# Modified Krasovskii's Method for Constructing Liapunov Function

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**Abstract**

In this paper, we modify the Krasovskii's method for constructing the Liapunov functions, which uses the numerical methods for solving ODE's to estimate the stability of systems via studying the stability of numerical method.

## 1- Introduction

In 1982, the Russian scientist A. M. Liapunov proposed a method for studying the stability of critical points of dynamical systems without solving the system.

The method involving number of steps and theorems which are concerned with the existence of function that satisfy certain conditions, these functions are called Liapunov functions. The method has a great advantage and vast applications of studying the stability of certain systems without evaluating the eigenvalues, which they have in sometimes zero real part, [1].

Several methods for constructing Liapunov functions had been given and discussed in several literatures that deal with stability quantitative theory. These methods are of great importance, so it is necessary in our work to discuss the most familiar and appropriate methods of them and one of such methods is the Krasovskii method, [3].

## 2- Stability of a Set of Matrices

With a set of matrices A, we associate the larger set $A'$ (the semigroup generated by matrices in A) consisting of all products of matrices in A, in general $A'$ is an infinite set.

***Definition (2.1), [2]:***

A matrix M is said to be stable if there exists a constant k, such that $\|M^i\| \leq k$, for all $i = 1, 2, \ldots, n$, and thus M is stable if and only if $|\lambda(M)| \leq 1$, and the eigenvalues of M lies on the unite circle are simple, where $\lambda(M)$ refers to the eigenvalue of M.

***Definition (2.2), [2]:***

A set of $n\times n$ matrices A is said to be stable if for every neighborhood of the origin $U \subset C^n$, there exists another neighborhood of the origin V, such that for each $M \in A'$, $MV \subset U$.

***Lemma (2.1), [4]:***

A set of matrices A is stable if and only if there is a bounded neighborhood of the origin W, such that for each $M \in A'$, $MW \subseteq W$. Furthermore, W may be chosen to be convex and balanced.

## 3- The Construction of Liapunov Function

Let $\Omega$ be an open set in the region D containing the origin. Suppose that V(x) is a scalar continuous function defined on $\Omega$, then the basic definition of Liapunov function can be summarized as follows:

***Definition (3.1), [1]:***

A scalar function $V(x) : \Omega \longrightarrow$ is said to be positive definite on the set $\Omega$ if $V(0) = 0$ and $V(x) > 0$, for all $x \neq 0$ and $x \in \Omega$.

***Definition (3.2), [1]:***

A scalar function $V(x) : \Omega \longrightarrow$ is said to be positive semi definite on the set $\Omega$ when V(x) has positive sign throughout $\Omega$, except at certain points (including the origin) where it is zero.

***Definition (3.3), [1]:***

A scalar function $V(x) : \Omega \longrightarrow$ is said to be negative definite (negative semi definite) on the set $\Omega$ if and only if $-V(x)$ is positive definite (positive semi definite) on $\Omega$.

***Definition (3.4), [1]:***

The real symmetric $n\times n$ matrix B is said to be positive definite if the leading principal minors of B are all positive.

***Definition (3.5), [1]:***

The real symmetric $n \times n$ matrix B is said to be negative definite if and only if $-B$ is positive definite.

The, among the simplest positive functions is the quadratic form, given by:

$$V(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j \text{ , i, j, = 1, 2, …, n}$$

with symmetric matrix $Q = q_{ij}$. To test the positive definiteness of V(x), we can apply the Sylvester's criteria (see [1]) which asserts that a necessary and sufficient condition for V(x) to be positive definite is that the determinants of all principal minors of the matrix Q are positive.

## 4- Modified Krasovskii's Method

Let us consider the following autonomous system of differential equations:

$$\dot{x} = f(x) \text{ ........................................ (1)}$$

where x is n-dimensional vector, such that $x \in {}^{n}$, $\dot{x} = \frac{dx}{dt}$, $t \in [0, \infty]$ and:

$$f(x) = \sum_{i=1}^{n} p_i x_i^r \text{ , i = 1, 2, …, n}$$

such that $p_i$, r are constants.

The method is begin by rewriting eq.(1) as:

$$\dot{x} = K(x)x \text{ ........................................ (2)}$$

where K(x) is a matrix chosen in such a way such that $K(x)x = f(x)$, where:

$K(x) = k_{ij}(x)$

where:

$$k_{ij} = \begin{cases} \dfrac{\partial f_i / \partial x_j}{r}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Recall that Euler's formula for solving the initial value problem:

$\dot{x} = f(t, x)$, $y(0) = x_0$

is given by:

$x_{n+1} = x_n + hf(t_n, t_n)$

which may be rewritten using (2) as:

$$x_{n+1} = x_n + h_n K(x_n) x_n \text{ .................................. (3)}$$

where $h_n = t_{n+1} - t_n$, which is called the step size. For every $x_n \in {}^{n}$, $K(x_n)$ will be a real $n \times n$ matrix. If we let $S_n$ to denote the set of all matrices obtained by varying $x_n$ over all values, then we can rewrite (3) equivalently as:

$$x_{n+1} = (I_n + h_n K_n) x_n,\ K_n \in S_n \text{ ....................... (4)}$$

where $I_n$ denotes the $n \times n$ identity matrix.

In [4] and [6] it is shown that $x = 0$ of (4) is stable (globally asymptotically stable) for all sequences $\{h_n\}$, $0 \leq h_n \leq h'$, for some $h' > 0$, then the equilibrium point $x = 0$ of (1) is stable (globally asymptotically stable).

The above result may be restated in an equivalent way which makes use of the stability properties of a class of matrices.

Let $S_n$ denote the set of $n \times n$ matrices with the property that for all $x \in {}^{n}$, there exists $K \in S$, such that $f(x) = Kx$. Suppose that the set of $n \times n$ matrices A is given by:

$$A = \{I_n + hS_n\} \text{ .......................................... (5)}$$

is stable (asymptotically stable) for some $h < 0$, then the equilibrium point $x = 0$ of (1) is stable (globally asymptotically stable).

A Liapunov function for this system is given by:

$$V(x) = f^T(x).f(x)$$
$$= f_1^2 + f_2^2 + \ldots + f_n^2$$

***Theorem (4.1), [2]:***

Consider the autonomous system (1), and assume that the origin is an equilibrium point, i.e., $f(0) = 0$, then if:

$x_{n+1} = x_n + h_n f(x_n)$

is stable for all sequences $\{h_n\}$, such that $0 \leq h_n \leq h'$, then the autonomous system (1) is stable.

## 5- Illustrative Examples

The proposed method in (4) has been applied to some examples. We present three examples to illustrate the results of this paper:

***Example (5.1):***

In order to study the stability of the zero solution of the system:

$$\dot{x} = x^2 + 2y^2$$
$$\dot{y} = x^3 - 5y^3$$

We consider the step length h =0.0009, and letting $\dot{x} = f_1$, $\dot{y} = f_2$.

First, choose K such that $Kx = f(x)$, such that K has the entries:

$$k_{ij} = \frac{\partial f_i / \partial x_j}{r}$$

we take $f_1$, such that r = 2

$$\frac{\partial f_1}{\partial x} = 2x,\ \frac{\partial f_1}{\partial y} = 4y$$

Then:

$$k_{11} = \frac{2x}{2} = x,\ k_{12} = \frac{4y}{2} = 2y$$

Now, take $f_2$ such that r =3

$$\frac{\partial f_2}{\partial x} = 2x^2,\ \frac{\partial f_2}{\partial y} = -15y^2$$

Then:

$$k_{21} = \frac{3x^2}{3} = x^2,\ k_{22} = \frac{-15y^2}{3} = -3y^2$$

Hence, one can constructs the following matrix:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

and upon applying Eule's method, one gets:

$$x_{n+1} = x_n + hK(x_n)$$
$$= (I + hk)x_n$$
$$= Bx_n$$

where:

$$B = \begin{bmatrix} 1+hk_{11} & hk_{12} \\ hk_{21} & 1+hk_{22} \end{bmatrix}$$

Now, to find the eigenvalues of B using certain computer programs, one get the following results, $\lambda_1 = 1$, $\lambda_2 < 1$, then from definition (2.1) B is stable, and by using theorem (4.1), we get the system is stable. Also, the Liapunov function is:

$$V(x) = f_1^2 + f_2^2$$
$$= (x^2 + 2y^2)^2 + (x^2 - 5y^3)^2$$

***Example (5.2):***

Consider the following system:

$$\dot{x}_1 = -x_1 - 3x_2 + x_3$$
$$\dot{x}_2 = x_1 - 2x_2 - x_3$$
$$\dot{x}_3 = -x_1 - 2x_2 - x_3$$

With step length h = 0.0001.

Let $\dot{x}_1 = f_1$, $\dot{x}_2 = f_2$ and $\dot{x}_3 = f_3$. Since r = 1 in all functions.

Now, by using the presented method, we get:

$$\frac{\partial f_1}{\partial x_1} = -1,\ \frac{\partial f_1}{\partial x_2} = -3,\ \frac{\partial f_1}{\partial x_3} = 1$$
$$\frac{\partial f_2}{\partial x_1} = 1,\ \frac{\partial f_2}{\partial x_2} = -2,\ \frac{\partial f_2}{\partial x_3} = -1$$
$$\frac{\partial f_3}{\partial x_1} = -1,\ \frac{\partial f_3}{\partial x_2} = -2,\ \frac{\partial f_3}{\partial x_3} = -1$$

Hence:

$$K = \begin{bmatrix} -1 & -3 & 1 \\ 1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1-h & -3h & h \\ h & 1-2h & -h \\ -h & -2h & 1-h \end{bmatrix}$$

Now, to find the eigenvalues of B by using exciting computer program, one get the following results $\lambda_1 = 1$, $\lambda_2 > 1$ and $\lambda_3 > 0$, then by definition (2.1) B is unstable and hence the system is unstable.

The Liapunov function is:

$$V(x) = f_1^2 + f_2^2 + f_3^2$$
$$= (-x_1 - 3x_2 + x_3)^2 + (x_1 - 2x_2 - x_3)^2 + (-x_1 - 2x_2 - x_3)^2$$

***Example (5.3):***

To study the stability of the zero solution of the system:

$$\dot{x} = -x - y$$
$$\dot{y} = x^2 - y^2 + z^2$$
$$\dot{z} = -z^3 - y^3$$
$$\dot{w} = -w + x - y + z$$

Consider the step length h = 0.005 and by using the presented method, we get:

$\dot{x} = f_1$, r = 1

$$\frac{\partial f_1}{\partial x} = -1,\ \frac{\partial f_1}{\partial y} = -1,\ \frac{\partial f_1}{\partial z} = 0,\ \frac{\partial f_1}{\partial w} = 0$$

$\dot{y} = f_2$, r = 2

$$\frac{\partial f_2}{\partial x} = 2x,\ \frac{\partial f_2}{\partial y} = -2y,\ \frac{\partial f_2}{\partial z} = 2z,\ \frac{\partial f_2}{\partial w} = 0$$

$\dot{z} = f_3$, r = 3

$$\frac{\partial f_3}{\partial x} = 0,\ \frac{\partial f_3}{\partial y} = -3y^2,\ \frac{\partial f_3}{\partial z} = -3z^2,\ \frac{\partial f_3}{\partial w} = 0$$

$\dot{w} = f_4$, r = 1

$$\frac{\partial f_4}{\partial x} = 1,\ \frac{\partial f_4}{\partial y} = -1,\ \frac{\partial f_4}{\partial z} = 1,\ \frac{\partial f_4}{\partial w} = -1$$

Hence:

$$K = \begin{bmatrix} -1 & -1 & 0 & 0 \\ x & -y & z & 0 \\ 0 & -y^2 & -z^2 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Also:

$$B = \begin{bmatrix} 1-h & -h & 0 & 0 \\ xh & 1-yh & zh & 0 \\ 0 & -y^2h & 1-z^2h & 0 \\ h & -h & h & 1-h \end{bmatrix}$$

Now, to find the eigenvalues of B by using exciting computer program, one can get the following results $\lambda_1 = 1$, $\lambda_2 < 1$, $\lambda_3 < 1$ and $\lambda_4 < 0$, then by definition (2.1) B is stable, and by theorem (4.1) we get the system is stable

The Liapunov function is:

$$V(x) = f_1^2 + f_2^2 + f_3^2 + f_4^2$$
$$= (-x - y)^2 + (x^2 - y^2 + z^2)^2 + (-z^3 - y^3)^2 + (-w + x - y + z)^2$$

## 6- References


[1] Adil H., "Vector Liapunov Function of Large Scale System", M.Sc. Thesis, University of Technology, Baghdad, Ira1, 1995.

[2] Brayton K. and Tong H., "Stability of Dynamical System: A Constructive Approach", IEEE Trans. Circuits Systems", Vol.CAS.26, No.1, April, 1979, pp.224-234.

[3] Kaddoura I H., “Stability of Ordinary Differential Equations to Study Dynamic Systems, “M.Sc. Thesis, University of Baghdad, Baghdad, Iraq, 1986.

[4] Michel A. N., Miliier R. K. and Nan B. H., “Stability Analysis of Interconnected Systems Using Computer Generated Liapunov Functions”, IEEE Trans. Circuits System, Vol CAS-29, No.7, July, 1982, p. 431.

[5] Ogata K., “State Space Analysis of Control Systems”, Prentice-Hall, Inc. Englewood Cliffs, USA, 1967.

[6] Robert K. Brayton, "Constructive Stability and Asymptotic Stability of Dynamical Systems", IEEE Trans. On Circuits Systems, Vol.CAS.27, No.11, November, 1980, pp.1120-1130.



**الخلاصة**

في هذا البحث طورت طريقة (كاروسفسكي) لتكوين دوال ليابانوف لانظمة المعادلات التفاضلية. وذلك بأستخدام وا دخال طرق عددية تستخدم في حل المعادلات التفاضلية الاعتيادية لايجاد أستقرارية النظام من خلال أستقرارية الطريقة العددية.