Approximation of Functions by Some Types of Szasz-mirakjan Operators

Sahib Al-Saidy* and Salim Dawood**
*Department of Mathematics, College of Science, Al-Mustansiriya University, Bagdad-Iraq.
**Department of Mathematic, College of Education, Al-Mustansiriya University, Bagdad-Iraq.
E-mail: dr_salim@yahoo.com.

Abstract

In this paper, we study the approximation of continuous functions by using some types of Szasz-Mirakjan operators (modified Szasz-Mirakjan operator and modified mulit Szasz-Mirakjan operator) defined on some normed space.

Keywords: ((Positive linear operators, normed space, continuous functions, Szasz-Mirakjan operator))

1-Introduction

Approximation theory represents an old field of mathematical research. In the fifties, a new breath over it has been brought by the studying of linear methods of approximation which are given by linear operators. Approximation problem for real valued continuous function on a closed interval $[a,b]$is considered in many literatures like [2]. Ispir N. and Atakut C. in 2002, [3], studied the best approximation of real valued continuous function f on $[0,\infty)$such that $\omega_\alpha . f$ is uniformly continuous and bounded on$[0,\infty)$,

where $\omega_\alpha(x)=(1+x^\alpha)^{-1}, \alpha \geq 1$ *and* $\omega_0=1$, Gadjive A. and Aral A. in 2007, [1] obtains a Korovkin type approximation by positive linear operators result for the functions in weight space $L_{p,\omega}(R)$.

In this paper, we will study the approximation of continuous functions of one and mulitple variable by some types of positive linear operators as modified Szasz-Mirakjan operators and modified multi- Szasz-Mirakjan operators.

2-Definitions and Notations

Here, we recall some basic definition and proposition that will needed in this paper.

Definition (2.1),[3]:

Let $f:[0,\infty)\rightarrow R$ be any function and the function $\omega_\alpha:[0,\infty)\rightarrow R^+$ is defined by $\omega_\alpha(x)=\overset{\alpha x}{e}, \alpha \geq 1$the modified Szasz-Mirakjan operator $S_n : L_{p,\alpha} \longrightarrow L_{p,\alpha}$ is an operator defined by:

$$S_n(f;x)=e^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}f\left(\frac{k}{b_n}\right), x\in[0,\infty) \quad n\in N,$$

where $\{a_n\}$, $\{b_n\}$ are given increasing sequence of positive integer numbers such that $\lim_{n\to\infty}\frac{1}{b_n}=0$ and $\frac{a_n}{b_n}=1+o(\frac{1}{b_n})$.

Also the following proposition give some properties of the operator S_n.

Proposition (2.2), [3]:

For any $x\in[0,\infty)$ and $n\in N$, the following statements hold:

(1) $S_n(f;x)=1$, where $f(x)=1$.

(2) $S_n(f;x)=\frac{a_n}{b_n}x$, where $f(x)=x$.

(3) $S_n(f;x)=\frac{a_n^2}{b_n^2}x^2+\frac{a_n}{b_n^2}x$, where $f(x)=x^2$.

3. Approximation of Functions of One Variable by Modified Szasz-Mirakjan Operator

Here, we approximate any continuous function defined on $[0,\infty)$ by the modified Szasz-Mirakjan operator S_n.

Lemma (3.1):

For each $1\leq p<\infty$, $L_{p,\alpha}=\left\{ f \,\middle|\, f:[0,\infty)\longrightarrow R \right.$ is a continuous function such that $\left.\int_0^\infty\left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx<\infty \right\}$ is a normed space where $\omega_\alpha(x)=e^{\alpha x}, \alpha$ is a positive real number.

Proof:

It is easy to check $0 \in L_{p,\alpha}$. Therefore $L_{p,\alpha} \neq \phi$. Define + and . on $L_{p,\alpha}$ by

$(f+g)(x) = f(x) + g(x) \quad \forall \quad f, g \in L_{p,\alpha}$ and

$(c.f)(x) = cf(x) \quad \forall f \in L_{p,\alpha}, \quad c \in \mathbb{C}.$

Let f and $g \in L_{p,\alpha}$ Then $\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx < \infty$ and $\int_0^\infty \left|\frac{g(x)}{\omega_\alpha(x)}\right|^p dx < \infty.$

By using [4, p. 236], one can have: $\int_0^\infty \left|\frac{(f+g)(x)}{\omega_\alpha(x)}\right|^p dx \le 2^p \int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx + 2^p \int_0^\infty \left|\frac{g(x)}{\omega_\alpha(x)}\right|^p dx < \infty.$

Thus $f+g \in L_{p,\alpha}$. Moreover; since

$$\int_0^\infty \left|\frac{(cf)(x)}{\omega_\alpha(x)}\right|^p dx = \int_0^\infty \left|\frac{(c)f(x)}{\omega_\alpha(x)}\right|^p dx$$

$$= |c|^p \int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx < \infty.$$

So, $\int_0^\infty \left|\frac{(cf)(x)}{\omega_\alpha(x)}\right|^p dx < \infty.$ Then $c.f \in L_{p,\alpha}$. The other conditions for $L_{p,\alpha}$ to be a vector space is easy to be verified, thus we omitted them.

and Define $\|.\|_{p,\alpha} : L_{p,\alpha} \longrightarrow R^+ \cup \{0\}$ by

$\|f\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$. We prove $\|.\|_{p,\alpha}$ is a norm on $L_{p,\alpha}$. To do this, we must prove the following conditions:

(i) If $f = 0$ then $\|f\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} = 0$. Conversely if

$\|f\|_{p,\alpha} = 0, \quad$ then $\left|\frac{f(x)}{\omega_\alpha(x)}\right|^p = 0 \quad \forall x \in [0,\infty)$

and hence $f(x) = 0, \ \forall x \in [0,\infty)$. Therefore; $f = 0.$

(ii) Let $f, g \in L_{p,\alpha}$ then

$\|f+g\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{f(x)+g(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$ and by using [4, pp. 236], we get

$$\|f+g\|_{p,\alpha}$$

$$\le \left(2^p \int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx + 2^p \int_0^\infty \left|\frac{g(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}}$$

also, by using Minkowski's inequality, one can have: $\|f+g\|_{p,\alpha}$

$$\le 2\left[\left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}} + \left(\int_0^\infty \left|\frac{g(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}} \right]$$

So, $\|f+g\|_{p,\alpha}$

$$\le \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_0^\infty \left|\frac{g(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$= \|f\|_{p,\alpha} + \|g\|_{p,\alpha}.$$

(iv) Let $\lambda \in \mathbb{C}$ and $f \in L_{p,\alpha}$ then

$$\|\lambda \ f\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{(\lambda f)(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^\infty \left|\frac{(\lambda)f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^\infty |\lambda^p| \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

Therefore; $\|\lambda \ f\|_{p,\alpha} = |\lambda| \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$

$= |\lambda| \ \|f\|_{p,\alpha}$. Therefore; $L_{p,\alpha}$ is a normed space.

<u>Lemma (3.2), [1]:</u>

Let L_n be any a uniformly bounded sequence of positive linear operators from $L_{p,\alpha}$ into itself satisfying the condition $\lim_{n\to\infty} \|L_n(f) - f\|_{p,\alpha} = 0, \quad$ where

$f(x)=1,\ x,\ x^2$ then for every $f \in L_{p,\alpha}$, $\lim_{n\to\infty}\|L_n(f)-f\|_{p,\alpha}=0.$

Theorem (3.3):

For $f \in L_{p,\alpha}$, then $S_n f \longrightarrow f \quad as \quad n\to\infty.$

Proof:

Since

$$S_n(\alpha f+\beta g;x)=e^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}(\alpha f+\beta g)\left(\frac{k}{b_n}\right),$$

$$x\in[0,\infty),\quad n\in N$$

$$=e^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\left(\alpha f\left(\frac{k}{b_n}\right)+\beta g\left(\frac{k}{b_n}\right)\right), x\in[0,\infty)\ , n\in N$$

$$=\alpha e^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}f\left(\frac{k}{b_n}\right)$$

$$+\beta e^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}g\left(\frac{k}{b_n}\right), x\in[0,\infty)\ , n\in N$$

$$=\alpha S_n(f,x)+\beta S_n(g,x) \qquad , x\in[0,\infty)\quad n\in N.$$

Thus, S_n linear operator. Also, since $\{a_n\}$, $\{b_n\}$ are increasing sequences of positive integer numbers and $n\geq 1$, so $S_n(f;x)\geq 0$, then S_n is positive operator , and by using [3], we can have S_n is uniformly bounded. Let $f(x)=1,\ \forall x\in[0,\infty)$. Then from proposition (2.2) one can have: $\lim_{n\to\infty}\|S_n f-f\|_{p,\alpha}=0.$ Also, for $f(x)=x \quad \forall x\in[0,\infty)$, one can have:

$$\lim_{n\to\infty}\|S_n f-f\|_{p,\alpha}=\lim_{n\to\infty}\left(\int_0^{\infty}\left|\frac{S_n f(x)-f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$=\lim_{n\to\infty}\left(\int_0^{\infty}\left|\frac{\frac{a_n x}{b_n}-x}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$=\left(\int_0^{\infty}\left|\frac{x}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}\lim_{n\to\infty} o\left(\frac{1}{b_n}\right)=0.$$

Moreover, for $f(x)=x^2,\ \ \forall x\in[0,\infty)$, one can have:

$$\lim_{n\to\infty}\|S_n f-f\|_{p,\alpha}=$$

$$\lim_{n\to\infty}\left(\int_0^{\infty}\left|\frac{\frac{a_n^{\,2}}{b_n^{\,2}}x^2+\frac{a_n}{b_n^{\,2}}x-x^2}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

Therefore;

$$\lim_{n\to\infty}\|S_n f-f\|_{p,\alpha}$$

$$=\lim_{n\to\infty}\left(\int_0^{\infty}\left|\frac{(\frac{a_n^{\,2}}{b_n^{\,2}}-1)x^2+\frac{a_n}{b_n^{\,2}}x}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq\left(\int_0^{\infty}\left|\frac{x^2}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}\lim_{n\to\infty}\left(\frac{a_n^{\,2}}{b_n^{\,2}}-1\right)+\left(\int_0^{\infty}\left|\frac{x}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n^{\,2}}=0\,,$$

Thus, $\lim_{n\to\infty}\|S_n f-f\|_{p,\alpha}=0$ where $f(x)=x^2$. Then by using lemma (3.2), one can get desired result.

4. Approximation of functions of multiple variables by multi-Szasz-Mirakjan operator.

Here, we generalized the results that are given in the pervious section to be valid for the modified multi-Szasz-Mirakjan operator and we approximate any continuous function of m independent variables on $[0,\infty)^m$ by these operators. For any $(x_1,x_2,\ldots x_m)\in[0,\infty)^m$ and $n_1,n_2,\ldots n_m\in N$, we define the modified multi-Szasz-Mirakjan operator $S_{n_1,n_1,\ldots,n_m}: L_{q,\alpha}\longrightarrow L_{q,\alpha}$ by:

$$S_{n_1,n_2,\dots,n_m}(f;x_1,x_2,\dots,x_m)=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\sum_{k_m=0}^{\infty}\sum_{k_{m-1}=0}^{\infty}\cdots\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(a_{n_i}x_i)^{k_i}}{k_i\,!}.$$
$$f\left(\frac{k_1}{b_{n_1}},\frac{k_2}{b_{n_2}},\dots,\frac{k_m}{b_{n_m}}\right).$$

where $\{a_{n_i}\}$ and $\{b_{n_i}\}$ are families of increasing sequence of positive integer numbers such that $\lim_{n_i\to\infty}\frac{1}{b_{n_i}}=0$ and $\frac{a_{n_i}}{b_{n_i}}=1+o(\frac{1}{b_{n_i}})$ for each $i=1,2,\dots,m.$

Lemma (4.1):

For each $1\le q<\infty$, $L_{q,\alpha}=\left\{\ f \mid f:[0,\infty)^m \longrightarrow R\right.$ is a continuous function with that $\left.\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m<\infty\ \right\}$ is a normed space, where $\omega_\alpha(x_1,x_2,\dots,x_m)=e^{\alpha\sum_{i=1}^{m}x_i}$, α is a positive real number.

Proof:

It is easy to check that $L_{q,\alpha}$ is a vector space.

Define $\|\,.\,\|_{q,\alpha}:L_{q,\alpha}\longrightarrow R^+\cup\{0\}$ by:

$$\|f\|_{q,\alpha}=\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}.$$

Then we prove $\|\,.\,\|_{q,\alpha}$ is a norm on $L_{q,\alpha}$. To do this, we must prove the following conditions:

(i) If $f=0$ then

$$\|f\|_{q,\alpha}=\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}=0.$$

Conversely let $\|\ f\ \|_{q,\alpha}=0$ then $\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q=0$ and hence $f(x_1,x_2,\dots,x_m)=0\quad\forall x_i\ge 0, i=1,2,\dots,m.$

Therefore $f=0.$

(ii) Let $f,g\in L_{q,\alpha}$ then

$$\|f+g\|_{q,\alpha}\le\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}+$$
$$\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{g(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$
$$=\|f\|_{q,\alpha}+\|g\|_{q,\alpha}$$

(iii) Let $\lambda\in\not\subset$ and $f\in L_{q,\alpha}$ then

$$\|\lambda f\|_{q,\alpha}=\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{(\lambda f)(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$

$$=\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left|\frac{(\lambda)f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$

$$=\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty|\lambda|^q\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$

$$=(|\lambda|^q)^{\frac{1}{q}}\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty|\lambda|^q\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$
$$=|\lambda|\ \ \|f\|_{q,\alpha}.$$

Therefore; $L_{q,\alpha}$ is a normed space.

Lemma (4.2):

Let $L_{n_1,n_2,\dots,n_m}$ be a uniformly bounded sequence of positive linear operators from such that $L_{q,\alpha}(R^m)$ into itself satisfying the condition $\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}}\left\|L_{n_1,n_2,\dots,n_m}f-f\right\|_{q,\alpha}=0.$

where $f(x_1,x_2,...,x_m)=1, x_j, \sum_{i=1}^{m} x_i^2$ for some $j\in\{1,2,...,m\}$ thus for every $f\in L_{q,\alpha}(R^m)$,

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| L_{n_1,n_2,...,n_m}(f)-f\right\|_{q,\alpha}=0.$$

The following proposition gives some properties of the operator $S_{n_1,n_2,...,n_m}$.

Proposition (4.3):

For any $x\in[0,\infty)^m$ and $n_1,n_2,...n_m\in N$, the following statements hold:

(1) $S_{n_1,n_2,...,n_m}(f;x_1,x_2,...,x_m)=1$, where $f(x_1,x_2,...x_m)=1$.

(2) $S_{n_1,n_2,...,n_m}(f;x_1,x_2,...,x_m)=\dfrac{a_{n_j}x_j}{b_{n_j}}$, where $f(x_1,x_2,...x_m)=x_j$ for some $j\in\{1,2,...,m\}$.

(3)

$$S_{n_1,n_2,...,n_m}(f;x_1,x_2,...,x_m)=\sum_{i=1}^{m}\left(\frac{a_{n_i}^{\,2}}{b_{n_i}^{\,2}}x_i^{\,2}+\frac{a_{n_i}}{b_{n_i}^{\,2}}x_i\right),$$

where $f(x_1,x_2,...x_m)=\sum_{i=1}^{m} x_i^2$.

Proof:

(1) Since $f(x_1,x_2,...x_m)=1$, then

$$S_{n_1,n_2,...,n_m}(f;x_1,x_2,...,x_m)=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\sum_{k_m=0}^{\infty}\sum_{k_{m-1}=0}^{\infty}...\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(a_{n_i}x_i)^{k_i}}{k_i!}$$

$$=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\cdot\prod_{i=1}^{m}e^{a_{n_i}x_i}$$

$$=1.$$

(2) Since $f(x_1,x_2,...x_m)=x_j$ for some $j\in\{1,2,...,m\}$, then

Therefore;

$$S_{n_1,n_2,...,n_m}(f;x_1,x_2,...,x_m)=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\sum_{k_m=0}^{\infty}\sum_{k_{m-1}=0}^{\infty}...\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(a_{n_i}x_i)^{k_i}}{k_i!}f\left(\frac{k_1}{b_{n_1}},\frac{k_2}{b_{n_2}},...,\frac{k_m}{b_{n_m}}\right)$$

$$=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\left[\prod_{\substack{i=1\\ i\neq j}}^{m}\sum_{k_i=0}^{\infty}\frac{(a_{n_i}x_i)^{k_i}}{k_i!}\sum_{k_j=0}^{\infty}\left(\frac{a_{n_j}x_j}{k_j!}\right)^{k_i}\left(\frac{k_j}{b_j}\right)\right]$$

$$=e^{-\sum_{\substack{i=1\\ i\neq j}}^{m}a_{n_i}x_i}\left[\prod_{\substack{i=1\\ i\neq j}}^{m}e^{a_{n_i}x_i}\frac{a_{n_j}x_j}{b_{n_j}}\right]$$

$$=\frac{a_{n_j}x_j}{b_{n_j}} \text{ for some } j\in\{1,2,...,m\}$$

(3) Since $f(x_1,x_2,...x_m)=\sum_{i=1}^{m} x_i^2$, then

$$S_{n_1,n_2,...,n_m}(f;x_1,x_2,...,x_m)=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\sum_{k_m=0}^{\infty}\sum_{k_{m-1}=0}^{\infty}...\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(a_{n_i}x_i)^{k_i}}{k_i!}f\left(\frac{k_1}{b_{n_1}},\frac{k_2}{b_{n_2}},...,\frac{k_m}{b_{n_m}}\right)$$

$$=e^{-\sum_{i=1}^{m}a_{n_i}x_i}\sum_{k_m=0}^{\infty}\sum_{k_{m-1}=0}^{\infty}...\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(a_{n_i}x_i)^{k_i}}{k_i!}\left[\left(\frac{k_1}{b_{n_1}}\right)^2+\left(\frac{k_2}{b_{n_2}}\right)^2+...+\left(\frac{k_m}{b_{n_m}}\right)^2\right]$$

$$=e^{-\sum_{i=2}^{m}a_{n_i}x_i}\prod_{i=2}^{m}\sum_{k_i=0}^{\infty}\left(\frac{a_{n_i}x_i}{k_i!}\right)^{k_i}\left[e^{-a_{n_1}x_1}\sum_{k_2=0}^{\infty}\left(\frac{a_{n_1}x_1}{k_1!}\right)^{k_1}\left(\frac{k_1}{b_{n_1}}\right)^2\right]+$$

$$e^{-\sum_{\substack{i=1\\ i\neq 2}}^{m}a_{n_i}x_i}\prod_{\substack{i=1\\ i\neq 2}}^{m}\sum_{k_i=0}^{\infty}\left(\frac{a_{n_i}x_i}{k_i!}\right)^{k_i}\left[e^{-a_{n_2}x_2}\sum_{k_2=0}^{\infty}\left(\frac{a_{n_2}x_2}{k_2!}\right)^{k_2}\left(\frac{k_2}{b_{n_2}}\right)^2\right]+$$

$$...+e^{-\sum_{i=1}^{m-1}a_{n_i}x_i}\prod_{i=1}^{m-1}\sum_{k_i=0}^{\infty}\left(\frac{a_{n_i}x_i}{k_i!}\right)\left[e^{-a_{n_m}x_m}\sum_{k_m=0}^{\infty}\left(\frac{a_{n_m}x_m}{k_m!}\right)^{k_m}\left(\frac{k_m}{b_{n_m}}\right)^2\right]$$

$$=e^{-\sum_{i=2}^{m}a_{n_i}x_i}\prod_{i=2}^{m}e^{a_{n_i}x_i}\left[\frac{a_{n_1}^{\,2}}{b_{n_1}^{\,2}}x_1^{\,2}+\frac{a_{n_1}}{b_{n_1}^{\,2}}x_1\right]+$$

$$e^{-\sum_{\substack{i=1\\ i\neq 2}}^{m}a_{n_i}x_i}\prod_{\substack{i=1\\ i\neq 2}}^{m}e^{a_{n_i}x_i}\left[\frac{a_{n_2}^{\,2}}{b_{n_2}^{\,2}}x_2^{\,2}+\frac{a_{n_2}}{b_{n_2}^{\,2}}x_2\right]+$$

$$...e^{-\sum_{i=1}^{m}a_{n_i}x_i}\prod_{i=1}^{m-1}e^{a_{n_i}x_i}\left[\frac{a_{n_m}^{\,2}}{b_{n_m}^{\,2}}x_m^{\,2}+\frac{a_{n_m}}{b_{n_m}^{\,2}}x_m\right]$$

$$=\sum_{i=1}^{m}\left[\frac{a_{n_i}^{\,2}}{b_{n_i}^{\,2}}x_i^{\,2}+\frac{a_{n_i}}{b_{n_i}^{\,2}}x_i\right].$$

Now, the following theorem shows that the operator $S_{n_1,n_2,...,n_m}$ converges to f.

Theorem (4.4):

Let $f \in L_{q,\alpha}$, then $S_{n_1,n_2,...,n_m} f \longrightarrow f$ as $n_1, n_2, ...n_m \to \infty$.

Proof:

Let $f(x_1, x_2, ...x_m) = 1, \quad \forall\ x \in [0,\infty)^m$ then by using proposition (4.3) one can have: $\lim\limits_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| S_{n_1,n_2,...,n_m} f - f \right\|_{q,\alpha} = 0.$ Also for

$f(x_1, x_2, ...x_m) = x_j$, for some $j \in \{1,2,...,m\}$ one can have:

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| S_{n_1,n_2,...,n_m} f - f \right\|_{q,\alpha} = \sum_{i=1}^{m} \left(\int_0^\infty \int_0^\infty ... \int_0^\infty \left| \frac{x_j}{\omega_\alpha(x_1, x_2, ..., x_n)} \right|^q dx_1 dx_2 ... dx_m \right)^{\frac{1}{q}}$$

$$\lim_{n_j \to\infty} \left(\frac{a_{n_j}}{b_{n_j}} - 1\right) = 0.$$

Moreover, consider $f(x_1, x_2, ...x_m) = \sum\limits_{i=1}^{m} x_i^2$ one can have:

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| S_{n_1,n_2,...,n_m} f - f \right\|_{q,\alpha} =$$

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left(\int_0^\infty \int_0^\infty ... \int_0^\infty \left| \frac{S_{n_1,n_2,...,n_m} f(x) - f(x)}{\omega_\alpha(x_1, x_2, ..., x_m)} \right|^q dx_1 dx_2 ... dx_m \right)^{\frac{1}{q}}$$

$$= \lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left(\int_0^\infty \int_0^\infty ... \int_0^\infty \left| \frac{\sum\limits_{i=1}^{m} \frac{a_{n_i}^{\ 2}}{b_{n_i}^{\ 2}} x_i^2 + \frac{a_{n_i}}{b_{n_i}^{\ 2}} x_i - \sum\limits_{i=1}^{m} x_i^2}{\omega_\alpha(x_1, x_2, ..., x_m)} \right|^q dx_1 dx_2 ... dx_m \right)^{\frac{1}{q}}$$

$$\le \sum_{i=1}^{m} \left(\int_0^\infty \int_0^\infty ... \int_0^\infty \left| \frac{x_i^{\ 2}}{\omega_\alpha(x_1, x_2, ..., x_m)} \right|^q dx_1 dx_2 ... dx_m \right)^{\frac{1}{q}}$$

$$\lim_{n_i\to\infty} \left(\frac{a_{n_i}^{\ 2}}{b_{n_i}^{\ 2}} - 1 \right)$$

$$+ \sum_{i=1}^{m} \left(\int_0^\infty \int_0^\infty ... \int_0^\infty \left| \frac{x_i}{\omega_\alpha(x_1, x_2, ..., x_m)} \right|^q dx_1 dx_2 ... dx_m \right)^{\frac{1}{q}}$$

$$\lim_{n_i\to\infty} \left(\frac{a_{n_i}}{b_{n_i}^{\ 2}} \right) = 0.$$

Thus, $\lim\limits_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| S_{n_1,n_2,...,n_m} f - f \right\|_{q,\alpha} = 0.$

By using lemma (4.2), one can get $S_{n_1,n_2,...,n_m} f \longrightarrow f$ as $n_1, n_2, ...n_m \to \infty$.

References

[1] Gadjiev A. and Aral A., "Weighted L_p-Approximation with Positive Linear Operators on Unbounded Sets", Appl. Math. Letters, Vol. 20, 2007, pp. 1046-1051.

[2] Ispir N., "On Modified Baskakov Operators On Weighted Space" Turk. J. Math. Vol. 26, No. 3, 2001, pp.355-365.

[3] Ispir N. and Atakut C., "Approximation by modified Szasz- Mirakjan Operators on Weighted Spaces", Pro. Indian Acad. Sci. Math., Vol.122, No.4, 2002, pp. 571-578.

[4] Sharma J. and Vasishtha A., "Functional Analysis", Published, Meerut. Eleventh edition, 1989, p. 263.

[5] Volkov V., On the Converges of Sequence of Linear Positive Operators in The Space of Continuous Functions of Two Variables" Math. Sb. N. Vol. 43, No. 52, 1957p.504.

[6] Tulin C. "Weighted Approximation of Unbounded Continuous Funcations by Sequences of Linear Positive Operators", Indian J. Pure. Appl. Math. Vol. 34, No. 3, 2003, pp. 477-485.

الخلاصة

في هذا البحث تم دراسة تقريب الدول المستمرة باستخدام بعض انواع من مؤثرات ساز- مارجن (مؤثرساز- مارجن المطورومؤثر ساز- مارجن المطور المتعدد) والمعرفة على بعض الفضاءات المعيارية.