Lie Ideals and Jordan q-Centralizers of Prime Rings

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Abstract

The main purpose of this paper is to extend the result of B.Zalar about centralizers to θ-centralizers on Lie ideals.

Keywords : Prime ring, left (right) centralizer, centralizer, Jordan centralizer , left (right)θ - centralizer, θ - centralizer, Jordan θ - centralizer.

Introduction

Throughout this paper, R will represent an associative ring with the center Z . For any x,y ∈ R , the symbol [x,y] will represent the commutator xy-yx , R is called prime if aRb = (0) implies a = 0 or b = 0 [6:p47], and semiprime if aRa = (0) implies a = 0 [12]. R is called 2-torsion free in case 2x=0 ,x ∈ R implies x=0 [6:p47]. A mapping D: R→R is called derivation if D(xy) = D(x)y+xD(y) holds for all x,y ∈ R [5]. A left (right) centralizer of R is an additive mapping T: R→R which satisfies T(xy) = T(x)y (T(xy) = xT(y)) for all x,y ∈ R. A centralizer of R is an additive mapping which is both left and right centralizer[12] . If a ∈ R, then La(x) = ax is a left centralizer and Ra(x) = xa is a right centralizer[12].

A mapping D :R→R is called (θ,θ) - derivation if D(xy) = D(x)θ(y)+ θ(x)D(y) holds for all x,y ∈ R[11]. A left (right) θ-centralizer of R is an additive mapping T: R→R which satisfies T(xy) = T(x)θ(y) (T(xy) = θ(x)T(y)) for all x,y ∈ R. A θ-centralizer of R is an additive mapping which is both left and right θ-centralizer[4] . If a ∈ R, then La(x) = aθ(x) is a left θ-centralizer and Ra(x) = θ(x)a is a right θ-centralizer[4].

A mapping D: R→R is called Jordan (θ,θ)-derivation if $D(x^2)$ = D(x)θ(x)+ θ(x)D(x) holds for all x ∈ R[11]. A Jordan left (right) θ-centralizer of R is an additive mapping T: R→R which satisfies $T(x^2)$ = T(x)θ(x) ($T(x^2)$ = θ(x)T(x)) for all x ∈ R[4]. A Jordan θ-centralizer of R is an additive mapping which is Jordan both left and right θ-centralizer [4] .

If R is a ring with involution *, then every additive mapping E: R→R which satisfies $E(x^2)$ = E(x)x* + xE(x) for all x ∈ R is called Jordan *-derivation[12]. These mappings are closely connected with a question of representability of quadratic forms by bilinear forms. Some algebraic properties of Jordan *-derivations are considered in [2], where further references can be found. For quadratic forms see [10].

In [3] M. BreŠar and B. Zalar obtained a representation of Jordan *-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space. They arrived at a problem whether an additive mapping T which satisfies a weaker condition $T(x^2)$ = T(x)x is automatically a left centralizer. They proved in [3] that this is in fact so if R is a prime ring (generally without involution). In [12] B. Zalar generalized this result on semiprime rings. In [7] A.H.Majeed and H.A.Shaker extended the results of B.Zalar [12]. We generalized results of Zalar[12] and A.H.Majeed and H.A.Shaker [7] to left θ-centralizer in [8],[9].

An easy computation shows that every centralizer T is satisfies T(xoy)=T(x)oy= xoT(y). B.Zalar in [12] proved that every additive mapping T: R→R which satisfies T(xoy)=T(x)oy= xoT(y) of a semiprime ring is a centralizer.

An easy computation shows that every θ-centralizer T is satisfies T(xoy)=T(x)oθ(y)= θ(x)oT(y). In [8] we proved that every additive mapping $T:R \to R$ which satisfies $T(xoy)=T(x)o\theta(y)= \theta(x)oT(y)$ is θ-centralizer.

In the present paper we generalize our results in [8] to lie ideal where R is 2-torsion free prime ring.

1. The First Result

To prove our first result, we need two lemmas which we now state.

Lemma 1.1 [1]:

If $U \not\subset Z$ is Lie ideal of a 2-torsion free prime ring R and $a , b \in R$ such that $aUb = \{0\}$, then a=0 or b=0 **.**

Lemma 1.2 :

Let R be 2-torsion free prime ring and U be a Lie ideal of R. Suppose that $A,B:R\times R\to R$ biadditive mappings. If $A(x,y)w B(x,y) = 0$ for all $x, y, w \in U$, then $A(x,y) w B(u,v) = 0$ for all $x, y, u, v, w \in U$.

Proof:

$A(x,y) w B(x,y) = 0$, for all $x,y,w \in U$

Replace x with x + u , we have

$A(x + u, y)wB(x + u, y)=0$ for all $x,y,w,u \in U$

$A(x,y) w B(u,y) = -A(u,y)wB(x,y)$

We used the biadditivity of A and B

$2^4(A(x,y) w B(u,y)) z (A(x,y) w B(u,y)) =$

$-2^4A(u,y) w B(u,y) z A(x,y) w B(x,y) = 0$

$2^4(A(x,y) w B(u,y)) z (A(x,y) w B(u,y)) = 0,$

for all $x, y, u, z, w \in U$ (*)

If $U \not\subset Z(R)$, by Lemma 1.1, we get

$A(x,y) w B(u,y) = 0$ for all $x, y, u, w \in U$

If $U\subset Z(R)$, right multiplication of relation (*) by r z , where $r \in R$, we get $2^4A(x,y)wB(u,y)z\ r\ A(x,y)wB(u,y)z = 0$, for all $x, y,u,z,w \in U, r \in R$

By primness of R, we have

$2^4A(x,y)wB(u,y)z = 0$ for all $x, y, u, z, w \in U$

Right multiplication of the above relation by r, where $r \in R$, and since R is prime , we get

$2^2A(x,y) w B(u,y) = 0$ for all $x, y, u, w \in U$

Now, we replace y by y + v and obtain the assertion of the lemma with a similar approach as above.

Theorem 1.3 :

Let R be 2-torsion free prime ring, and U a square closed Lie ideal of R, $\theta,T: R\to R$ are additive mappings, if $T(x^2) = T(x)\theta(x)$ **(**$T(x^2)= \theta(x)T(x)$**)** for all $x \in U$, then $T(xy)= T(x)\theta(y)$ **(**$T(xy)= \theta(x)T(y)$**)** for all $x,y \in U$, where θ be surjective endomorphism of U, and $T(u) \in U$ for all $u\in U$.

Proof :

$T(x^2) = T(x)\ \theta(x)$ for all $x\in U$ .... (1)

If we replace x by x + y, we get

$T(xy+yx)=T(x)\theta(y)+T(y)\theta(x)$ for all $x,y\in U$ .... (2)

By replacing y with xy + yx and using (2), we arrive at

$T(x(xy+yx)+(xy+yx)x)=T(x)\theta(xy)+2T(x)\theta(yx) + T(y)\theta(x^2)$ .... (3)

But this can also be calculated in a different way.

$T(x^2y + yx^2) + 2T(xyx) = T(x)\ \theta(xy) + T(y)\ \theta(x^2) + 2T(xyx)$ .... (4)

Comparing (3) and (4), we obtain

$T(xyx) = T(x)\ \theta(yx)$ for all $x,y\in U$ .... (5)

If we linearize (5), we get

$T(xyz + zyx) = T(x)\ \theta(yz) + T(z)\ \theta(yx)$ for all $x,y\in U$ .... (6)

Now we shall compute $j = 2^4T(xyzyx + yxzxy)$ in two different ways. Using (5), we have

$j = 2^4\ T(x)\ \theta(yzyx) + 2^4\ T(y)\ \theta(xzxy)$ for all $x,y,z\in U$ .... (7)

Using (6), we have

$j = 2^4\ T(xy)\ \theta(zyx) +2^4\ T(yx)\ \theta(zxy)$ .... for all $x,y,z\in U$ .... (8)

Comparing (7) and (8) and introducing a biadditive mapping $B(x,y) = T(xy) - T(x)\,\theta(y)$, we arrive at
$B(x,y)\,\theta(zyx) + B(y,x)\,\theta(zxy) = 0$,

for all $x,y,z\in U$ .... (9)

Equality (2) can be rewritten in this notation as $B(x, y) = -B(y, x)$. Using this fact and equality (9), we obtain

$B(x,y)\theta(z)[\theta(x),\theta(y)]=0$, for all $x,y,z\in U$ ... (10)

Using Lemma 1.2, we have

$B(x,y)\theta(z)[\theta(u),\theta(v)]=0$,

for all $x,y,z,u,v\in U$ .... (11)

(i) If U is a non commutative Lie ideal

Using θ is onto and Lemma 1.1, we have

$B(x,y) = 0$ for all $x,y \in U$

(ii) If U is a commutative Lie ideal and $U \not\subset Z(R)$

Now, we shall compute $N = 2^4T(xyzyx)$ in two different ways. Using (5), we have

$N= 2^4T(x)\,\theta(yzyx)$ for all $x,y,z\in U$ .... (12)

$N= 2^4T(xy)\,\theta(zxy)$ for all $x,y,z\in U$ .... (13)

Comparing (12) and (13), we arrive at

$B(x,y)\,\theta(z)\,\theta(yx) = 0$ for all $x,y,z\in U$ .... (14)

Let $\Phi(x,y)=\theta(x)\theta(y)$, it's clear that Φ is a biadditive mapping, therefore

$B(x, y)\,\theta(z)\,\Phi(y,x) = 0$, for all $x,y,z \in U$

Using θ is onto and Lemma 1.2, we have

$B(x, y)\,\theta(z)\,\Phi(u,v) = 0$, for all $x,y,z,u,v \in U$

Implies that

$B(x,y)\,\theta(z)\,\theta(uv) = 0$, or all $x,y,z,u,v \in U$

.... (15)

Using θ is onto, replacing $\theta(v)$ with $4B(x,y)\,\theta(z)$, and Lemma 1.1, we have
$B(x,y)\,\theta(z) = 0$, for all $x,y,z \in U$ .... (16)

Using θ is onto and Lemma 1.1, we have
$B(x,y) = 0$, for all $x,y \in U$
(iii) If $U \subset Z(R)$

Right multiplication of relation (15) by r, where $r \in R$, and since R is prime, we get
$B(x,y) = 0$ for all $x,y \in U$

If $T(x^2)= \theta(x)T(x)$, we obtain the assertion of the theorem with similar approach as above , the proof is complete.

2. The Second Result.

We again divide the proof in few lemmas.

Lemma 2.1:
Let R be 2-torsion free prime ring, U be a square closed Lie ideal of R, $\theta,D: R\to R$ are additive mappings, such that $D(x^2) = D(x)\theta(x) + \theta(x)D(x)$ for all $x \in U$, and $a \in U$ some fixed element , where θ be surjective endomorphism of U
(i) $D(x)D(y) = 0$ for all $x, y \in U$ implies $D = 0$.
(ii) $a\theta(x) - \theta(x)a \in Z(U)$ for all $x \in U$ implies $a \in Z(U)$.

Proof :
(i)$D(x)\theta(y)D(x)=D(x)D(yx)-D(x)D(y)\,\theta(x) = 0$

If $U \not\subset Z(R)$, using θ is onto and Lemma 1.1.

we have $D = 0$ on U

If $U \subset Z(R)$, we get

$D(x)\,w\,D(y) = 0$, for all $x,y \in U$, $w \in R$

By primness of R we have $D = 0$ on U

(ii) Define $D(x) = a\theta(x) - \theta(x)a$. It is easy to see that D is a (θ,θ)-derivation. Since $D(x) \in Z(U)$, for all $x \in U$, we have $D(y)\theta(x) = \theta(x)D(y)$ and also $2D(yz)\theta(x) = 2\theta(x)D(yz)$.
Hence

$$D(y)\theta(zx)+\theta(y)D(z)\theta(x)=\theta(x)D(y)\theta(z)+\theta(xy)D(z)$$

$$D(y)[\theta(z),\theta(x)] = D(z)[\theta(x), \theta(y)]$$

Since θ is surjective take $a=\theta(z)$. Obviously $D(z) = 0$, so we obtain

$$0 = D(y)[a,\theta(x)] = D(y)D(x)$$
From (i) we get $D = 0$ and hence $a \in Z(U)$.

Lemma 2.2 :
Let R be 2-torsion free prime ring, U be a square closed Lie ideal of R, $\theta,T: R\to R$ are

additive mappings, and $a \in U$ some fixed element. If $T(x) = a\theta(x)+\theta(x)a$, and $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all $x, y \in U$, then $a \in Z(U)$, where θ be surjective endomorphism of U

Proof :

$T(xy + yx) = T(x)\theta(y) + \theta(y)T(x)$

gives us

$$a\theta(xy) + a\theta(yx) + \theta(xy)a + \theta(yx)a = (a\theta(x) + \theta(x)a)\theta(y) + \theta(y)(a\theta(x) + \theta(x)a)$$

Implies that

$$a\theta(yx) + \theta(xy)a - \theta(x)a\theta(y) - \theta(y)a\theta(x) = 0$$

$$= (a\theta(y) - \theta(y)a)\,\theta(x) - \theta(x)\,(a\theta(y) - \theta(y)a)$$

The second part of Lemma 2.1 (ii) gives us $a \in Z(U)$.

<u>Lemma 2.3:</u>

Let R be 2-torsion free prime ring, U be a square closed Lie ideal of R. $\theta, T: R \to R$ are additive mappings, such that $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$, for all $x, y \in U$, then $T(z) \in Z(U)$, for all $z \in Z(U)$, where θ is a surjective endomorphism of U.

Proof :

Take any $c \in Z(U)$ and denote $a = T(c)$.

$2T(cx) = T(cx + xc) = T(c)\theta(x) + \theta(x)T(c) = a\theta(x) + \theta(x)a$

A straightforward verification shows that $S(x) = 2T(cx)$ is satisfies $S(xoy) = S(x)o\theta(y) = \theta(x)oS(y)$ for all $x, y \in U$.

By Lemma 2.2, we have $T(c) \in Z(U)$.

<u>Theorem 2.4:</u>

Let R be 2-torsion free prime ring and U a square closed Lie ideal of R. and $\theta, T: R \to R$ are additive mappings, suth that $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all $x, y \in U$. Then $T(xy) = T(x)\theta(y) = \theta(x)T(y)$ for all $x,y \in U$, where θ be surjective endomorphism of U, $T(u) \in U$ for all $u \in U$, and $\theta(Z(U)) = Z(U)$

Proof :

$2T(xy + yx) = 2T(x)\theta(y) + 2\theta(y)T(x) =$

$2\theta(x)T(y) + 2T(y)\theta(x)$ .... (**)

If U is a commutative Lie ideal, we have

$$T(x^2) = T(x)\theta(x) = \theta(x)T(x)$$

If U is a non commutative Lie ideal

Replace y by 2xy + 2yx in (**), we get

$4T(x)\,\theta(xy+yx) + 4\theta(xy+yx)\,T(x) =$

$4T(xy+yx)\,\theta(x) + 4\theta(x)T(xy + yx) = 4(T(x)$

$\theta(y)+\theta(y)T(x))\theta(x)+4\theta(x)(T(x)\theta(y)+ \theta(y)T(x))$

Now it follows that $[T(x),\theta(x)]\theta(y) = \theta(y)[T(x),\theta(x)]$ holds for all $x, y \in U$, but θ is surjective, then we get $[T(x), \theta(x)] \in Z(U)$

Next, we show that $[T(x),\theta(x)] = 0$ holds. Take any $c \in Z(U)$

$$4T(cx) = 2T(cx + xc) = 2T(c)\theta(x) + 2\theta(x)T(c)$$

$$= 2T(x)\theta(c) + 2\theta(c)T(x)$$

Using Lemma 2.3, we get

$$T(cx) = T(x)\theta(c)= T(c)\theta(x)$$

$$4[T(x),\theta(x)]\theta(c) = 4T(x)\theta(xc) - 4\theta(x)T(x)\theta(c)$$

$$= 4T(c)\theta(x^2) - \theta(x)T(c)\theta(x) = 0$$

Since $\theta(Z(U)) = Z(U)$, and $[T(x), \theta(x)] \in Z(U)$, by Lemma 2.1.2 we get our goal .

$2T(x^2) = T(xx + xx) = T(x)\theta(x) + \theta(x)T(x) = 2T(x)\theta(x) = 2\theta(x)T(x)$

Theorem 1.3 now concludes the proof.

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الخلاصة

الهدف الرئيس من البحث هو توسيع نتائج بورت زلار حول التمركزات الى تمركزات θ على مثاليات لي.