Properties of the Characteristic Polynomials of P_n and C_n

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Abstract

In this paper, the computational formula for the generalized characteristic polynomials of graphs P_n and C_n are found. Also, we show the relation between the vertices and edges of the graphs and the coefficients of the characteristic polynomials. Finally, the eigenvalues of the polynomials are computed and their properties are studied.

Keywords: Characteristic polynomials, algebraic graph, eigenvalues of graphs.

1. Introduction

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics and its results have application in many areas of the computing and natural sciences.

Techniques from group theory and linear algebra assist in studying the structure and enumeration of graphs [2].

Depending on the specific problems and personal favor graph theorists use different kinds of matrices to represent a graph [5]. The algebraic properties of the matrix are used as a bridge between different kinds of structural properties of the graph [6].

The relation between the structural properties of the graph and the algebraic ones of the corresponding matrix is a very interesting one.

Structural properties of the graph can be derived from the characteristic polynomial [5].

Many researches are interested in the study of the characteristic polynomials of the graphs, for more details see [1, 4, 5].

1. Algebraic Graph Theory

Consider a finite, simple, undirected graph G with the set of vertices $V(G)$ such that the order of the graph is the number of the vertices in $V(G)$, the set of edges $E(G)$ such that $e \in E$ if $e = v_i v_j$ where v_i and $v_j \in V(G)$ and the degree of $v_i \in V(G)$ is the number of incidence edges on v_i denoted by $deg(v_i)$ [6,p. 25].

Note that for a set M\{x} we will frequently write M-x. Likewise, G-v denotes the subgraph of G=(V,E) induced by the vertices V-v.

Definition 2.1, [3, p.65] :

If e=*uv* is an edge of G, then *contraction* of e is the operation of replacing *u* and *v* by a single vertex *w* whose incident edges are the edges other than e that were incident to *u* or *v*. The resulting graph denoted G-e, has one less edge than G, see figure (1).

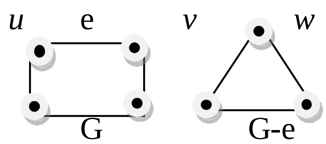


Fig. (1).

Let V= $\{v_1, v_2, \dots, v_n\}$ and write $v_i \sim v_j$ if the vertices v_i and v_j are adjacent. Then we define the *adjacency* matrix $A(G) = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & if \;\; v_i \sim v_j \\ deg(v_i) & if \;\; v_i = v_j \\ 0 & else \end{cases}$$

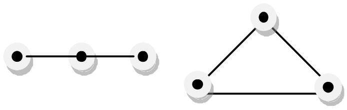
Note that for the undirected graph the matrix $A(G)$ is symmetric [6].

Remark:

All the previous works are with adjacency matrix of the from, $A(G) = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & if \;\; v_i \sim v_j \\ 0 & else \end{cases}$$

A *walk* of length l between two vertices v and u is a sequence of vertices $v = v_0, v_1, \dots v_s = u$ (not necessarily distinct), such that for any i the vertices v_i and v_{i+1} are adjacent. If all vertices are distinct then the walk is called a *path*, if the path has n vertices and m= (n-1) edges then it is denoted by P_n. If the first vertex of P_n (say v_0) equal the end one v_m (v_0=v_m) then the path is closed and called *cycle*, if the cycle with n vertices and m=n edges it is denoted by C_n for example see figure(2), [3,p. 14].



path (P_3) cycle (C_3)

Fig. (2).

The *distance* $d(u,v)$ is the length of a shortest path in G from vertex u to vertex v.

Further, the diameter diam(G) of a graph G is the length of a longest path in G, [3, p. 54].

2. Properties of the Characteristic Polynomial

Let us denote the characteristic polynomial of an $n \times n$ matrix A as follows:

$$\varphi(A,\lambda) = \sum_{i=0}^{n} b_i \lambda^i \text{ ....} (1)$$

In this section we explained the properties of $\varphi(P_n,\lambda)$ and $\varphi(c_n,\lambda)$ as follow:

1) The path with three vertices and two edges which is P_3 having the adjacency matrix A as follow:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\varphi(P_3,\lambda) = det[A - \lambda I] = -\lambda^3 + 4\lambda^2 - 3\lambda$$

By repeating the above procedure we get the following table of the coefficients of $\varphi(P_n,\lambda)$, for $n = 2,3,\dots,7$.

Table (1).

n	b_7	b_6	b_5	b_4	b_3	b_2	b_1
2	0	0	0	0	0	1	-2
3	0	0	0	0	-1	4	-3
4	0	0	0	1	-6	10	-4
5	0	0	-1	8	-21	20	-5
6	0	1	-10	36	-56	35	-6
7	-1	12	-55	120	-126	56	-7

From these results we conclude the recurrence relation of P_n:

$$\varphi(P_{n+1},\lambda) = (2-\lambda)\,\varphi(P_n,\lambda) - \varphi(P_{n-1},\lambda) \text{ ....} (2)$$

The properties of the characteristic polynomial according to the adjacency matrix are explained by the following proposition.

<u>Proposition (1):</u>

Let P_n be a path with n vertices, $\varphi(P_n,\lambda)$ be its characteristic equation which satisfy eq.(2) and λ be its eigenvalue then $\varphi(P_n,\lambda)$ satisfy the following properties:

i) The degree of $\varphi(P_n,\lambda)$ is n.

ii) $b_n = (-1)^n$.

iii) $b_{n-1} = 2(n-1)(-1)^{n-1}$.

iv) $b_{n-2} = (2n^2 - 7n + 6)(-1)^{n-2}$.

v) $b_1 = -n$.

vi) $b_0 = 0$.

vii) The coefficients of $\varphi(P_n,\lambda)$ are alternating.

Proof:

We prove this proposition using mathematical induction hypotheses on the number of the vertices n (order of P_n**).**

For $n = 2$ we have

with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

then

$$\varphi(P_2,\lambda) = \lambda^2 - 2\lambda \text{ ....} (3)$$

For $n = 3$ we have

P_3

with $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

then

$\varphi(P_3, \lambda) = -\lambda^3 + 4\lambda^2 - 3\lambda$ ....(4)

from above result (i) is satisfied, since $n = 2$ and the degree of eq.(3) is 2, and when $n = 3$ the degree of eq. (4) is 3.

for (ii) we have in eq.(3)

$b_2 =$ (+1) which is $(-1)^2$ and for eq.(4) $b_2 = (-1)$ which is $(-1)^3$ so the result is hold.

for (iii) b_1 in eq.(3) is

−2=2 (2−1)(−1) and b_2 in eq.(4) is 4= 2(3−1) $(-1)^2$.

for (iv) b_1 in eq.(4) is

−3= [(2(3)² −7(3) +6] (−1).

In eq.(3) and eq.(4) b_1 is −2 and −3 respectively so (v) is hold.

for (vi) b_0 in eq.(3) and eq.(4) are zero.

for (vii) the coefficients of eq.(3) and eq.(4) are alternating so its hold.

We assume that the properties of this proposition from (i) to (vii) hold for all characteristic equation of order equal or less than (n−1).

Now to prove the properties from (i) to (vii) for characteristic equation of order n.

The characteristic equation of order n is characteristic equation of P_n, so if we remove end vertex with its edge from P_n we obtain P_{n-1} which has characteristic equation of order (n−1) such that it is satisfy the properties from (i) to (vii) by induction. So we have the equation

$$\varphi(P_{n-1},\lambda) = (-1)^{n-1}\lambda^{n-1} + (-1)^{n-2}2(n-2)\lambda^{n-2} + (-1)^{n-3}[2(n-1)^2 - 7(n-1) + 6]\lambda^{n-3} + (-1)^{n-4}\lambda^{n-4} \ldots - (n-1)\lambda .$$

When we remove another end vertex with its edge from P_{n-1} we obtain P_{n-2} which is satisfy properties from (i) to (vii) by induction with characteristic equation

$$\varphi(P_{n-2},\lambda) = (-1)^{n-2}\lambda^{n-2} + 2(-1)^{n-3}(n-3)\lambda^{n-3} + (-1)^{n-4}[2(n-3)^2 - 7(n-3) + 6]\lambda^{n-4} + (-1)^{n-5}\lambda^{n-5} - \ldots - (n-2)\lambda .$$

using eq.(2) we have

$$\begin{aligned} \varphi(P_n,\lambda) &= (2-\lambda)\varphi(P_{n-1},\lambda) - \varphi(P_{n-2},\lambda) \\ &= (2-\lambda)[(-1)^{n-1}\lambda^{n-1} + (-1)^{n-2}2(n-2)\lambda^{n-2} + (-1)^{n-3}[2(n-1)^2 - 7(n-1) + 6]\lambda^{n-3} + (-1)^{n-4}\lambda^{n-4} + \cdots - (n-1)\lambda] - [(-1)^{n-2}\lambda^{n-2} + 2(-1)^{n-3}(n-3)\lambda^{n-3} + (-1)^{n-4}[2(n-3)^2 - 7(n-3) + 6]\lambda^{n-4} + (-1)^{n-5}\lambda^{n-5} + \cdots - (n-2)\lambda] \\ &= (-1)^n\lambda^n + 2(-1)^{n-1}(n-1)\lambda^{n-1} + (-1)^{n-2}[2n^2 - 7n + 6]\lambda^{n-2} + \cdots - n\lambda \end{aligned}$$

The last equation satisfies all properties of this proposition from (i) to (vii) so the proposition is hold. ∎

2) The cycle C_3 with three vertices has the adjacency matrix A as follow:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\varphi(C_3,\lambda) = det[A - \lambda I] = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

By repeating the above procedure we get the coefficients of the characteristic polynomials of C_n, with n=3, 4, 5, 6 which are shown in the following table:

Table (2).

n	b_6	b_5	b_4	b_3	b_2	b_1	b_0
3	0	0	0	-1	6	-9	4
4	0	0	1	-8	20	-16	0
5	0	-1	10	-35	50	-25	4
6	1	-12	54	-112	105	-36	0

From above results we conclude the recurrence relation

$$\varphi(C_{n+1},\lambda) = (2-\lambda)\,\varphi(C_n,\lambda) - \varphi(C_{n-1},\lambda) \quad ....(5)$$

eq.(5) holds whenever, $b_1 = -n^2$, and b_0 is 4 if n is an odd number.

The properties of the characteristic polynomial according to the adjacency matrix can be explained in the following proposition.

Proposition (2):

Let C_n be a cycle with n vertices and $\varphi(C_n,\lambda)$ its characteristic equation and all the coefficients of this characteristic polynomial except b_1 and b_0 satisfy eq.(5), then $\varphi(C_n,\lambda)$ posses the following properties.

i) The degree of $\varphi(C_n,\lambda)$ is n.

ii) $b_n = (-1)^n$.

iii) $b_{n-1} = 2n(-1)^{n-1}$.

iv) $b_{n-2} = (2n^2 - 3n)(-1)^{n-2}$.

v) $b_1 = -n^2$.

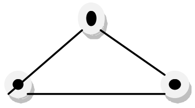
vi) $b_0 = \begin{cases} 0 & \text{if } \text{n even} \\ 4 & \text{if } \text{n odd} \end{cases}$

vii) The coefficients of $\varphi(C_n,\lambda)$ are alternating.

Proof:

We prove this proposition using mathematical induction hypotheses on the number of the vertices n (order of C_n).

For $n = 3$ we have



C_3

with $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

then

$$\varphi(C_3,\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

For (i) the degree of $\varphi(C_3,\lambda)$ is 3.

for (ii) $b_3 = (-1)^3 = -1$.

for (iii) $b_2 = 6 = 2 \times 3 \times (-1)^2$.

for (iv) $b_1 = -9$

$= [(2 \times 3^2) - (3 \times 3)](-1)^1$.

for (v) $b_1 = -(3^2)$, since n=3, which is odd integer then $b_0 = 4$, and the polynomial with alternating coefficients, such that all properties of the proposition from (i) to (vii) are hold, assume that these properties are hold for all graphs of order equal or less than (n-1).

Now we have to prove these properties for all graphs with n vertices.

If we contract the edge e= *uv* in C_n, we obtain another graph with (n-1) vertices and (n-1) edges which is C_{n-1} and by induction its satisfies the properties from (i) to (vii) such that

$$\varphi(C_{n-1},\lambda) = (-1)^{n-1}\lambda^{n-1} + (-1)^{n-2}2(n-1)\lambda^{n-2} + (-1)^{n-3}[2(n-1)^2 - 3(n-1)]\lambda^{n-3} + \cdots - (n-1)^2\lambda + 4.$$

By repeating the above reduction on C_{n-1} we obtain C_{n-2} with (n-2) vertices and edges such that

$$\varphi(C_{n-2},\lambda) = (-1)^{n-2}\lambda^{n-2} + 2(-1)^{n-3}(n-2)\lambda^{n-3} + (-1)^{n-4}[2(n-2)^2 - 3(n-1)]\lambda^{n-4} + \ldots - (n-2)^2\lambda.$$

Note that if (n-1) is odd then (n-2) even, by using eq.(4) we have

$$\varphi(C_n,\lambda) = (2-\lambda)o(C_{n-1},\lambda) - \varphi(C_{n-2},\lambda)$$

$$= [(-1)^{n-1}2\lambda^{n-1} + (-1)^{n-2}4(n-1)\lambda^{n-2} + (-1)^{n-3}2[2(n-1)^2 - 3(n-1)]\lambda^{n-3} + \cdots - 2(n-1)^2\lambda + 8 - [(-1)^{n-1}\lambda^{n} + (-1)^{n-2}2(n-1)\lambda^{n-1} + (-1)^{n-3}[2(n-1)^2 3(n-1)]\lambda^{n-2} + \cdots - (n-1)^2\lambda^2 + 4\lambda] - [(-1)^{n-2}\lambda^{n-2} + (-1)^{n-3}2(n-2)\lambda^{n-3} + (-1)^{n-4}[2(n-2)^2 3(n-1)]\lambda^{n-4} + - (n-2)^2\lambda\ ]$$

$$= (-1)^n\lambda^n + (-1)^{n-1}2n\lambda^{n-1} + (-1)^{n-2}[2n^2 - 3n]\lambda^{n-2} + \cdots + (-1)n^2\lambda + \begin{cases} 0 & \text{if } n \text{ even} \\ 4 & \text{if } n \text{ odd} \end{cases}$$

The last equation satisfy the properties from (i) to (vii), in table(2) the sequence of b_1, is −9, −16, −25, −36… have nth term $(-n^2)$.

Since the sequence of b_0 in table (2) is, 4, 0, 4, 0, 4, 0… it's clear that if n is odd then this coefficients is 4 otherwise equal zero. ■

3. Spectral Radius of the Graphs P_n and C_n

In this section we introduce some properties of the eigenvalues of the graphs P_n and C_n according to their characteristic polynomials.

In the following table the values of the eigenvalues of $\varphi(p_n, \lambda)$ for n= 3, 4,…, 8 are given

Table (3).

P_n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
P_3	0	1	3					
P_4	0	0.585	2	3.414				
P_5	0	0.382	1.382	2.618	3.618			
P_6	0	0.282	1.0	2	3	3.732		
P_7	0	0.198	0.753	1.555	2.445	3.247	3.801	
P_8	0	0.152	0.585	1.234	2	2.765	3.414	3.84

The following table contains the values of the eigenvalues of C_n for n= 3, 4,…, 7

Table (4).

C_n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
C_3	1	1	4				
C_4	0	2	2	4			
C_5	0.382	0.382	2.618	2.618	4		
C_6	0	1	1	3	3	4	
C_7	0.198	0.198	1.555	1.555	3.247	3.247	4

Now, let $\rho(G)$ denotes the spectral radius of the graph G. From tables (3) and (4) we conclude that

1) $\rho(p_n) \leq \rho(C_n) \leq 4$.
2) All the eigenvalues are real.
3) Every eigenvalues of C_n is an eigenvalues of p_n except the maximum one.
4) The distinct number of the eigenvalues of p_n and C_n equal to diam(G)+1.
5) $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ for all eigenvalues of p_n and C_n.
6) The eigenvalues of p_n is a subset of the eigenvalues of p_{3n} and so on, we satisfy
$\{\lambda_{p_n}\} \subseteq \{\lambda_{p_{3n}}\} \subseteq \{\lambda_{p_{6n}}\} \subseteq \cdots$
and
$\{\lambda_{C_n}\} \subseteq \{\lambda_{C_{3n}}\} \subseteq \{\lambda_{C_{9n}}\} \subseteq \cdots$ for n= 1,2,…
where $\{\lambda_{p_n}\}$ means the set of all eigenvalues of p_n.

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الخلاصة

في هذا البحث، تم إيجاد صيغة حسابية للمتعدد المميز العام للبيانات P_n و C_n. كما بينا العلاقة بين رؤوس وحافات البيان ومعاملات المتعدد المميز. كما تم حساب القيم الخاصة التابعة للمتعدد ودراسة خواصها.