Mathematical Structure Analysis of Game Problems

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Abstract

In this paper, we are present a novel direction to identify efficient and inefficient strategies of the game problems, based on the features of system of linear inequalities. Definitions and simple illustrated example have been presented.

Keywords: Game Theory, Linear Programming, Redundancy.

1. Introduction

There are many problems in economic policies, warfare and others requires decisions to be made in conflicting or competitive situations. The theory of games is an area of applied mathematics that attempts to analyze conflict situations and provides a basis for rational decision making. A game is a competitive situation in which each of a number of players is pursuing his objective in direct conflict with the other players. Each player is doing every thing to gain as much as possible for him self.

The most widely known real-world illustration of game theory is that reported by [5], in which the decision of the first player is based on the *capabilities* of the other opponent player rather than the opponent player intentions. In other words, the decision making is depend on the basis of the estimate of what the opponent player is *able to do* in response, rather than on the basis of the estimate of what the opponent is *going to do*. Therefore, in real games problems, such as war problems, the input data are estimated for both players, and the unexpected information may not be considered, will effect the solutions results obtained by any computational methods. These problems difficulties lead most of the decisions makers, preferring to know what strategies can be chosen, in order to avoid maximum losses. Therefore, in this paper, since game theory can be formulated as a linear inequalities system, so we are presented some new definitions to identifies the efficient strategies for each player, based on some features of linear inequalities system, prior to solve it by any mathematical techniques (such as linear programming problems), gaining benefits in reducing size problem, and time cost.

2. Problem Formulation [7]

A matrix game is a two-player game defined as follows. Each player first select, independently of other, an action from a finite set of strategies. If we let i denote the first player's strategy, j be the second player's strategy, and $a_{ij}>0$ denote the cost that be paid to the first player, otherwise, if $a_{ij}<0$ then the first player must be paid. The array of possible payments $A=[a_{ij}]$ is called *payoff matrix*, presumed know to both players before the game start. Also, we shall refer to first player as the row player with finite number of strategies (i = 1, 2, …, m) to be choose. Similarly, for the second player as the column player with finite number of strategies (j=1,…,n) to be choose.

Since the game has the property that every deterministic strategy can be foiled by an intelligent opponent, then randomized behavior will be remain appropriate, with the best probabilities are no longer uniformly. Therefore, if we let p_i, $1 \leq i \leq m$, be the probability that the first player chooses the i^{th} row (strategy) of A, and q_j, $1 \leq j \leq n$, be the probability that the second player chooses the j^{th} column (strategy) of A, then the *expected payoff* $E(P,Q)$ of the game to the first player is written in the matrix form as PAQ, where $P=(p_1,\ldots,p_m)$, and $Q=(q_1,\ldots,q_n)$.

To formulate the above problem as a system of linear inequalities, suppose that the first player is play, and the second player plays his first strategy, then the expected payoff to the first player is

$a_{11}p_1 + \ldots + a_{i1}p_i$ .... (1)

and so on, if the second player plays his n^{th} strategy, then the expected payoff to the first player is

$a_{1n}p_1 + \cdots + a_{in}p_n$ .... (2)

If Z_{min} is the minimum of the expected payoffs, can found as row minima, then we have the following system of linear inequalities

$a_{11}p_1 + \ldots + a_{m1}p_m \geq Z_{min}$ .... (3)

$a_{1n}p_1 + \cdots + a_{mn}p_m \geq Z_{min}$

and

$p_1 + \ldots + p_m = 1$

$p_1 \geq 0, \ldots, p_m \geq 0, Z_{min} \geq 0$

Similarly, one can formulate the dual of the above system as in the following system of linear inequalities

$a_{11}q_1 + \cdots + a_{1n}q_n \leq W_{max} \ldots$ .... (4)

$a_{m1}q_1 + \cdots + a_{mn}q_n \leq W_{max}$

and

$q_1 + \ldots + q_n = 1$

$q_1 \geq 0, \ldots, q_n \geq 0, W_{max} \geq 0$

Where W_{max} the maximum expected payoff, can be found as column maxima.

3. Definitions

Before we propose our approach, first, we present some essential definitions concerning redundancy in linear systems that are required in our approach based on [2], [4] and [7]. We consider the feasible region Ω define as following

$\Omega = \{x \in R^n : A_i^T x \leq b_i, i \in I\}$,

where $A_i^T \leq b_i$ is refer to the i-th inequality. The region represented by all but the i-th inequality is given by

$\Omega_j = \{x \in R^n : A_j^T x \leq b_j, i \in I \backslash \{j\}\}$,

where $I \backslash \{j\}$ is the set I with the element j removed.

Definition (1):

The i-th inequality is said to be inefficient in the description of Ω if $\Omega = \Omega_j$, and otherwise is said to be efficient.

Definition (2):

The j-th inequality is ε-efficient at x_k if $0 \leq b_j - A_j^T x_k \leq \varepsilon$, for some scalar ε > 0.

Definition (3):

The j-th inequality is $\varepsilon - active\ at\ x_k$ if ∄ dis $(x_k, H_r) <$ dis $(x_k, H_j) \leq \varepsilon$, for some scalar ε>0, where dis (x_k, H_j) is the distance from x_k to the hyperplane:

$H_j = \{x \in R^n : A_j^T x = b_j\}$

Definition (4):

The projection $P_j(x_k)$ of the point x_k onto the hyperplane:

$H_j = \{x \in R^n : A_j^T x = b_j\}$,

is defined by:

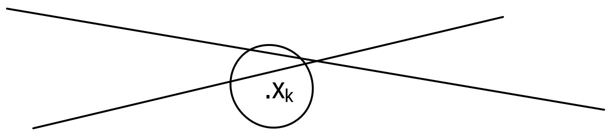
$P_j(x_k) = x_k + A_j(b_j - A_j^T x_k)$.

Consequently, we have:

$\| P_j(x_k) - x_k \| = A_j(b_j - A_j^T x_k)$

$= \mathrm{dis}(x_k, H_j)$

In [2], a definition of local active inequality is presented, which is of no use, in identifying whether the inequality is active or not, since local inactive inequality may be active in another local feasible region, as illustrated in the following figure:



(An illustration of a locally active inequality. Inequality 1 is locally active at x_k)

Therefore, due to the above mathematical representation, we prefer to define a local efficient inequality, which its existence is necessary to keep the hole feasible region of the problem unchanged. In doing so, we set δ>0, and define:

$U_\delta(x) = \{y \in R^n : \| y - x \| \leq \delta\}$

We use the usual definition of the distance function **dis(.,.)** between the point and set of linear equations. Suppose that "ε" the infimum distance between the point x and the set of the inequalities Ω at a local region, so, one can develop the following definition؛

Definition (5):

The r-th inequality is ***locally efficient*** at x if there exist an open set $\delta(\varepsilon), such\ that\ A_r \in \Omega \cap \bar{\delta}(\varepsilon)$ & $\Omega_r \cap \bar{\delta}(\varepsilon) = \emptyset, where\ \bar{\delta}(\varepsilon) is\ the\ closuer\ of\ \delta.$

We can state the above definition into another way:

Definition (6):

The r-th inequality is **locally efficient** at *x*, if for some (ε),

$\Omega \cap \bar{\delta}(\varepsilon) \neq \Omega_r \cap \bar{\delta}(\varepsilon)$. **Otherwise, it is inefficient.**

4. Proposed Procedure

The correct identification of active inequalities is important from both a theoretical and a practical point of view. Such an identification, by removing the difficult combination aspect of the problem. Theoretically, the identification of the active inequalities is not difficult. For more details see [1], [2], [4] and [6]. However, as far as we are aware of, to date no technique can successfully identify all active inequalities. To do this, we denote the following:

Å= The set of indices of active inequalities.

int (x_k) = The k-th interior feasible point.

dist (***int***(x_k), A_i) = The distance between the ***int***(x_k) and the i-th inequality A_i.

Our procedure, start by constructing the feasible interior point ***int***(x_k), and then according to the definition (4), calculate $\boldsymbol{minimum_j}\ \mathbf{dis}(\boldsymbol{x_k}, \boldsymbol{H_j})$. The index of the correspond inequality is added to the set Å. Repeat this processing with different feasible interior point, until no more indices can be added to the set Å for certain number iterations. Therefore, the remaining unidentified indices are considered as almost inefficient.

5. Illustrated Example

To illustrate our theories, we are presenting the following simple matrix game [3]:

5	20
35	10
10	15
25	5

where the entries in the matrix are the payoffs from the second player to the first player, in which each row represent the strategy of the first player, while each column represent the strategy of the second player. Since $Z_{min} = 10$ and $W_{max} = 35$, we can easily formulate systems (3) and (4). We are selected the system (3), to perform our procedure, identified that the first player strategies 1, 3, and 4 are efficient, while the strategy (2) is inefficient.

6. Conclusions

To verify our theories, we solve the system before and after removing the inequality (2), we get the same results. Also, we can insure our results by drawing the system graphically, we see the inequality (2) is out of the feasible solutions, and that means redundant, whether the inequality (2) remaining in the system or removing it from the system will not effected to the feasible solutions, getting more simple mathematical representation to be solved and having fastest decision.

Finally, we believe that, more other different tests approaches, could be developed to have minimal structure representation, and identifying the one who strictly efficient inequality.

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الخلاصة

تم في هذا البحث عرض أفكار جديدة في تحديد المسالك (الستراتيجيات) الكفؤة وغير الكفؤة لمشاكل المباريات مستنداً على بعض خواص نظم المتراجحات الخطية. وتم تقديم بعض التعاريف مع مسألة توضيحية.