N-C-Compactness

Ahmed Ibrahem Nasir and Rana Bahjat Esmaeel
Department of Mathematics, Ibn-Al-Haitham, College of Education,
University of Baghdad.

Abstract

In this paper, we defi ne another types of Compactness on bitopological spaces, namely "N-C-Compactness", "S-C-Compactness" and "pair-wise C-Compactness", and we review some remarks, propositions and examples are also reviewed about these types.

Also, we discussed the relationships among them.

1-Introduction and preliminaries

In 1963, the term of "bitopological space" was used for the first time by Kelly[1].A set equipped with two topologies is called a" bitopological space" and is denoted by $(X,\tau,\acute{\tau})$,where (X,τ), $(X,\acute{\tau})$ are two topological spaces.

In 2005,A.I.Nasir[2] introduced N-open, S-open sets, A subset A of a bitopological space $(X,\tau,\acute{\tau})$ is said to be an "N-open set" if and only if it is open in the space $(X,\tau\vee\acute{\tau})$, where $\tau\vee\acute{\tau}$ is the supremum topology on X contains τ and $\acute{\tau}$, and A is said to be "S-open set" if and only if it is $\tau-open$ or $\acute{\tau}$-open.

The complement of an N-open(S-open, respectively) set in a bitopological spaces $(X,\tau,\acute{\tau})$ is called "N-closed"(S-closed, respectively) set.

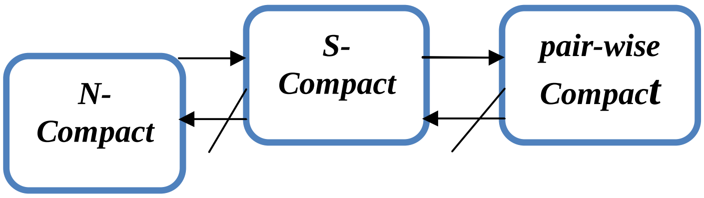
Obviously each S-open set in a bitopological space$(X,\tau,\acute{\tau})$ is N-open. But the converse is not true,[2].

A bitopological spaces $(X,\tau,\acute{\tau})$ is called "N-compact", (S-compact, respectively), space if and only if every N-open(S-open, respectively) cover of X has a finite subcover. [2],[3].

In 1996, Mrsevic and I.L.Reilly[3] introduced the concept of "pair-wise open cover". An S-open cover of a subset A of a space $(X,\tau,\acute{\tau})$ is called a "pair-wise open cover" if it is contains at least one non-empty element from τ and at least one non-empty element from$\acute{\tau}$.

A bitopological space $(X,\tau,\acute{\tau})$ is called a "pair-wise compact space" if every pair-wise open cover of X has a finite subcover, [3].

N-compactness, C-compactness and pair-wise compactness are discussed in[2], and the following diagram shows the relations among these different types of compactness



A topological spaces (X,τ) is said to be "C-compact space" if for each closed set $A\subseteq X$, each open cover of A contains a finite subfamily W such that $\{cl\,V : V\in W\}$ covers A , [4], [5].

Clearly, every compact space is C-compact, but the converse is not true[4].

The main goal of this paper is to give types of compactness on a bitopological space $(X,\tau,\acute{\tau})$, namely N-C-compactness, S-C-compactness and pair-wise c-compactness. We have studied the properties of such spaces and the relationships among them. We also gave many counter examples to improve the ones which are invalid and put conditions to the invalid direction true.

Definition (1.1):

A bitopological space $(X,\tau,\acute{\tau})$ is said to be "N-C-compact space" if for each N-closed set $A\subseteq X$, each N-open cover for

A has a finite subfamily W such that $\{N - cl\, U: U \in W\}$ covers A.

$N - cl\, U =$ the N-closure of U (the smallest N-closed set contain U).

In proposition (1.2) and remark (1.3) below we discuss the relationship between N-compact and N-C-compact spaces.

Proposition (1.2):

Every N-compact space is N-C-compact.

Proof: Clear

The opposite direction of Proposition (1.2) may be false, for example:

In the bitopological spaces (N, τ_I, τ) where τ_I is the indescrete topology on N and $\tau = \{U_n | U_n = \{1,2,\cdots,n\}, n \in N\} \cup \{N, \emptyset\}$

Then (N, τ_I, τ) is not N-compact space since, $\{U_n | U_n = \{1,2,\cdots,n\}, n \in N\}$ is an N-open cover for N which has no finite subcover. But (N, τ_I, τ) is N-C-compact space since the N-closure of any non-empty set in any N-open cover is N.

In proposition (1.3) and remark (1.4) below we discuss the relationship between the space $(X, \tau, \acute{\tau})$ is N-C-compact and the two spaces (X, τ) and $(X, \acute{\tau})$are C-compact.

Proposition (1.3):

If a space $(X, \tau, \acute{\tau})$ is N-C-compact then both (X, τ) and $(X, \acute{\tau})$ are C-compact spaces.

Proof:

Let A be a closed subset of a space (X, τ) and $\{U_\alpha : \alpha \in \Lambda\}$ be a τ- open cover for A, implies, A is N- closed subset of X and $\{U_\alpha : \alpha \in \Lambda\}$ be an N- open cover for A, which has a finite subfamily W such that $A \subseteq \cup \{N - cl\, u : u \in W\}$,and by virtue of $(N - cl\, u \subseteq \tau - cl\, u)$, then (X, τ) is C-compact space. ■

Similarity, one can prove that $(X, \acute{\tau})$ is C-compact space.

Remark (1.4):

The converse of proposition (1.3) is not true in general, for example: A space $(N, \tau, \acute{\tau})$ where, $\tau = \mathbb{P}(O^+) \cup \{N\}$ and $\acute{\tau} = \mathbb{P}(E^+) \cup \{N\}$

Then both (N, τ) and $(N, \acute{\tau})$ are compact spaces, so they are C-compact spaces. But $(N, \tau, \acute{\tau})$ is not N-C-compact space, since $\{\{n\} | n \in N\}$ is an N- open cover for N which has no finite subfamily W such that $N \subseteq \cup \{N - cl\, u_\alpha : u_\alpha \in W\}$. ■

In the following proposition we add some condition to make the opposite direction of proposition (1.3) hold.

Proposition (1.5):

If τ is a subfamily of $\acute{\tau}$ then $(X, \tau, \acute{\tau})$ is N-C-compact if and only if both (X, τ) and $(X, \acute{\tau})$ are C-compact.

Proof: clear. ■

If $(X, \tau, \acute{\tau})$ is an N-C-compact space and Y be a subset of X, then $(Y, \tau_Y, \acute{\tau}_Y)$ need not to be N-C-compact space, where τ_Y is the induced topology on Y with respect to τ, and $\acute{\tau}_Y$ is the induced topology on Y with respect to $\acute{\tau}$. As the following remark shows:

Remark (1.6):

N-C-compactness is not a hereditary property. For example :

Let $X = N \cup \{0\}$, τ_I be the indescret topology on X. And

$\tau = \mathbb{P}(N) \cup \{H \subseteq X | 0 \in H \wedge X - H \text{ is finite}\}$

Then $(X, \tau, \acute{\tau})$ is N-C-compact space, but $(N, \tau_I, \acute{\tau}_N)$ is not N-C-compact. ■

In the following Proposition we shall explain that N-C-compactness is weakly hereditary property.

Proposition (1.7):

The N-closed subset of an N-C-compact space is N-C-compact.

Proof: clear. ■

Definition (1.8),[2]:

A function $f:(X,\tau,t) \to (Y,T,\acute{T})$ is said to be "N-continuous function" if and only if the inverse image of each N-open subset of Y is an N-open subset of X.

Proposition (1.9):

The N-continuous image of an N-C-compact space is N-C-compact.
Proof: clear. ■

2. S-C-Compactness

Definition (2.1):

A bitopological space (X,τ,t) is said to be "S-C-compact space" if for each S-closed set $A \subseteq X$ and each S-open cover W for A, there is a finite subcover $\{u_i : i = 1,2,\cdots,n\}$ such that $A \subseteq \bigcup_{i=1}^{n} s - cl\ u_i$ where
S-c l U= the S-closure of U.
= the smallest S-closed set that contains U.

In proposition (2.2) and remark (2.3) below we discuss the relationship between S-compactness and S-C-compactness.

Proposition (2.2):

Every S-compact space is S-C-compact.
Proof: Follows from (the S-closed subset of an S-compact space is S-compact [2]) ■

Remark (2.3):

The opposite direction of Proposition (2.2) may be un true.

See the example of proposition (1.2), (N,τ_I,τ) is S-C-compact space which is not S-compact. ■

In proposition (2.4) and remark (2.5) below we discuss the relationship between the space (X,τ,t) is S-C-compact and both (X,τ) and (X,t) are C-compact.

Proposition (2.4):

If (X,τ,t) is an S-C-compact space, then both (X,τ) and (X,t) are C-compact spaces.
Proof: clear. ■

Remark (2.5):

The opposite direction of Proposition (2.4) is not true in general, for example:

Let $X = [0,1]$

$\tau = \{X, \emptyset, \{0\}\} \cup \{[0,\frac{1}{n}) | n \in N\}$ and

$t = \{X, \emptyset, (0.1]\} \cup \{(\frac{1}{n}, 1] | n \in N\}$

Then both (X,τ) and (X,t) are compact spaces, so C-compact spaces. but (X,τ,t) is not S-C-compact, since the S- open cover $\{(\frac{1}{n},1] | n \in N\}$ of an S-closed set $A = (0,1]$ has no finite subfamily W such that $A \subseteq \cup \{S - cl\ u_\propto : u_\propto \in W\}$.

The opposite direction of Proposition (2.4) becomes valid in a special case, as the following proposition shows:

Proposition (2.6):

If τ is a subfamily of t, then (X,τ,t) is S-C-compact if and only if both (X,τ) and (X,t) are C-compact spaces.
Proof: clear. ■

Remark (2.7):

A subset of an S-C-compact space need not to be S-C-compact. For example: see the example of remark (1.6).

Then (X,τ,t) is an S-C-compact space .But N is a subset of X which is not an S-C-compact set, since $\{\{n\} | n \in N\}$ is an S-open cover for N which has no finite subfamily W such that $\{S - cl\ u_\propto : u_\propto \in W\}$ covers N.

Remark (2.7) shows that the S-C-compactness is not a hereditary property, but it is weakly hereditary property, as the following proposition shows:

Proposition (2.8):

An S-closed subset of an S-C-compact space is an S-C-compact

Proof: clear. ■

Now, we will discuss the image of S-C-compact spaces under a bicontinuous functions.

Definition (2.9),[6]:

Let $f:(X,\tau,t) \to (Y,T,\acute{T})$ be a function,then f is said to be a "bicontinuous functions" if and only if $f^{-1}(u) \in \tau$, for each $u \in T$, and $f^{-1}(v) \in t$, for each $v \subset \acute{T}$.

Proposition (2.10):

A bicontinuous image of an S-C-compact spaces is S-C-compact.
Proof: clear. ■

Proposition (2.11):

If A and B are two S-C-compact subsets of $(X,\tau,\acute{\tau})$, then $A\cup B$ is S-C-compact.

Proof: clear. ■

Remark (2.12):

If A and B are two S-C-compact subsets of a bitopological space $(X,\tau,\acute{\tau})$, then $A\cap B$ need not to be S-C-compact.

For example: let $X=N\cup\{0,-1\}$

$\tau=\mathbb{P}(N)\cup\{H\subseteq X|0\in H\wedge X-H\ is\ finite\}$

and

$\acute{\tau}=\mathbb{P}(N)\cup\{H\subseteq X|-1\in H\wedge X-H\ is\ finite\}$,

let $A=N\cup\{0\}$ and $B=N\cup\{-1\}$, then both A and B are S-C-compact sets, but $A\cap B=N$

Which is not S-C-compact set since, the S-open cover $\{\{n\}|n\in N\}$ for N has no finite subfamily W such that $\{S-cl\ u_\propto:u_\propto\in W\}$ covers N, since $(S-cl\{n\}=\{n\}\ \forall n\in N)$.

3.Pair-wiseC-compactness

Definition (3.1):

A bitopological spaces $(X,\tau,\acute{\tau})$ is said to be" Pair-wise C-compact space" if for each τ-closed set $A\subset X$, every Pair-wise open cover for A has a finite subfamily W such that $\{\acute{\tau}-cl\ u_\propto:u_\propto\in W\}$ covers A.

Proposition (3.2):

Every Pair-wise compact space is Pair-wise C-compact.

Proof: clear. ■

Remark (3.3):

The implication in proposition (3.2) is not revisable.

For example: $(N,\tau_D,\acute{\tau})$, where $\tau=\tau_D$ the desecrate topology on N and $\acute{\tau}=\{u_n=\{1,2,\cdots,n\}|n\in N\}\cup\{N,\emptyset\}$

then $(N,\tau,\acute{\tau})$ is Pair-wise C-compact space. Since every Pair-wise open cover for any τ- closed set $A\subseteq N$, has at least one non- empty element $U\in\acute{\tau}$ and $\acute{\tau}\quad cl\ U=N$, but $(N,\tau,\acute{\tau})$ is not Pair-wise compact space since $W=\{\{1,2\},\{3\},\{4\},\{5\},\cdots\}$ is a Pair-wise open cover for N which has no finite subcover.

Remark (3.4):

i. If both $(X,\tau),(X,\acute{\tau})$ are C-compact spaces then $(X,\tau,\acute{\tau})$ may be not Pair-wise C-compact space. For example:

Let $X=N\cup\{0,-1\}$, $\tau=\mathbb{P}(N)\cup$ $\{H\subseteq X|0\in H\wedge X-H\ is\ finite\ set\}\cup$ $\{H\subseteq X|0\notin H\wedge -1\in H\}$

and $\acute{\tau}=\mathbb{P}(N)\cup$ $\{H\subseteq X|-1\in H\wedge X-H\ is\ finite\ set\}\cup$ $\{H\subseteq X|0\in H\wedge -1\notin H\}$

Both $(X,\tau),(X,\acute{\tau})$ are compact spaces so they are C-compact.but $(X,\tau,\acute{\tau})$ is not Pair-wise C-compact space since the Pair-wise open cover $\{\{n\},\{0\},\{-1\}|n\in N\}$ for X has no finite sub family W such that $\{\acute{\tau}-cl\ u:u\in W\}$ covers X.

ii. Both $(X,\tau),(X,\acute{\tau})$ are not C-compact spaces whenever $(X,\tau,\acute{\tau})$ is Pair-wise C-compact. For example:

See the example of remark (3.3), (X,τ_D) is not C-compact.

In remark (3.5) and proposition (3.6) below we will explain that a Pair-wise C-compactness is not hereditary property but it is weakly hereditary property.

Remark (3.5):

A Pair-wise C-compactness is not hereditary property. For example:

$X=N\cup\{0\}$, $\tau=\mathbb{P}(N)\cup\{X\}$ and $\acute{\tau}=\mathbb{P}(N)\cup$

$\{U\subset X|0\in U\wedge X-U\ is\ finite\}$, then $(X,\tau,\acute{\tau})$ is a Pair-wise C-compact space.

Since for every τ- closed set $A\subseteq X$, then $0\in A$ and every Pair-wise open cover for A has at least one element U and $0\in U$. So

$X - U$ is finite set and $\acute{\tau} - cl\,(U) = U$ and U with a finite member of that is cover will be covers A.

Now let $B = N$, then $\tau_B = \mathbb{P}(N)$ and $\acute{\tau}_B = \mathbb{P}(N)$. So $(B, \tau_B, \acute{\tau}_B)$ is not Pair-wise C-compact.

Proposition (3.6):

A τ- closed subset of a Pair-wise C-compact space $(X, \tau, \acute{\tau})$ is Pair-wise C-compact.

Proof: clear. ■

In the following proposition we will discuss the image of a Pair-wise C-compact spaces under the bicontinuous functions.

Proposition (3.7):

The bicontinuous image of a Pair-wise C-compact spaces is Pair-wise C-compact.

Proof: clear. ■

Proposition (3.8):

If A and B are two Pair-wise C-compact subset of $(X, \tau, \acute{\tau})$, then $A \cup B$ is Pair-wise C-compact

Proof: clear. ■

Remark (3.9):

If A and B are two Pair-wise C-compact subset of $(X, \tau, \acute{\tau})$, then $A \cap B$ need not to be.

For example:

Let $X = N \cup \{0, -1\}, \tau = \mathbb{P}(N) \cup \{H \subseteq X | 0, -1 \in H \wedge X - H \text{ is finite set}\}$ and, $\acute{\tau} = \{H \subseteq X | 0, -1 \in H\} \cup \{\{1\}, \emptyset\}$ Let $A = N \cup \{0\}$ and $B = N \cup \{-1\}$ then both $(A, \tau_A, \acute{\tau}_A)$ and $(B, \tau_B, \acute{\tau}_B)$ are Pair-wise C-compact spaces. since for every τ_A-closed set $C \subseteq A$, each Pair-wise open cover for C from τ_A and $\acute{\tau}_A$ must contain at least one element U such that $0 \in U$ and $\acute{\tau}_A - cl\ U = A$.

Similarly, B is Pair-wise C-compact.

But $A \cap B = N$ which is not Pair-wise C-compact since $\tau_N = \mathbb{P}(N) = \acute{\tau}_N$.

Now we will discuss the relationships among the concepts N-C-,S-C- and Pair-wise C-compactness.

Lemma (3.10):

Let $(X, \tau, \acute{\tau})$ be a bitopological space and $A \subseteq X$. then

1) N-closur $A \subseteq S - closur\ A$.
2) N-closur $A \subseteq \tau - closur\ A$. and N-closur $A \subseteq \acute{\tau} - closur\ A$.

Proof:

(1) Let $x \notin S - cl\,(A)$, then there is an S-closed set B such that $A \subseteq B$ and $x \notin B$ and since every S-closed set is N-closed, implies $x \notin N - cl\,(A)$.

(2) Let $x \notin \tau - cl\,(A)$, then there is a closed set B such that $A \subseteq B$ and $x \notin B$ and since every closed set is N-closed, implies $x \notin N - cl\,(A)$.

Similarity, one can prove that $N - cl\,(A) \subseteq \acute{\tau} - cl\,(A)$

Proposition (3.11):

Every N-C- compact space is S-C- compact.

Proof :

Since every S-open set is an N-open set and in virtue of Lemma (3.10), the proof is over. ■

Remark (3.12):

The implication in proposition (3.11) is not reversible. For example:

The bitopological space $(N, \tau, \acute{\tau})$ where, $\tau = \mathbb{P}(E^+) \cup \{N\}$ and $\acute{\tau} = \mathbb{P}(O^+) \cup \{N\}$. then, the family of all closed sets in a space (X, τ) is $F = \{B \subseteq N | O^+ \subseteq B\} \cup \{\emptyset\}$ and the family of all closed sets in a space $(X, \acute{\tau})$ is $\acute{F} = \{B \subseteq N | E^+ \subseteq B\} \cup \{\emptyset\}$.

Now, $(N, \tau, \acute{\tau})$ is S-C- compact space. Since for each an S-closed set $A \subseteq X$, then there are three probabilities:

(i) If $A \subseteq E^+$, then every S-open cover W for A must contain an element U such that there exist $n \in U \cap E^+$. So $E^+ \subseteq S - cl\ U$ since $S - cl\{n\} = E^+$ for each $n \in E^+$.

(ii) Similarly if $A \subseteq O'$, then $O^{+} \subset S - cl\ U$. when U is an element in any S-open cover for A and $\exists\ m \in U \cap O^{+}$.

(iii) If $A \cap O^{+} \neq \emptyset$, and $A \cap E^{+} \neq \emptyset$. Then for every S-open cover W for A must contain at least two elements U and V such that $U \cap E^{+} \neq \emptyset$ and $V \cap O^{+} \neq \emptyset$. So $E^{+} \subseteq S - cl\ U$and $O^{+} \subseteq S - cl\ V$, therefore $A \subseteq S - cl\ U \cup S - cl\ V = N$

But, $(N, \tau, \acute{\tau})$ is not N-C- compact space. since $\tau \vee \acute{\tau} = \mathbb{P}(N)$.

Proposition (3.13):

Every N-C-compact space is Pair-wise C-compact.

Proof:

Since, every Pair-wise open cover is an N-open cover and in virtue of lemma (3.10) the proof is over. ■

Remark (3.14):

The implication in Proposition (3.13) is not reversible. For example:

The bitopological space (N, ι_D, ι_I) is Pair-wise C-compact space which is not N-C-compact.

Remark (3.15):

The concepts S-C- compactness and Pair-wise C-compactness are independent. For examples:

i. The example of remark (3.14) is not S-C- compact The example of remark (3.12) is not Pair-wise C-compact. Since the Pair-wise open cover of the τ -closed set N $\{\{n\} | n \in N\}$ has no finite subfamily W such that $\{\acute{\tau} - cl\ u : u \in W\}$ covers N, because

$$\acute{\tau} - cl\{n\} = \begin{cases} E^{+} & \text{if } n \in E^{+} \\ E^{+} \cup \{n\} & \text{if } n \in O^{+} \end{cases}$$

4- Conclusion and Recommendations

Our conclusions in this paper, that an N -compact space is N-C-, S-C- and pair-wise C-Compact. But any one of them is not reversible. So we have to give the relationships among N-C-,S-C- and pair-wise C-compactness.

For future works, we shall study N-α-C-compactness, S-α- C-compactness and pair-wise α-C-compactness.

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الخلاصة

في هذا البحث، قمنا بتعريف انواع اخرى من التراص على الفضاءات الثنائية، اسميناها "الفضاءات المرصوصة- N-C" و "الفضاءات المرصوصة- S-C" و"الفضاء الثنائي الاولي المتراص-C". ودراسة بعض خواص هذه الفضاءات وناقشنا العلاقات بينها.