The Homotopy Perturbation Method for Solving Some Types of Nonlocal Problems with Some Real Life Applications

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Abstract

In this paper, we use the homotopy perturbation method for finding the solutions of the nonlocal problems that consist of the one-dimensional heat equation together with nonlocal conditions and the nonlocal problems that consist of the parabolic linear integro-differential equations together with nonlocal conditions. Also, some numerical examples are presented to explain the efficiency of this method. Moreover, some real life applications for the nonlocal problems are solved via this method.

Keywords: homotopy perturbation method, parabolic integro-differential equations, nonlocal condition, diffusion equation, nonlocal problem.

1. Introduction

It is seen that in the modeling of many real life applications systems in various fields of physics, ecology, biology, etc, an integral term over the spatial domain is appeared in some part or in the whole boundary, [5]. Such boundary value problems are known as nonlocal problems. The integral term may appear in the boundary conditions. Nonlocal conditions appear when values of the function on the boundary are connected to values inside the domain, [1]

Many researchers studied the nonlocal problems, say [4] used Galerkin method for solving the nonlocal problem for diffusion equations, [2] discussed the existence of the solutions for the nonlocal problem of the one-dimensional wave equations, [7] used Fourier method to establish the existence of the solution for a class of linear hyperbolic equations with nonlocal conditions,

In this paper, we first use the homotopy perturbation method to solve the nonlocal problem that consists of the one-dimensional non-homogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad (x,t) \in \Omega \quad \text{......................................(1.a)}$$

Together with the initial condition:

$$u(x,0) = r(x),\ 0 \le x \le \ell, \quad \text{......................... (1.b)}$$

The non-homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \alpha(t),\ t \ge 0 \quad \text{...........................(1.c)}$$

and the non-homogeneous nonlocal (integral) condition:

$$\int_0^{\ell} u(x,t)dx = \beta(t),\ t \ge 0 \quad \text{........................... (1.d)}$$

Where c is a known nonzero constant, f is a known function of x and t, $\Omega = \{(x,t) | 0 < x < \ell,\ t>0\}$, r, α, β are given functions that must satisfy the compatibility conditions:

$$r'(0) = \alpha(0)$$

and

$$\int_0^{\ell} r(x)dx = \beta(0).$$

Second, we use the homotopy perturbation method to solve the nonlocal problem that consists of the parabolic integro-differential equation:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x,t) = \int_0^t k(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \quad \text{....................................(2.a)}$$

Together with initial condition:

$$u(x,0) = r(x),\ 0 \le x \le \ell, \quad \text{.......................... (2.b)}$$

The non-homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \alpha(t),\ t \ge 0 \quad \text{...........................(2.c)}$$

and the non-homogeneous nonlocal condition:

$$\int_0^{\ell} u(x,t)dx = \beta(t),\ t \geq 0 \quad \text{.......................... (2.d)}$$

where ρ is a known constant, f is a known function of x and t, $\Omega = \{(x,t) | 0 < x < \ell,\ t>0\}$, k is a known function of t, s, $0 \leq t \leq s,$, r, α, β are defined similar to the previous.

2. The Homotopy Perturbation Method, [6]

Consider the following nonlinear differential equation:

$$A(u) - f(r) = 0,\ r \in \Omega \quad \text{................................ (3)}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \Gamma \quad \text{.................................. (4)}$$

where A is a general differential operator, B is a boundary operator, f is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts, which are L and N where L is a linear operator and N is a non-linear operator. Therefore equation (3) becomes:

$$L(u) + N(u) - f(r) = 0 \quad \text{................................ (5)}$$

By the homotopy technique, we construct a homotopy $v: \Omega \times [0,1] \longrightarrow \Re$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0$$

or equivalently

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad \text{................................ (6)}$$

where $p \in [0,1]$ is an embedding parameter, u_o is an initial approximation of equation (3), which satisfies the boundary conditions given by equation (4). Obviously, from equation (6) one can have:

$$H(v,0) = L(v) - L(u_0) = 0$$

$$H(v,1) = A(v) - f(r) = 0$$

and the changing process of p from zero to unity is just that of $v(r,p)$ from $u_o(r)$ to $u(r)$. In a topology, this is called deformation, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic.

Next, we assume that the solution of equation (6) can be written as a power series in p:

$$v(r,p) = \sum_{i=0}^{\infty} p^i v_i(r)$$

Therefore the approximated solution of the boundary value problem given by equations (3)-(4) can be obtained as follows:

$$u(r) = \lim_{p \to 1} v(r,p) = \sum_{i=0}^{\infty} v_i(r) \quad \text{........................ (7)}$$

The convergence of the series given by equation (7) has been given in [6].

3. Solutions of the One-Dimensional Heat Equation with Homogeneous NonLocal Conditions:

Consider the one-dimensional non-homogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t) \in \Omega \quad \text{.....................................(8.a)}$$

together with initial condition:

$$u(x,0) = r(x),\ 0 \leq x \leq \ell, \quad \text{.......................... (8.b)}$$

the homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0,\ t \geq 0 \quad \text{...............................(8.c)}$$

and the homogeneous nonlocal condition:

$$\int_0^{\ell} u(x,t)dx = 0,\ t \geq 0 \quad \text{............................... (8.d)}$$

where c is a known nonzero constant, f is a known function of x and t, $\Omega = \{(x,t) | 0 < x < \ell,\ t>0\}$, r is a given function that must satisfy the compatibility condition:

$$r'(0) = \int_0^{\ell} r(x)dx = 0$$

This nonlocal problem is a special case of the nonlocal problem given by equations (1) in case $\alpha(t) = \beta(t) = 0,\ t \geq 0$.

To solve this nonlocal problem by the homotopy perturbation method, we rewrite equation (8.a) as

$$A(u) - f(x,y) = 0$$

where $A(u) = \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2}$. Then the operator A can be divided into two parts L and N such that equation (8.a) becomes:

$$L(u) + N(u) - f(x,t) = 0$$

where $L(u) = \frac{\partial u}{\partial t}$ and $N(u) = -c^2 \frac{\partial^2 u}{\partial x^2}$.

According to [6], we can construct a homotopy $v:\Omega\times[0,1]\longrightarrow\Re$ which satisfies equation (6). In other words we can construct a homotopy v which satisfies

$$H(v,p)=\frac{\partial v(x,t,p)}{\partial t}-\frac{\partial u_0(x,t)}{\partial t}+p\frac{\partial u_0(x,t)}{\partial t}+p\left[-c^2\frac{\partial^2 v(x,t,p)}{\partial x^2}-f(x,t)\right]=0 \quad \text{.................................} (9)$$

where $p\in[0,1]$ is an embedding parameter and u_o is the initial approximation to the solution of equation (8.a) which satisfies the initial condition, the Neumann condition and the nonlocal condition given by equations (8.b)-(8.d).

Next, we assume that the solution of equation (9) can be expressed as

$$v(x,t,p)=\sum_{i=0}^{\infty}p^i v_i(x,t) \quad \text{............................} (10)$$

Therefore the approximated solution of the nonlocal problem given by equations (8) can be obtained as follows:

$$u(x,t)=\lim_{p\to 1}v(x,t,p)=\sum_{i=0}^{\infty}v_i(x,t) \quad \text{............} (11)$$

By substituting the approximated solution given by equation (10) into equation (9) one can get:

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial v_i(x,t)}{\partial t}-\frac{\partial u_0(x,t)}{\partial t}+p\frac{\partial u_0(x,t)}{\partial x}+p\left[-c^2\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial x^2}-f(x,t)\right]=0$$

Then by equating the terms with identical powers of p one can have:

$$p^0:\frac{\partial v_0(x,t)}{\partial t}-\frac{\partial u_0(x,t)}{\partial t}=0 \quad \text{...................} (12.a)$$

$$p^1:\frac{\partial v_1(x,t)}{\partial t}+\frac{\partial u_0(x,t)}{\partial t}-c^2\frac{\partial^2 v_0(x,t)}{\partial x^2}-f(x,t)=0 \quad \text{............................} (12.b)$$

$$p^j:\frac{\partial v_j(x,t)}{\partial t}-c^2\frac{\partial^2 v_{j-1}(x,t)}{\partial x^2}=0,\quad j=2,3,... \quad (12.c)$$

For simplicity, we take $v_0(x,t)=u_0(x,t)$. In this case equation (12.a) is automatically satisfied. Let $u_0(x,t)=r(x)$ then

$$u_0(x,0)=r(x),\ 0\le x\le \ell\ ,$$

$$\left.\frac{\partial u_0(x,t)}{\partial x}\right|_{x=0}=r'(0)=0,\ t\ge 0$$

and

$$\int_0^{\ell}u_0(x,t)dx=\int_0^{\ell}r(x)dx=0,\ t\ge 0.$$

Therefore u_o satisfies the the initial condition, the Neumann condition and the nonlocal condition given by equations (8.b)-(8.d). Therefore by substituting t=0 in equation (11) one can have:

$$u(x,0)=\sum_{i=0}^{\infty}v_i(x,0)$$

But $v_0(x,0)=r(x)$ and $u(x,0)=r(x)$, hence $v_i(x,0)=0$, i=1,2,.... By substituting $v_0(x,0)=u_0(x,t)=r(x)$ into equation (12.b) one can get:

$$\frac{\partial v_1(x,t)}{\partial t}=c^2 r''(x)+f(x,t)$$

By integrating both sides of the above differential equation with respect to t and by using the initial condition $v_1(x,0)=0$ one can obtain:

$$v_1(x,t)=c^2 r''(x)t+\int_0^t f(x,\tau)d\tau$$

By substituting j=2 and v_1 into equation (12.c) and by solving the resulting first order linear partial differential equation together with the initial condition $v_2(x,0)=0$ one can get $v_2(x,t)$. In a similar manner one can get $v_i(x,t),\ i=3,4,...$. By substituting $v_i(x,t),\ i=0,\ 1,...$ into equation (11) one can get the approximated solution of the nonlocal problem given by equations (8).

4. Solutions of the One-Dimensional Heat Equation with Non-Homogeneous NonLocal Conditions:

Consider the nonlocal problem that consisting of the one-dimensional non-homogeneous heat equation (1.a) together with the initial condition, the non-homogeneous Neumann condition and the nonhomogeneous nonlocal condition given by equations (1.b)-(1.d).

To solve this nonlocal problem by the homotopy perturbation method, we first transform this nonlocal problem into another nonlocal problem, but with homogeneous Neumann condition and homogeneous nonlocal condition. To do this we use the transformation that appeared in [1]:

$$w(x,t) = u(x,t) - z(x,t),\ (x,t) \in \Omega \qquad (13)$$

where $z(x,t) = \alpha(t)\left[x - \frac{\ell}{2}\right] + \frac{\beta(t)}{\ell}$. Then

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial w(x,t)}{\partial t} + \frac{\partial z(x,t)}{\partial t}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial x^2}.$$

Therefore the nonlocal problem given by equations (1) is transformed to the one-dimensional non-homogeneous heat equation:

$$\frac{\partial w(x,t)}{\partial t} = c^2 \frac{\partial^2 w(x,t)}{\partial x^2} + g(x,t), \qquad (x,t) \in \Omega \qquad \text{.............................(14.a)}$$

Together with the initial condition:

$$w(x,0) = q(x),\ 0 \le x \le \ell, \quad \text{..................... (14.b)}$$

The homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0,\ t \ge 0 \quad \text{.............................(14.c)}$$

and the homogeneous nonlocal condition:

$$\int_0^\ell w(x,t)dx = 0,\ t \ge 0 \quad \text{............................ (14.d)}$$

where $g(x,t) = f(x,t) - \frac{\partial z(x,t)}{\partial t}$ and $q(x) = r(x) - z(x,0)$.

To solve this nonlocal problem by the homotopy perturbation method, we can construct a homotopy $v : \Omega \times [0,1] \longrightarrow \Re$ which satisfies

$$H(v,p) = \frac{\partial v(x,t,p)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} + p\frac{\partial w_0(x,t)}{\partial t} + p\left[-c^2 \frac{\partial^2 v(x,t,p)}{\partial x^2} - g(x,t)\right] = 0 \quad \text{.............................. (15)}$$

where $p \in [0,1]$ is an embedding parameter and w_0 is the initial approximation to the solution of equation (14.a) which satisfies the initial condition, the Neumann condition and the nonlocal condition given by equations (14.b)-(14.d).

Next, we assume that the solution of equation (15) can be expressed as in equation (10). Therefore the approximated solution of the nonlocal problem given by equations (14) can be obtained as follows:

$$w(x,t) = \lim_{p \to 1} v(x,t,p) = \sum_{i=0}^{\infty} v_i(x,t) \quad \text{........... (16)}$$

By substituting the approximated solution given by equation (10) into equation (15) one can get:

$$H(v,p) = \sum_{i=0}^{\infty} p^i \frac{\partial v_i(x,t)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} + p\frac{\partial w_0(x,t)}{\partial x} + p\left[-c^2 \sum_{i=0}^{\infty} p^i \frac{\partial^2 v_i(x,t)}{\partial x^2} - g(x,t)\right] = 0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : \frac{\partial v_0(x,t)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} = 0 \quad \text{...................(17.a)}$$

$$p^1 : \frac{\partial v_1(x,t)}{\partial t} + \frac{\partial w_0(x,t)}{\partial t} - c^2 \frac{\partial^2 v_0(x,t)}{\partial x^2} - g(x,t) = 0 \quad \text{............................ (17.b)}$$

$$p^j : \frac{\partial v_j(x,t)}{\partial t} - c^2 \frac{\partial^2 v_{j-1}(x,t)}{\partial x^2} = 0, \quad j=2,3,... \quad \text{.............................(17.c)}$$

For simplicity, we take $v_0(x,t) = w_0(x,t)$. In this case equation (17.a) is automatically satisfied. Let $w_0(x,t) = q(x)$ then

$$w_0(x,0) = q(x),\ 0 \le x \le \ell\ ,$$

$$\left.\frac{\partial w_0(x,t)}{\partial x}\right|_{x=0} = q'(0) = r'(0) - \alpha(0) = 0,\ t \ge 0$$

and

$$\int_0^\ell w_0(x,t)dx = \int_0^\ell r(x)dx - \int_0^\ell z(x,0)dx = \beta(0) - \beta(0) = 0,\ t \ge 0.$$

Therefore w_o satisfies the initial condition, the Neumann condition and the nonlocal condition given by equations (15.b)-(15.d). Therefore by substituting t=0 in equation (16) one can have:

$w(x,0) = \sum_{i=0}^{\infty} v_i(x,0)$. But $v_0(x,0) = q(x)$ and $w(x,0) = q(x)$, hence $v_i(x,0) = 0$, i=1,2,....

By substituting $v_0(x,0) = w_0(x,t) = q(x)$ into equation (17.b) one can get:

$$\frac{\partial v_1(x,t)}{\partial t} = c^2 q''(x) + g(x,t)$$

By integrating both sides of the above differential equation with respect to t and by using the initial condition $v_1(x,0)=0$ one can obtain:

$$v_1(x,t)=c^2q''(x)t+\int_0^t g(x,\tau)d\tau$$

By substituting j=2 and v_1 into equation (17.c) and by solving the resulting first order linear partial differential equation together with the initial condition $v_2(x,0)=0$ one can get $v_2(x,t)$. In a similar manner one can get $v_i(x,t),\ i=3,4,\ldots$. By substituting $v_i(x,t),\ i=0,1,\ldots$ into equation (16) one can get the approximated solution w of the nonlocal problem given by equations (14). Therefore from equation (13):

$$u(x,t)=w(x,t)+z(x,t)$$

$$=\sum_{i=1}^{\infty} v_i(x,t)+z(x,t),\ (x,t)\in\Omega$$

is the solution of the original nonlocal problem given by equations (1).

5. Solutions of the Parabolic Integro-differential Equations with Non-homogeneous NonLocal Conditions:

Consider the nonlocal problem that consists of the parabolic integro-differential equation (2.a) together with initial condition, the non-homogeneous Neumann condition and the non-homogeneous nonlocal condition given by equations (2.b)-(2.d). This nonlocal problem is introduced in [1] with its solution via the variational iteration method. Here we use the homotopy perturbation method to solve this nonlocal problem, we first transform this nonlocal problem into another nonlocal problem, but with homogeneous Neumann condition and homogeneous nonlocal condition. To do this we use the transformation given by equation (13). Therefore the nonlocal problem given by equations (19) is transformed to the parabolic integro-differential equation:

$$\frac{\partial w(x,t)}{\partial t}-\frac{\partial^2 w(x,t)}{\partial x^2}+\rho w(x,t)=\int_0^t k(t,s)w(x,s)ds+g(x,t),\quad (x,t)\in\Omega \quad \text{.............................(18.a)}$$

together with the initial condition:

$$w(x,0)=q(x),\ 0\le x\le \ell, \quad \text{..................... (18.b)}$$

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0}=0,\ t\ge 0 \quad \text{............................(18.c)}$$

and the homogeneous nonlocal condition:

$$\int_0^{\ell} w(x,t)dx=0,\ t\ge 0 \quad \text{............................ (18.d)}$$

where

$$g(x,t)=f(x,t)-\frac{\partial z(x,t)}{\partial t}-\rho z(x,t)+\int_0^t k(t,s)z(x,s)ds$$

and $q(x)=r(x)-z(x,0)$..

To solve this nonlocal problem by the homotopy perturbation method, we construct a homotopy $v:\Omega\times[0,1]\longrightarrow\Re$ which satisfies

$$H(v,p)=\frac{\partial v(x,t,p)}{\partial t}-\frac{\partial w_0(x,t)}{\partial t}+p\frac{\partial w_0(x,t)}{\partial t}+p\left[-\frac{\partial^2 v(x,t,p)}{\partial x^2}+\rho v(x,t,p)-\int_0^t k(t,s)v(x,s,p)ds-g(x,t)\right]=0 \quad \text{...(19)}$$

where $p\in[0,1]$ is an embedding parameter and w_0 is the initial approximation to the solution of equation (18.a) which satisfies the initial condition, the Neumann condition and the nonlocal condition given by equations (18.b)-(18.d).

Next, we assume that the solution of equation (19) can be expressed as in equation (10). Therefore the approximated solution of the nonlocal problem given by equations (18) is given by equation (16).

By substituting the approximated solution given by equation (10) into equation (19) one can get:

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial v_i(x,t)}{\partial t}-\frac{\partial w_0(x,t)}{\partial t}+p\frac{\partial w_0(x,t)}{\partial t}+p\left[-\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial x^2}+\rho\sum_{i=0}^{\infty}p^i v_i(x,t)-\int_0^t k(t,s)\sum_{i=0}^{\infty}p^i v_i(x,s)ds-g(x,t)\right]=0$$

Then by equating the terms with identical powers of p one can have:

$$p^0: \frac{\partial v_0(x,t)}{\partial t} - \frac{\partial w_0(x,t)}{\partial t} = 0 \quad \text{..................(20.a)}$$

$$p^1: \frac{\partial v_1(x,t)}{\partial t} + \frac{\partial w_0(x,t)}{\partial t} - \frac{\partial^2 v_0(x,t)}{\partial x^2} + \rho v_0(x,t) - \int_0^t k(t,s) v_0(x,s) ds - g(x,t) = 0$$

$$\text{............................ (20.b)}$$

$$p^j: \frac{\partial v_j(x,t)}{\partial t} - \frac{\partial^2 v_{j-1}(x,t)}{\partial x^2} + \rho v_{j-1}(x,t) - \int_0^t k(t,s) v_{j-1}(x,s) ds = 0, \quad j=2,3,....$$

$$\text{.............................(20.c)}$$

Similar to the previous, we take $v_0(x,t) = w_0(x,t) = q(x)$. In this case equation (20.a) is automatically satisfied. By substituting it into equation (20.b) one can get:

$$\frac{\partial v_1(x,t)}{\partial t} = q''(x) - \rho q(x) + q(x) \int_0^t k(t,s) ds + g(x,t)$$

By integrating both sides of the above differential equation with respect to t and by using the initial condition $v_1(x,0) = 0$ one can obtain:

$$v_1(x,t) = [q''(x) - \rho q(x)]t + q(x) \int_0^t \int_0^\tau k(\tau,s) ds d\tau + \int_0^\tau g(x,\tau) d\tau$$

In a similar manner one can get $v_i(x,t),\ i = 2,3,....$ By substituting $v_i(x,t),\ i = 0,1,...$ into equation (16) one can get the approximated solution w of the nonlocal problem given by equations (18). Therefore from equation (13):

$$u(x,t) = w(x,t) + z(x,t)$$

$$= \sum_{i=1}^{\infty} v_i(x,t) + z(x,t), \ (x,t) \in \Omega$$

is the solution of the original nonlocal problem given by equations (2).

6. Solutions of Some Real life Applications of Nonlocal Problems Via the Homtopy Perturbation Method Parabolic:

Consider the one-dimensional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \ (x,t) \in (0,1) \times (0,T]$$

$$\text{.............................(21.a)}$$

together with initial condition:

$$u(x,0) = r(x), \ 0 \le x \le 1, \ \text{........................ (21.b)}$$

and the non-homogeneous nonlocal conditions:

$$\int_0^1 u(x,t) dx = \alpha(t), \ 0 \le t \le T \ \text{....................(21.c)}$$

and

$$\int_0^1 x u(x,t) dx = \beta(t), \ 0 \le t \le T \ \text{................. (21.d)}$$

where f is a known function of x and t and r, α, β are defined similar to the previous. This mathematical model, recently studied in [3], described the quasistatic flexure of a thermoelastic rod, where the nonlocal conditions represent the average and weighted average of the entropy u. To solve this nonlocal problem by the homotopy perturbation method, we first transform this nonlocal problem into another nonlocal problem, but with homogeneous nonlocal conditions. To do this we use the transformation that appeared in [8]:

$$w(x,t) = u(x,t) - z(x,t), \ (x,t) \in \Omega \ \text{........... (22)}$$

where

$$z(x,t) = 6[2\beta(t) - \alpha(t)]x - 2[3\beta(t) - 2\alpha(t)].$$

Then the nonlocal problem given by equations (23) is transformed to the one-dimensional non-homogeneous heat equation:

$$\frac{\partial w(x,t)}{\partial t} - \frac{\partial^2 w(x,t)}{\partial x^2} = g(x,t), \ (x,t) \in (0,1) \times (0,T]$$

$$\text{.............................(23.a)}$$

together with the initial condition:

$$w(x,0) = q(x), \ 0 \le x \le 1, \ \text{....................... (23.b)}$$

and the homogeneous nonlocal conditions:

$$\int_0^1 w(x,t) dx = 0, \ 0 \le t \le T \ \text{.......................(23.c)}$$

and

$$\int_0^1 x w(x,t) dx = 0, \ 0 \le t \le T \ \text{.................... (23.d)}$$

where $g(x,t) = f(x,t) - \frac{\partial z(x,t)}{\partial t}$ and $q(x) = r(x) - z(x,0)$.

To solve this nonlocal problem by the homotopy perturbation method, we can construct a homotopy $v: (0,1) \times (0,T] \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p)=\frac{\partial v(x,t,p)}{\partial t}-\frac{\partial w_0(x,t)}{\partial t}+p\frac{\partial w_0(x,t)}{\partial t}+ p\left[-\frac{\partial^2 v(x,t,p)}{\partial x^2}-g(x,t)\right] =0 \quad \text{.............................. (24)}$$

where $p\in[0,1]$ is an embedding parameter and w_0 is the initial approximation to the solution of equation (23.a) which satisfies the initial condition and the nonlocal conditions given by equations (23.b)-(23.d).

Next, we assume that the solution of equation (24) can be expressed as in equation (10). Therefore the approximated solution of the nonlocal problem given by equations (23) is given by equation (16). By substituting the approximated solution given by equation (10) into equation (24) one can get:

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial v_i(x,t)}{\partial t}-\frac{\partial w_0(x,t)}{\partial t}+p\frac{\partial w_0(x,t)}{\partial x}+ p\left[\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_i(x,t)}{\partial x^2}-g(x,t)\right] =0$$

Then by equating the terms with identical powers of p one can obtain equations (17), where $c^2=1$. For simplicity, we take $v_0(x,t)=w_0(x,t)$. In this case equation (17.a) is automatically satisfied. Let $w_0(x,t)=q(x)$ then $w_0(x,0)=q(x),\ 0\le x\le 1$,

$$\int_0^1 w_0(x,t)dx=\int_0^1 r(x)dx-\int_0^1 z(x,0)dx$$
$$=\alpha(0)-\int_0^1 6[2\beta(0)-\alpha(0)]xdx+2\int_0^1[3\beta(0)-2\alpha(0)]dx$$
$$=\alpha(0)-3[2\beta(0)-\alpha(0)]+2[3\beta(0)-2\alpha(0)]$$
$$=0$$

and

$$\int_0^1 xw_0(x,t)dx=\int_0^1 xr(x)dx-\int_0^1 xz(x,0)dx$$
$$=\beta(0)-\int_0^1 6[2\beta(0)-\alpha(0)]x^2dx+2\int_0^1[3\beta(0)-2\alpha(0)]xdx$$
$$=\beta(0)-2[2\beta(0)-\alpha(0)]+[3\beta(0)-2\alpha(0)]$$
$$=0.$$

Therefore w_o satisfies the initial condition and the nonlocal conditions given by equations (23.b)-(23.d).

By substituting $v_0(x,0)=w_0(x,t)=\ q(x)$ into equation (17.b) one can get:

$$\frac{\partial v_1(x,t)}{\partial t}=q''(x)+g(x,t)$$

By integrating both sides of the above differential equation with respect to t and by using the initial condition $v_1(x,0)=0$ one can obtain:

$$v_1(x,t)=q''(x)t+\int_0^t g(x,\tau)d\tau$$

By substituting v_1 and $c^2=1$ into equation (17.c) and by solving the resulting first order linear partial differential equation together with the initial condition $v_2(x,0)=0$ one can get $v_2(x,t)$. In a similar manner one can get $v_i(x,t),\ i=3,4,\ldots$. By substituting $v_i(x,t),\ i=0,1,\ldots$ into equation (16) one can get the approximated solution w of the transformed nonlocal problem given by equations (23). Therefore from equation (22):

$$u(x,t)=w(x,t)+z(x,t)$$
$$=\sum_{i=1}^{\infty}v_i(x,t)+z(x,t),\ (x.t)\in(0,1)\times(0,T]$$

is the solution of the original nonlocal problem given by equations (21).

7. Numerical Examples

In this section we present four examples of the nonlocal problems that are solved by the homotopy perturbation method.

Example (1):

Consider the one-dimensional homogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2},\quad 0<x<1,\ t>0 \ \text{...(25.a)}$$

together with initial condition:

$$u(x,0)=\cos(2\pi x),\ 0\le x\le 1,\ \text{................. (25.b)}$$

the homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=0,\ t\ge 0\ \text{.............................(25.c)}$$

and the homogeneous nonlocal condition:

$$\int_0^{\ell} u(x,t)dx = 0,\ t \ge 0 \ \ldots\ldots\ldots\ldots\ (25.d)$$

It is easy to check that the compatibility conditions are satisfied for this nonlocal problem. We use the homotopy perturbation method to solve this example. To do this, let $v_o(x,t) = u_o(x,t) = u(x,0) = \cos(2\pi x)$. From equation (12.b) one can have:

$$\frac{\partial v_1(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\cos(2\pi x) = -4\pi^2\cos(2\pi x).$$

Therefore

$$v_1(x,t) = -4\pi^2 t\cos(2\pi x).$$

From equation (12.c) one can have:

$$\frac{\partial v_2(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left[-4\pi^2 t\cos(2\pi x)\right] = \left(4\pi^2\right)^2 t\cos(2\pi x).$$

Therefore

$$v_2(x,t) = \frac{\left(4\pi^2 t\right)^2}{2!}\cos(2\pi x).$$

By continuing in this manner, one can deduce that:

$$v_i(x,t) = \frac{\left(-4\pi^2 t\right)^i}{i!}\cos(2\pi x),\ i=0,1,\ldots.$$

Thus

$$u(x,t) \cong \sum_{i=0}^{\infty} v_i(x,t) = \left(1 - 4\pi^2 t + \frac{(4\pi^2)^2}{2!}t^2 - \cdots\right)\cos(2\pi x) = \sum_{i=0}^{\infty}\frac{\left(-4\pi^2 t\right)^i}{i!}\cos(2\pi x) = e^{-4\pi^2 t}\cos(2\pi x)$$

which is the exact solution of the nonlocal problem given by equations (25).

Example (2):

Consider the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (3t^2 - t^3)e^{-x},\ 0 < x < 1,\ t>0$$

together with initial condition:

$$u(x,0) = 0,\ 0 \le x \le 1,$$

the nonhomogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = -t^3,\ t \ge 0$$

and the nonhomogeneous nonlocal condition:

$$\int_0^1 u(x,t)dx = t^3(1 - e^{-1}),\ t \ge 0$$

We use the homotopy perturbation method to solve this example. To do this, we transform this nonlocal problem into one but with homogeneous Neumann condition and homogeneous nonlocal condition. Therefore, consider the transformation given by equation (13). In this case:

$$z(x,t) = t^3\left(\frac{3}{2} - x - e^{-1}\right).$$

and the nonlocal problem given by equations (14) consisting of the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial^2 w(x,t)}{\partial x^2} + (3t^2 - t^3)e^{-x} + 3t^2\left(x - \frac{3}{2} + e^{-1}\right),\ 0 < x < 1,\ t>0 \ \ldots\ldots\ldots\ (26.a)$$

together with initial condition:

$$w(x,0) = 0,\ 0 \le x \le 1, \ \ldots\ldots\ldots\ (26.b)$$

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0,\ t \ge 0 \ \ldots\ldots\ldots\ (26.c)$$

and the homogeneous nonlocal condition:

$$\int_0^1 w(x,t)dx = 0,\ t \ge 0 \ \ldots\ldots\ldots\ (26.d)$$

To solve this nonlocal problem by using the homotopy perturbation method, let

$$v_0(x,t) = w_0(x,t) = w(x,0) = 0.$$

From equation (17.b) one can have:

$$\frac{\partial v_1(x,t)}{\partial t} = g(x,t) = (3t^2 - t^3)e^{-x} + 3t^2\left(x - \frac{3}{2} + e^{-1}\right)$$

Therefore

$$v_1(x,t) = \int_0^t\left[(3\tau^2 - \tau^3)e^{-x} + 3\tau^2\left(x - \frac{3}{2} + e^{-1}\right)\right]d\tau = \left(t^3 - \frac{1}{4}t^4\right)e^{-x} + t^3\left(x - \frac{3}{2} + e^{-1}\right).$$

From equation (17.c) one can have:

$$\frac{\partial v_2(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left[\left(t^3 - \frac{1}{4}t^4\right)e^{-x} + t^3\left(x - \frac{3}{2} + e^{-1}\right)\right] = \left(t^3 - \frac{1}{4}t^4\right)e^{-x}.$$

Therefore

$$v_2(x,t)=\left(\frac{1}{4}t^4-\frac{1}{20}t^5\right)e^{-x}.$$

By continuing in this manner, one can deduce that:

$$v_i(x,t)=\left(\frac{1}{(4)(5)...(i+2)}t^{i+2}-\frac{1}{(4)(5)...(i+3)}t^{i+3}\right)e^{-x},\quad i=2,3,....$$

Thus

$$w(x,t)=\sum_{i=0}^{\infty}v_i(x,t)$$
$$=\left(t^3-\frac{1}{4}t^4\right)e^{-x}+t^3\left(x-\frac{3}{2}+e^{-1}\right)+\left(\frac{1}{4}t^4-\frac{1}{20}t^5\right)e^{-x}+\left(\frac{1}{20}t^5-\frac{1}{120}t^6\right)e^{-x}+\cdots$$
$$=t^3e^{-x}+t^3\left(x-\frac{3}{2}+e^{-1}\right).$$

is the exact solution of the transformed nonlocal problem given by equations (26). Hence

$$u(x,t)=w(x,t)+z(x,t)$$
$$=t^3e^{-x}+t^3\left(x-\frac{3}{2}+e^{-1}\right)+t^3\left(\frac{3}{2}-x-e^{-1}\right)$$
$$=t^3e^{-x}$$

is the exact solution of the original nonlocal problem.

Example (3):

Consider the parabolic integro-differential equation:

$$\frac{\partial u(x,t)}{\partial t}-\frac{\partial^2 u(x,t)}{\partial x^2}+2u(x,t)=\int_0^t(t-s)u(x,s)ds+3\sin x+3+6t-\frac{1}{2}t^3-\frac{1}{2}t^2\sin x,\quad 0<x<1,\ t>0$$

together with initial condition:

$u(x,0)=\sin x,\ 0\le x\le 1,$

the nonhomogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=1,\ \ t\ge 0$$

and the nonhomogeneous nonlocal condition:

$$\int_0^1 u(x,t)dx=1+3t-\cos 1,\ \ t\ge 0$$

We shall use the homotopy perturbation method to solve this example. To do this, we transform this nonlocal problem into one but with homogeneous Neumann condition and homogeneous nonlocal condition. Therefore, consider the transformation given by equation (13). In this case:

$$z(x,t)=x+\frac{1}{2}-\cos 1+3t.$$

and the nonlocal problem given by equations (18) consisting of the parabolic integro-differential equation:

$$\frac{\partial w(x,t)}{\partial t}-\frac{\partial^2 w(x,t)}{\partial x^2}+2w(x,t)=\int_0^t(t-s)w(x,s)ds-3\sin x-2x-1+2\cos 1-\frac{1}{2}t^2\sin x+\frac{1}{2}t^2x+\frac{1}{4}t^2-\frac{1}{2}t^2\cos 1,\ \ 0<x<1,\ \ t>0$$

..............................(27.a)

together with initial condition:

$$w(x,0)=\sin x-x-\frac{1}{2}+\cos 1,\quad 0\le x\le 1,$$

.... (27.b)

the homogeneous Neumann condition:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0}=0,\ t\ge 0\quad \text{.............................(27.c)}$$

and the homogeneous nonlocal condition:

$$\int_0^1 w(x,t)dx=0,\ t\ge 0\quad \text{.............................(27.d)}$$

To solve this nonlocal problem by using the homotopy perturbation method, let

$$v_0(x,t)=w_0(x,t)=w(x,0)=\sin x-x-\frac{1}{2}+\cos 1.$$

From equation (20.b) one can have:

$$\frac{\partial v_1(x,t)}{\partial t}=-\frac{\partial}{\partial x}\left(\sin x-x-\frac{1}{2}+\cos 1\right)+\frac{\partial^2}{\partial x^2}\left(\sin x-x-\frac{1}{2}+\cos 1\right)-2\left(\sin x-x-\frac{1}{2}+\cos 1\right)-\left(\sin x-x-\frac{1}{2}+\cos 1\right)\int_0^t(t-s)ds-\left(-3\sin x-2x-1+2\cos 1-\frac{1}{2}t^2\sin x+\frac{1}{2}t^2x+\frac{1}{4}t^2-\frac{1}{2}t^2\cos 1\right)$$

After simple computations one can get:

$$v_1(x,t)=0.$$

Hence

$$v_i(x,t)=0, \quad i=2,3,....$$

Thus

$$w(x,t)\cong\sum_{i=0}^{\infty}v_i(x,t)$$
$$=v_o(x,t)$$
$$=\sin x - x-\frac{1}{2}+\cos 1.$$

is the exact solution of the transformed nonlocal problem given by equations (27). Hence

$$u(x,t)=w(x,t)+z(x,t)$$
$$=\sin x - x-\frac{1}{2}+\cos 1+x+\frac{1}{2}-\cos 1+3t$$
$$=\sin x+3t$$

which is the exact solution of the original nonlocal problem.

Example (4):

Consider the one-dimensional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{1}{(x+1)}-\frac{2t}{(x+1)^3}, \quad 0<x<1, \ 0\le t\le 1$$

Together with initial condition:

$$u(x,0)=3, \ 0\le x\le 1,$$

and the nonhomogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx=\ln(2)t+3, \quad 0\le t\le 1$$

and

$$\int_0^1 xu(x,t)dx=\frac{3}{2}+(1-\ln 2)t, \quad 0\le t\le 1$$

We shall use the homotopy perturbation method to solve this example. To do this, we transform this nonlocal problem into one but with homogeneous Neumann condition and homogeneous nonlocal condition. Therefore, consider the transformation given by equation (22). In this case:

$$z(x,t)=\left(12-18\ln(2)\right)tx+3-6t+10\ln(2)t.$$

and the nonlocal problem given by equations (23) consisting of the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{1}{(x+1)}-\frac{2t}{(x+1)^3}-\left[12-18\ln(2)\right]x+6-10\ln(2), \quad 0<x<1, \ 0\le t\le 1 \quad ....(28.a)$$

together with initial condition:

$$w(x,0)=0, \ 0\le x\le 1, \quad .... (28.b)$$

and the homogeneous nonlocal conditions:

$$\int_0^1 w(x,t)dx=0, \ 0\le t\le 1 \quad ....(28.c)$$

and

$$\int_0^1 xw(x,t)dx=0, \ 0\le t\le 1 \quad ....(28.d)$$

To solve this nonlocal problem by using the homotopy perturbation method, let

$$v_0(x,t)=w_0(x,t)=w(x,0)=0.$$

From equation (17.b) one can have:

$$v_1(x,t)=\frac{t}{x+1}-\frac{t^2}{(x+1)^3}-(12-18\ln(2))xt+6t-10\ln(2)t.$$

From equation (17.c) one can have:

$$v_2(x,t)=\int_0^t\frac{\partial^2}{\partial x^2}\left[\frac{\tau}{x+1}+6\tau-10\ln(2)\tau-\frac{\tau^2}{(x+1)^3}-(12-18\ln(2))x\tau\right]d\tau$$
$$=\frac{1}{(x+1)^3}t^2-\frac{4}{(x+1)^5}t^3.$$

By continuing in this manner, one can deduce that:

$$v_i(x,t)=\left(\frac{(2i-2)!t^i}{(x+1)^{2i-1}i!}-\frac{(2i)!t^{i+1}}{(x+1)^{2i+1}(i+1)!}\right), \ i=2,3,....$$

Thus

$$w(x,t)\cong\sum_{i=0}^{\infty}v_i(x,t)$$
$$=\frac{t}{x+1}-\frac{t^2}{(x+1)^3}-(12-18\ln(2))xt+6t-10\ln(2)t+\sum_{i=0}^{\infty}\left(\frac{(2i-2)!t^i}{(x+1)^{2i-1}i!}-\frac{(2i)!t^{i+1}}{(x+1)^{2i+1}(i+1)!}\right)$$
$$=\frac{t}{x+1}-(12-18\ln(2))xt+6t-10\ln(2)t$$

is the exact solution of the transformed nonlocal problem given by equations (28). Hence

$$u(x,t) = w(x,t) + z(x,t)$$
$$= \frac{t}{x+1} - (12 - 18\ln(2))xt + 6t - 10\ln(2)t + (12 - 18\ln(2))tx + 3 - 6t + 10\ln(2)t$$
$$= \frac{t}{x+1} + 3$$

which is the exact solution of the original nonlocal problem.

8. Conclusions

In this paper, we deal with the approximated solutions of the linear nonlocal problems for the one-dimensional heat equation, parabolic integro-differential equations and some real life applications arising in thermoelastic using the homotopy perturbation method. This technique was tested on some examples and were seen to produce satisfactory results.

9. References

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الخلاصة

في هذا البحث استعملنا طريقة القلق الهوموتوبي لايجاد الحلول للمسائل اللامحلية والتي تتكون من معادلة الحرارة ذات البعد الواحد مع شروط لا محلية والمسائل اللامحلية والتي تتكون من المعادلات التكاميلية-التفاضلية الخطية مع شروط لامحلية. بعض الأمثلة العددية قدمت لتوضيح كفاءة هذه الطريقة. بالاضافة الى ذلك بعض المسائل الحياتية للمسائل اللامحلية حُلت باستخدام هذه الطريقة.