g(A)$ is pre-closed in Z.Thus $g: Y \to Z$ is an pre-closed function.

3) To prove that $f: X \to Y$ is pre-closed. Let A be a closed subset of X, since $g \circ f$ is pre-closed, then $(g \circ f)(A)$ is pre-closed in Z. Since g is pre-irresolute, then $g^{-1}(g \circ f(A)) = (g^{-1} \circ g)(f(A))$ is pre-closed in Y .Since g is one-to-one, then $f(A)$ is pre-closed in Y. Thus $f: X \to Y$ is an pre-closed function.

3.7. Theorem:

Let X and Y be two topological spaces and $f: X \to Y$ be an pre-closed function. Then for each open subset T of Y, $f_T: f^{-1}(T) \to T$ which is defined by $f_T(x) = f(x)$, $\forall\, x \in f^{-1}(T)$ is also pre-closed.

Proof:

Let F be a closed subset of $f^{-1}(T)$, then there is a closed subset F_1 of X such that $F = F_1 \cap f^{-1}(T)$. Since $f_T(F) = f(F_1) \cap T$ and $f(F_1)$ is pre-closed in Y and T is open in Y, then by (2.3) $f(F_1) \cap T$ is pre-closed in T.Thus f_T is an pre-closed function.

3.8. Theorem:

Let $f: X \to Y$ be an pre-closed Function and let F be a closed subset of X, then the restriction function $f \backslash F: F \to Y$ is an pre-closed function.

Proof:

Since F is a closed set in X, then the inclusion function $i: F \to X$ is a closed function. Since $f: X \to Y$ is an pre-closed function, then by (3.6) $f \circ i: F \to Y$ is an pre-closed function. But $f \circ i = f \setminus F$, thus the restriction function $f \setminus F: F \to Y$ is an pre-closed function.

3.9. Remark:

If $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ are two pre-closed functions .Then $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is not necessarily an pre-closed function.

Example:

Let $f_1: (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by: $f_1(x) = 0, \forall x \in \Re$. And Let $f_2: (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by: $f_2(x) = x, \forall x \in \Re$. Where f_2 is the identity function on $\Re$.

Clearly f_1 and f_2 are pre-closed functions, but $f_1 \times f_2: \Re \times \Re \to \Re \times \Re$, Such that $(f_1 \times f_2)(x, y) = (0, y), \forall (x, y) \in \Re \times \Re$ is not an pre-closed function, since the set $A = \{(x, y) \in \Re \times \Re : x\,y = 1\}$ is closed in $\Re \times \Re$, but $(f_1 \times f_2)(A) = \Re - \{0\}$ is not pre-closed in $\Re \times \Re$.

Now, we introduce the following theorem:

3.10. Theorem:

Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be two functions and if $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is pre-closed, then f_1 and f_2 are also pre-closed functions.

Proof:

Suppose that $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is an pre-closed function. To prove that $f_1: X_1 \to Y_1$ is an pre-closed function. Let F be a closed subset of X_1, to prove that $f_1(F)$ is pre-closed in Y_1.

Suppose that $G = f_1(F) \Rightarrow F \times X_2$ is closed in $X_1 \times X_2$. Since $f_1 \times f_2$ is pre-closed $\Rightarrow (f_1 \times f_2)(F \times X_2) = f_1(F) \times f_2(X_2)$

$= G \times f_2(X_2)$ is pre-closed in $Y_1 \times Y_2$.

i.e. $cl(int(G \times f_2(X_2))) \subseteq G \times f_2(X_2)$

$\Rightarrow cl(int(G)) \times cl(int(f_2(X_2))) \subseteq G \times f_2(X_2)$

$\Rightarrow cl(int(G)) \subseteq G$ & $cl(int(f_2(X_2))) \subseteq f_2(X_2)$.

Thus $G = f_1(F)$ is an pre-closed set in Y_1

Hence $f_1: X_1 \to Y_1$ is an pre-closed function.

By the same way we can prove that f_2 is an pre-closed function. Thus f_1 and f_2 are pre-closed functions.

4. Pre-proper functions

Here we introduce functions, which we call them pre-proper functions, which is weaker than proper functions, with some examples and theorems.

4.1. Definition:

Let X and Y be two topological spaces, and $f: X \to Y$ be a function. Then f is called an pr-proper function if **:-**

i) f is continuous .

ii) $f \times I_Z: X \times Z \to Y \times Z$ is pre-closed, for every topological space Z**.**

4.2. Examples:

1) Let $f: (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by: $f(x) = 0, \forall x \in \Re$.Notice that f is an pre-closed function, but f is not an pre- proper function, since for the usual topological space $(\Re, \mu)$,the function $f \times I_{\Re}: \Re \times \Re \to \Re \times \Re$,such that $(f \times I_{\Re})(x, y) = (0, y), \forall\ (x, y) \in \Re \times \Re$ is not an pre-closed function.

2) An inclusion function $i: F \to X$ is pre-proper iff F is an pre-closed set in X.

Since every closed function is an pre-closed function, then we have the following theorem.

4.3. Theorem:

Every proper function is an pre-proper function.

4.4. Remark:

The converse of (4.3) may not be true in general.

Example:

If F is pre-closed in X, then $i: F \to X$ is an pre-proper function ,but not a proper function.

4.5. Theorem:

Every pre-proper function is an pre-closed function.

Proof:

Let $f:X \to Y$ be an pre-proper function, then $f \times I_Z : X \times Z \to Y \times Z$ is pre-closed for every topological space Z. Let $Z = \{t\}$, then $X \times Z = X \times \{t\} \cong X$ and $Y \times Z = Y \times \{t\} \cong Y$ and we can replace $f \times I_Z$ by f .Thus $f:X \to Y$ is an pre-closed function.

4.6. Remark:

The converse of (4.5) may not be true in general.

Example:

In (4.2) $f:(\Re,\mu) \to (\Re,\mu)$ is an pre-closed function, but not an pre-proper function.

Now, we introduce the following definition.

4.7. Definition:

Let X and Y be two topological spaces. A function $f:X \to Y$ is called an pre-homeomorphism if :

i) f is continuous.
ii) f is one-to-one and onto.
iii) f is pre-closed (pre-open).

4.8. Theorem:

Let X and Y be two topological spaces, and $f:X \to Y$ be a continuous, one-to-one function. Then the following statements are equivalent:

i) f is pre-proper.
ii) f is pre-closed.
iii) f is an pre-homeomorphism of X onto an pre-closed subset of Y.

Proof:

By (4.5), $(i \Rightarrow ii)$.

$(ii \Rightarrow iii)$.Assume that $f:X \to Y$ is an pre-closed function. Since X is a closed set in X, then $f(X)$ is an pre-closed set in Y. Since f is continuous and one-to-one, then f is an pre-homeomorphism of X onto an pre-closed subset $f(X)$ of Y.

$(iii \Rightarrow i)$.Assume that f is an pre-homeomorphism of X onto an pre-closed subset U of Y. Now, let Z be any topological space, and W be any closed set in $X \times Z$, then $W = W_1 \times W_2$,where W_1 is closed in X and W_2 is closed in Z. Since $(f \times I_Z)(W) = (f \times I_Z)(W_1 \times W_2)$ $= f(W_1) \times W_2$ and f is an pre-homeomorphism, then $f(W_1)$ is an pre-closed set in U, thus $f(W_1) \times W_2$ is an pre-closed in $U \times Z$ .Since $U \times Z$ is an pre-closed set in $Y \times Z$,then by [10] $(f \times I_Z)(W)$ is pre-closed in $Y \times Z$. Hence $f \times I_Z : X \times Z \to Y \times Z$ is an pre-closed function. Thus $f:X \to Y$ is an pre-proper function.

4.9. Corollary:

Every pre-homeomorphism is an pre-proper function.

4.10.Remark:

The converse of (4.9) may not be true in general by the following example:-

Example:

Let $f:([0,1],\mu') \to (\Re,\mu)$ be a function which is defined by:

$f(x) = x$, $\forall\ x \in [0,1]$, where μ' is the relative usual topology on [0,1].

Clearly that f is an pre-proper function, but not pre-homeomorphism.

4.11. Theorem:

Let X and Y be two topological spaces and $f:X \to Y$ be an pre-proper function. Let T be any open subset of Y, then $f_T : f^{-1}(T) \to T$ is an pre-proper function.

Proof:

Since $f:X \to Y$ is continuous, then so is f_T .To prove that $f_T \times I_Z : f^{-1}(T) \times Z \to T \times Z$ is pre-closed for every topological space Z. Since f is pre-proper, then $f \times I_Z : X \times Z \to Y \times Z$ is pre-closed for every topological space Z. Since $f_T \times I_Z = (f \times I_Z)_{T \times Z}$ and $T \times Z$ is open in $Y \times Z$, then by (3.7) $f_T \times I_Z$ is pre-closed. Thus $f_T : f^{-1}(T) \to T$ is an pre-proper function.

4.12. Theorem:

Let X, Y, Z be three topological spaces, and $f:X \to Y, g:Y \to Z$ be two continuous functions. Then:-

i) If f is proper and g is pre-proper, then $g\circ f$ is pre-proper.

ii) If $g\circ f$ is pre-proper and f is onto, then g is pre-proper.

iii) If $g\circ f$ is pre-proper and g is one-to-one and pre-irresolute, then f is pre- proper.

Proof:

i) It is clear that $g\circ f : X \to Z$ is continuous. Let Z_1 be any topological space, we have : $(g\circ f)\times I_{Z_1} = (g\times I_{Z_1})\circ(f\times I_{Z_1})$. Since f is proper, then $f\times I_{Z_1} : X\times Z_1 \to Y\times Z_1$ is Closed. Since g is pre-proper, then $g\times I_{Z_1} : Y\times Z_1 \to Z\times Z_1$ is pre-closed. Hence by (3.6) $(g\circ f)\times I_{Z_1} : X\times Z_1 \to Z\times Z_1$ is pre-closed. Thus $g\circ f : X\to Z$ is an pre-proper function.

ii) Let Z_1 be any topological space. To prove that $g\times I_{Z_1} : Y\times Z_1 \to Z\times Z_1$ is pre-closed. Since $g\circ f$ is pre-proper, then:
$(g\circ f)\times I_{Z_1} = (g\times I_{Z_1})\circ(f\times I_{Z_1})$
is pre-closed. Since f is continuous and onto, then so is $f\times I_{Z_1}$, hence by (3.6) $g\times I_{Z_1}$ is pre-closed. Thus $g : Y\to Z$ is an pre-proper function.

iii) Let Z_1 be any topological space. To prove that $f\times I_{Z_1} : X\times Z_1 \to Y\times Z_1$ is pre-closed. Since $g\circ f$ is pre-proper, then $(g\circ f)\times I_{Z_1} = (g\times I_{Z_1})\circ(f\times I_{Z_1})$ is pre-closed. Since g is one-to-one and pre-irresolute, then so is $g\times I_{Z_1}$, hence by (3.6) $f\times I_{Z_1}$ is pre-closed. Thus $f : X\to Y$ is an pre-proper function.

4.13. Theorem:

Let X and Y be two topological spaces. If $f : X\to Y$ is an pre-proper function, then the restriction of f to a closed subset F of X is an pre-proper function of F into Y.

Proof:

Since F is a closed set in X, then the inclusion function $i : F\to X$ is a proper function .Since $f : X\to Y$ is an pre-proper function, then by (4.12) $f\circ i : F\to Y$ is an pre-proper function. But $f\circ i = f\setminus F$, thus the restriction function $f\setminus F : F\to Y$ is an pre-proper function.

4.14. Theorem:

If $f_1 : X_1 \to Y_1$ is a proper function and $f_2 : X_2 \to Y_2$ is an pre-proper function .Then $f_1\times f_2 : X_1\times X_2 \to Y_1\times Y_2$ is an pre-proper function.

Proof:

Let Z be any topological space.

We can write $f_1\times f_2\times I_Z$ by the composition of $I_{Y_1}\times f_2\times I_Z$ and $f_1\times I_{X_2}\times I_Z$. Since f_1 is proper, then $f_1\times I_{X_2}\times I_Z$ is closed. Since f_2 is pre-proper, then $I_{Y_1}\times f_2\times I_Z$ is pre-closed, hence by (3.6)

$(I_{Y_1}\times f_2\times I_Z)\circ(f_1\times I_{X_2}\times I_Z)$

is pre-closed. But

$f_1\times f_2\times I_Z = (I_{Y_1}\times f_2\times I_Z)\circ(f_1\times I_{X_2}\times I_Z)$

$\Rightarrow f_1\times f_2\times I_Z$ is pre-closed. Thus

$f_1\times f_2 : X_1\times X_2 \to Y_1\times Y_2$ is an pre-proper function.

4.15. Theorem:

Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two functions such that $f_1\times f_2$ is pre-proper. Then:

1) If $X_1 \neq \phi$, then f_2 is pre-proper.

2) If $X_2 \neq \phi$, then f_1 is pre-proper.

3) If $X_1 \neq \phi$ and $X_2 \neq \phi$, then both f_1 and f_2 are pre-proper.

Proof:

1) Let Z be any topological space.
To Prove that $f_2\times I_Z : X_2\times Z \to Y_2\times Z$ is pre-closed. Let F be a closed set in $X_2\times Z$ and $G = (f_2\times I_Z)(F)$. To prove that G is pre-closed in $Y_2\times Z$. Since $X_1 \neq \phi$, then $X_1\times F$ is closed in $X_1\times X_2\times Z$.Since $f_1\times f_2$ is pre-proper, then $(f_1\times f_2\times I_Z)(X_1\times F) = f_1(X_1)\times G$ is pre-closed in $Y_1\times Y_2\times Z$.
i.e $cl(int(f_1(X_1)\times G)) \subseteq f_1(X_1)\times G$.
$\Rightarrow cl(int(G)) \subseteq G$.Hence $G = (f_2\times I_Z)(F)$ is pre-closed in $Y_2\times Z$.Therefore $f_2\times I_Z$ is pre-closed. Thus f_2 is an pre-proper function.

2) Similar to (1).
3) Clear.

4.16. Theorem:

Let X and Y be two topological spaces, and $f : X \to Y$ be a continuous function .Then the following statements are equivalent:

i) f is an pre-proper function.

ii) If $(x_d)_{d\in D}$ is a net in X and $y \in Y$ is an pre-cluster point of $(f(x_d))_{d\in D}$, then there is a cluster point $x \in X$ of $(x_d)_{d\in D}$ such that $f(x) = y$.

Proof:

$(i \Rightarrow ii)$ .Claim $f^{-1}(y) \neq \phi$, if $f^{-1}(y) = \phi$, then $y \notin f(X) \Rightarrow y \in (f(X))^c$, since X is a closed set in X and f is pre-closed, then $f(X)$ is an pre-closed set in Y. Thus $(f(X))^c$ is an pre-open set in Y. Therefore $(f(x_d))_{d\in D}$ is frequently in $(f(X))^c$.But $f(x_d) \in f(X), \forall d$ Then $f(X) \cap (f(X))^c \neq \phi$, and this is a contradiction. Thus $f^{-1}(y) \neq \phi$.

Now, suppose that the statement (ii) is not true, that means for all $x_i \in f^{-1}(y)$ there exists an open set U_{x_i} in X contains x_i such that $(x_d)_{d\in D}$ is not frequently in U_{x_i}.Notice that $f^{-1}(y) = \bigcup_{x_i \in f^{-1}(y)} \{x_i\} \subseteq \bigcup_{i\in I} U_{x_i}$ .But $(x_d)_{d\in D}$ is not frequently in $U_{x_i}, \forall\, i \in I$ .Thus $(x_d)_{d\in D}$ is not frequently in $\bigcup_{i\in I} U_{x_i}$,but $\bigcup_{i\in I} U_{x_i}$ is an open set in X, then $\bigcap_{i\in I} {U_{x_i}}^c$ is a closed set in X. Thus $f(\bigcap_{i\in I} {U_{x_i}}^c)$ is an pre-closed set in Y.

Claim $y \notin f(\bigcap_{i\in I} {U_{x_i}}^c)$, if $y \in f(\bigcap_{i\in I} {U_{x_i}}^c)$, then there exists $x \in \bigcap_{i\in I} {U_{x_i}}^c$ such that f(x)=y, thus $x \notin \bigcup_{i\in I} U_{x_i}$,but $x \in f^{-1}(y)$, therefore $f^{-1}(y)$ is not a subset of $\bigcup_{i\in I} U_{x_i}$,and this is a contradiction. Hence $y \notin f(\bigcap_{i\in I} {U_{x_i}}^c)$ and by [8] there is an pre-open set A in Y such that $y \in A$ and $A \cap f(\bigcap_{i\in I} {U_{x_i}}^c) = \phi$.

$\Rightarrow f^{-1}(A) \cap f^{-1}(f(\bigcap_{i\in I} {U_{x_i}}^c)) = \phi$.

$\Rightarrow f^{-1}(A) \cap (\bigcap_{i\in I} {U_{x_i}}^c) = \phi \Rightarrow f^{-1}(A) \subseteq \bigcup_{i\in I} U_{x_i}$

But $(f(x_d))_{d\in D}$ is frequently in A, then $(x_d)_{d\in D}$ is frequently in $f^{-1}(A)$ and then $(x_d)_{d\in D}$ is frequently in $\bigcup_{i\in I} U_{x_i}$, this is a contradiction. Thus there is a cluster point $x \in X$ of $(x_d)_{d\in D}$ such that f(x)=y.

$(ii \Rightarrow i)$, to prove that $f \times I_Z :$ $X \times Z \to Y \times Z$ is an pre-closed function for every topological space Z.Let F be a closed subset of $X \times Z$ and $(f \times I_Z)(F) = G$ .To prove that G is an pre-closed set in $Y \times Z$ .Let $(y,z) \in pcl(G)$, then by (2.2) there is a net $\{(y_d, z_d)\}$ in G such that $(y_d, z_d) \overset{p}{\propto} (y,z)$.Thus there is a net $\{(x_d, z_d)\}$ in F such that

$(f \times I_Z)(x_d, z_d) = (y_d, z_d), \forall d \in D$.

Since $(y_d, z_d) \overset{p}{\propto} (y,z)$,then by [8] $y_d \overset{p}{\propto} y$ and $z_d \overset{p}{\propto} z$, hence there is $x \in X$ such that $x_d \propto x$ & $f(x) = y$ .Since $z_d \overset{p}{\propto} z$, then by [8] $z_d \propto z$. Therefore $x_{d_u} \to x$ & $z_{d_u} \to z$.

$\Rightarrow (x_{d_u}, z_{d_u}) \to (x,z)$ .Since $\{(x_{d_u}, z_{d_u})\}$ is a net in F and F is closed, thus $(x,z) \in \bar{F} = F \Rightarrow (y,z) = (f \times I_Z)(x,z) \in G \Rightarrow$ $pcl(G) \subseteq G$ .Since $G \subseteq pcl(G) \Rightarrow G = pcl(G)$

Hence G is an pre-closed set in $Y \times Z$. Therefore $f \times I_Z : X \times Z \to Y \times Z$ is an pre-closed function for every topological space Z. Thus $f : X \to Y$ is an pre-proper function.

4.17. Theorem:

Let X be a topological space and $\{p\}$ be a space consisting of a single point. Then a function $f : X \to \{p\}$ is pre-proper if and only if X is compact.

Proof:

If X is compact, then by [1], $f : X \to \{p\}$ is proper, therefore by (4.3) $f : X \to \{p\}$ is pre-proper.

Conversely, let $f:X\rightarrow\{p\}$ be an pre-proper function and $(x_d)_{d\in D}$ be a net in X, then $f(x_d)=p$, $\forall\, d\in D \Rightarrow f(x_d)\overset{p}{\propto} p$, since f is pre-proper, then by (4.16) there is $x\in X$ such that $x_d \propto x$ and $f(x)=p$. Thus X is a compact space.

4.18. Theorem:

If $f:X\rightarrow Y$ is a proper function and $g:X\rightarrow Z$ is an pre-proper function, where X is Hausdorff .Then the diagonal function $f\,\Delta\, g:X\rightarrow Y\times Z$ is an pre-proper function.

Proof:

First we have $f\,\Delta\, g=(f\times g)\circ d$, where $d:X\rightarrow X\times X$ such that $d(x)=(x,x)$, $\forall x\in X$. Since X is Hausdorff, then by [1] $d(X)$ is closed in $X\times X$, hence d is a homeomorphism of X onto closed subset of $X\times X$, then by [1] $d:X\rightarrow X\times X$ is proper. Also, since f is proper and g is pre-proper, then by (4.14) $f\times g$ is pre-proper. Hence $(f\times g)\circ d$ is pre-proper. Thus the diagonal function $f\,\Delta\, g:X\rightarrow Y\times Z$ is an pre-proper function.

4.19. Theorem:

If X is any compact topological space and Y is any topological space, then the projection $pr_2:X\times Y\rightarrow Y$ is an pre-proper function.

Proof:

Since X is a compact space, then by [1]

$f:X\rightarrow\{p\}$ is proper, since $I_Y:Y\rightarrow Y$ is pre-proper, then by (4.14)

$$f\times I_Y : X\times Y\rightarrow\{p\}\times Y\cong Y$$

is pre-proper. But $pr_2=f\times I_Y$. Thus $pr_2:X\times Y\rightarrow Y$ is an pre-proper function.

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الخلاصة

في هذا البحث قدمنا نوع جديد من الدوال أسميناها بالدوال السديدة من النمط-pre (pre-proper functions) كتعميم للدوال السديدة (proper functions).كذلك نحن درسنا المكافئات والخواص الاساسية للدوال السديدة من النمط-pre (pre-proper functions) والدوال المغلقة من النمط -pre (pre-closed functions) بالاضافه إلى ذلك درسنا العلاقة بين الدوال السديد من النمط -pre وكل من الدوال السديدة والدوال المغلقة من النمط -pre والدوال المغلقة (closed functions) على التوالي مع أعطاء مثال للاتجاة الغير صحيح.