A Note on an R-Modules with Nearly Pure Intersection Property

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Abstract

Let R be a commutative ring with identity and let M be a left R–module. A submodule$N$ of an R-module M is said to be nearly pure of M (abbreviated by N- pure), if $N \cap IM = IN + J(M) \cap (N \cap IM)$, for each ideal $I$of R; where $J(M)$ is the Jacobson radical of M. The main purpose of this paper is to introduce the concept of modules with the Nearly Pure intersection property, andwe show that if $M = A \oplus B$, has the NPIP and $f: A \to B$ is an R −homomorphism for any N – pure submodules A and B, then kernel f is N-pure.

Keywords: Pure submodule, Nearly Pure submodule, module with the Nearly Pureintersection property.

Introduction

A submodule$N$ of an R- module M is said to be pure submodule, if for every finitely generated ideal $I$of R, $N \cap IM = IN$, [2]. Also Al-Bahraany in [1] observed that an R-module M has the pure intersection property (abbreviated by PIP), if the intersection of any two pure submodules is again pure.

In this paper we introduce the concept of an R–module M has nearly- pure intersection property (abbreviated by NPIP), *i.e.*, the intersection of any two nearly pure submodules is nearly pure. We show that for $M = \underset{i \in I}{\oplus} M_i$. If A_i is N- pure submodules of M_i for each $i \in I$, then $\oplus_{i \in I} A_i$ is N-pure submodule in M proposition (1.5).

This work consists of two sections. In section one, we study some properties of nearly pure submodules. In section two, we give the definition of modules with the Nearly Pure intersection property (NPIP) with some properties. We prove that if M is an R– module with the NPIP. Then for every decomposition $M = A \oplus B$ and for every R- homomorphism $f: A \to B$, $Kerf$ is N- pure in M, see cor. (2.8).

1- Properties of Nearly Pure Submodules:

Recall that a submodule N of an R-module M is called Nearly Pure (abbreviated byN- pure):

$$N \cap IM = IN + J(M) \cap (N \cap IM)$$

for each ideal I of R[3]. It is clear that each pure submoduleis $N - pure$, but the converse is not true. For example, it is easily checked that $\{\bar{0}, \bar{2}\}$ in Z_4 is N- pure but not pure.

Remarks (1.1):

1- It is clear that each summand of an R- module M is N- pure submodule of M.

2- Let M be an R -module and let N be a pure submodule of M. If K is N-pure submoduleof N, then K is N-pure submodule of M.

To show this, let I be an ideal of a ring R, since $IK + J(M) \cap (K \cap IM) \subseteq K \cap IM$, so we have to show only $K \cap IM \subseteq IK + J(M) \cap (K \cap IM)$. Since N is pure in M and K is N- pure in N, then $N \cap IM = IN$and $K \cap IN = IK + J(N) \cap (K \cap IN)$.

Now since $K \subseteq N$, therefor $K \cap IM \subseteq N \cap IM = IN$.Then$K \cap IM \subseteq IN \cap K = IK + J(N) \cap (K \cap IN)$, but $J(N) \cap (K \cap IN) \subseteq J(M) \cap (K \cap IM)$. Thus $K \cap IM \subseteq IK + J(M) \cap (K \cap IM)$ Which implies that K is N-pure in M.

Proposition (1.2):

Let M be an R-module and let N be N- pure submodule of M. If K is a submodule of M containing N and $J(M) \cap K = J(K)$, then N is N- pure submodule of K.

Proof:

Let I be an ideal of R, since N is N-pure in M, then $N \cap IM = IN + J(M) \cap (N \cap IM)$ since $K \subseteq M$.Then $N \cap IK \subseteq N \cap IM = IN + J(M) \cap (N \cap IM)$. Thus $N \cap IK \subseteq [IN + J(M) \cap (N \cap IM)] \cap IK = IN + J(M) \cap (N \cap IK) = IN + J(M) \cap (K \cap N \cap IK) =$

$IN+(J(M)\cap K)\cap(N\cap IK)$. Since $J(M)\cap K=J(K)$, then $N\cap IK\subseteq IN+J(K)\cap(N\cap IK)$. Also since $+J(K)\cap(N\cap IK)\subseteq N\cap IK$, then $N\cap IK=IN+J(K)\cap(N\cap IK)$. Hence N is N-pure in K.

Proposition (1.3):

Let M be an R- module and N be N- pure submodule of M.If K is a submodule of $N\cap J(M)$, then $\frac{N}{K}$ is N- pure submodule of $\frac{M}{K}$.

Proof:

Let I be an ideal of R, we have to show

$$\frac{N}{K}\cap I\left(\frac{M}{K}\right)\subseteq I\left(\frac{N}{K}\right)+J\left(\frac{M}{K}\right)\cap\left[\frac{N}{K}\cap I\left(\frac{M}{K}\right)\right]$$

Since N is N- pure in M, then$N\cap IM=IN+J(M)\cap(N\cap IM)$.

Now,$\frac{N}{K}\cap I\left(\frac{M}{K}\right)=\frac{N}{K}\cap\frac{IM+K}{K}=\frac{(N\cap IM)+K}{K}=\frac{IN+J(M)\cap(N\cap IM)+K}{K}=\frac{IN+K}{K}+\frac{J(M)\cap(N\cap IM)+K}{K}=I\left(\frac{N}{K}\right)+\frac{J(M)}{K}\cap\frac{(N\cap IM)+K}{K}=I\left(\frac{N}{K}\right)+\frac{J(M)}{K}\cap\left(\frac{N}{K}\cap\frac{IM+K}{K}\right)$.

Since $\frac{J(M)}{K}\subseteq J\left(\frac{M}{K}\right)$,

then$I\left(\frac{N}{K}\right)+\frac{J(M)}{K}\cap\left(\frac{N}{K}\right)\cap I\left(\frac{M}{K}\right)\subseteq I\left(\frac{N}{K}\right)+J\left(\frac{M}{K}\right)\cap\left[\left(\frac{N}{K}\right)\cap I\left(\frac{M}{K}\right)\right]$, on the other hand,

$I\left(\frac{N}{K}\right)+J\left(\frac{M}{K}\right)\cap\left[\left(\frac{N}{K}\right)+I\left(\frac{M}{K}\right)\right]\subseteq\frac{N}{K}\cap I\left(\frac{M}{K}\right)$.

Hence$\frac{N}{K}\cap I\left(\frac{M}{K}\right)=I\left(\frac{N}{K}\right)+J\left(\frac{M}{K}\right)\cap\left[\frac{N}{K}\cap I\left(\frac{M}{K}\right)\right]J$,

thus$\frac{N}{K}$ is N- pure in $\frac{M}{K}$.

Proposition (1.4):

Let M be an R-module,$N$and Kbe submodules of Msuch that$K\subseteq N\cap J(M)$. If K is N- pure submodule of M and $\frac{N}{K}$ is N- pure submodule of $\frac{M}{K}$, and $J\left(\frac{M}{K}\right)=\frac{J(M)}{K}$, then N is N- pure submodule of M.

Proof:

Let I be an ideal of a ring R. We have to show$N\cap IM=IN+J(M)\cap(N\cap IM)$

Since $\frac{N}{K}$ is N- pure in $\frac{M}{K}$, then$\frac{N}{K}\cap I\left(\frac{M}{K}\right)=I\left(\frac{N}{K}\right)+J\left(\frac{M}{K}\right)\cap\left[\left(\frac{N}{K}\right)\cap I\left(\frac{M}{K}\right)\right]$. Since $J\left(\frac{M}{K}\right)=\frac{J(M)}{K}$. Therefore, $\frac{N}{K}\cap I\left(\frac{M}{K}\right)=I\left(\frac{N}{K}\right)+\frac{J(M)}{K}\cap\left[\left(\frac{N}{K}\right)\cap I\left(\frac{M}{K}\right)\right]$, hence $\frac{N}{K}\cap\frac{IM+K}{K}=\frac{IN+K}{K}+\frac{J(M)}{K}\cap\left[\left(\frac{N}{K}\right)\cap\frac{IM+K}{K}\right]$, thus $\frac{N\cap(IM+K)}{K}=\frac{IN+K}{K}+\frac{J(M)\cap N\cap(IM+K)}{K}=\frac{(IN+K)+J(M)\cap N\cap(IM+K)}{K}$, then $(N\cap IM)+K=(IN+J(M)\cap N\cap IM)+K$

Now, let $x\in N\cap IM\subseteq N\cap(IM+K)=IN+J(M)\cap(N\cap IM)+K$, then $x=w+m+k$, where $w\in IN$ and $m\in J(M)\cap(N\cap IM)$ and $k\in K$, hence $k=x-w-m\in K\cap IM=IK+J(M)\cap(K\cap IM)\subseteq IN+J(M)\cap(N\cap IM)$.

Proposition (1.5):

Let $M=\oplus_{i\in I}M_i$. If A_i is N- pure submodules of M_i for each $i\in I$, then $\oplus_{i\in I}A_i$ is N- pure submodule in M.

Proof:

Let K be an ideal in a ring R and let $x=\sum_{j=1}^{n}r_jx_j\in(\oplus_{i\in I}A_i)\cap K(\oplus_{i\in I}M_i)$; $r_j\in K$ and $x_j\in\oplus_{i\in I}M_i$. Then $x_j=\sum_{i\in I}m_{ij}$; $m_{ij}\in M_i$ for each $i\in I$. Hence$x=\sum_{j=1}^{n}r_j\sum_{i\in I}m_{ij}=\sum_{i\in I}\sum_{j=1}^{n}r_j\,m_{ij}$.

Since$\sum_{j=1}^{n}r_jm_{ij}\in M_i$and $M=\oplus_{i\in I}M_i$,then$x$ can be written uniquely as $\sum_{i\in I}\sum_{j=1}^{n}r_jm_{ij}$, but $x\in\oplus_{i\in I}A_i$, then $\sum_{j=1}^{n}r_jm_{ij}\in A_i\forall i$,since$A_i$ is N- pure in M_i, then$\sum_{j=1}^{n}r_jm_{ij}\in A_i\cap KM_i=KA_i+J(M_i)\cap(A_i\cap KM_i)$.Thus $\sum_{j=1}^{n}r_j\,m_{ij}=\sum_{j=1}^{n}r_j\,a_{ij}+w_i$; $a_{ij}\in A_i$for each $j$and $w_i\in J(M_i)\cap(A_i\cap KM_i)$, Hence $x=\sum_{i\in I}\left(\sum_{j=1}^{n}r_{ij}a_{ij}+w_i\right)=\sum_{j=1}^{n}r_j\sum_{i\in I}a_{ij}+\sum_{i\in I}w_i\in K(\oplus_{i\in I}A_i)+\oplus_{i\in I}J(M_i)\cap[(\oplus_{i\in I}A_i)\cap K(\oplus_{i\in I}M_i)]$. Since $J(M)=J(\oplus_{i\in I}M_i)=\oplus_{i\in I}J(M_i)$, thus$x=\sum_{j=1}^{n}r_j\sum_{i\in I}a_{ij}+\sum_{i\in I}w_i\in K(\oplus_{i\in I}A_i)+J(\oplus_{i\in I}M_i)\cap[(\oplus_{i\in I}A_i)\cap K(\oplus_{i\in I}M_i)]$, hence $\oplus_{i\in I}A_i$ is N- pure in $\oplus_{i\in I}M_i$.

2-The Nearly Pure Intersection Property:

In this section we introduced the concept of Nearly Pureintersection property (NPIP), and we illustrated it by some examples and some basic properties.

Definition (2.1):

An R-module has the Nearly Pure intersection property (abbreviated byNPIP) if the intersection of any two N- pure submodule is again N-pure.

Examples and Remarks (2.2):

1. If an R–modules M has the PIP, then it has the (NPIP).
2. An R- modules M is said to have N- regular if every submodule of M is N- pure, [3], so every N- regular R- module has the NPIP. In fact Q as Z - module has the NPIP.
3. If an R-modules M has the NPIP, then every N-pure submodule of M has the NPIP.
4. Consider the module $M = Z_4 \oplus Z_2$ as a Z - module, let $A = Z_4 \oplus 0$ and $B = Z(1,1)$, the submodule generated by $(1,1)$. It is clear that each of A and B is summand of M and hence A and $B$are N-pure of M. But $A \cap B = \{(0,0),(2,0)\}$ is not N-pure in M.

Proposition (2.3):

Let N be N-pure submodule of an R-module M, such that $N \subseteq J(M)$. M has the NPIP if and only if $\frac{M}{N}$ has the NPIP and $J\left(\frac{M}{N}\right) = \frac{J(M)}{N}$.

Proof:

Suppose that M has the NPIP and let $\frac{A}{N}, \frac{B}{N}$ be two N - pure submodules of $\frac{M}{N}$ .Since N is N - pure submodules of M, then by prop.(1.4) A and $B$are N - pure submodules of M. But M has the NPIP, then $A \cap B$ is N - pure in M, hence by prop.(1.3) $\frac{A\cap B}{N} = \frac{A}{N} \cap \frac{B}{N}$ is N- pure in $\frac{M}{N}$, which implies that $\frac{M}{N}$ has the NPIP.

Conversely, let A and B be N - pure submodules of M, let N be a submodule of A and Brespectively, then by prop.(1.3) $\frac{A}{N}$ and $\frac{B}{N}$ are N - pure submodules of $\frac{M}{N}$. Since $\frac{M}{N}$.has the NPIP, then $\frac{A}{N} \cap \frac{B}{N} = \frac{A\cap B}{N}$ is N- pure in $\frac{M}{N}$. Thus by prop.(1.4)$A \cap B$ is N - pure submodule of M.

The following theorem gives a another property of modules have the NPIP.

Theorem (2.4):

Let Mbe an R- module, if$M$ has the NPIP, then$I(A\cap B) + J(M) \cap [(A\cap B)\cap IM] = (IA\cap IB) + J(M)\cap[(A\cap B)\cap IM]$ for every ideal I of R and for every N-pure submodule $A$and $B$of$M$.

Proof:

Suppose that M has the NPIP and let A and B be N-pure submodules of M, then $A \cap B$ is N-pure in M. Let I be an ideal in R, then $(A\cap B)\cap IM = I(A\cap B) + J(M)\cap[(A\cap B)\cap IM]$, hence $I(A\cap B) + J(M)\cap[(A\cap B)\cap IM] \subseteq (IA\cap IB) + J(M)\cap[(A\cap B)\cap IM]$, and$(IA\cap IB) + J(M)\cap[(A\cap B)\cap IM] \subseteq A\cap(B\cap IM) = (A\cap B)\cap IM = I(A\cap B) + J(M)\cap[(A\cap B)\cap IM]$. Thus $(IA\cap IB) + J(M)\cap[(A\cap B)\cap IM] = I(A\cap B) + J(M)\cap[(A\cap B)\cap IM]$.

Now, we have the following Theorem.

Theorem (2.5):

Let M be an R-module with the NPIP, then for every N - pure submodules A and B of M and for every R-homomorphism $f: A\cap B \to M$ such that $A \cap Imf = \{0\}$ and $A + Imf$ is pure in M, $\ker f$ is N pure in M.

Proof:

Let A and B be N – pure submodule of M, let $T = \{x + f(x) : x \in A\cap B\}$. We claim that T is N-pure submodule of M. Let $y = \sum_{i=1}^{n} r_i m_i \in T\cap IM$; $r_i \in I = \langle r_i | i \rangle, m_i \in M, i = 1, \dots, n$; then $y = \sum_{i=1}^{n} r_i m_i = x + f(x)$,hence$y \in (A\cap B) + Imf \subseteq A + Imf$. But $A + Imf$ is pure in Mthus $y \in (A + Imf)\cap IM = I(A + Imf)$. Hence $y = \sum_{i=1}^{n} r_i(x_i + y_i)$; $x_i \in A$, $y_i \in Imf$,for all $i = 1, \dots, n$. Thus $\sum_{i=1}^{n} r_i(x_i + y_i) = x + f(x)$. Then $x - \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i y_i - f(x) \in (A\cap Imf) = \{0\}$, then $x = \sum_{i=1}^{n} r_i x_i \in (A\cap B)\cap IA \subseteq (A\cap B)\cap IM$. But $A\cap B$ is N – pure in M, hence$(A\cap B)\cap IM = I(A\cap B) + J(M)\cap[(A\cap B)\cap IM]$,thus

$$x \in I(A\cap B) + J(M)\cap[(A\cap B)\cap IM]$$

Hence $x = \sum_{i=1}^{n} r_i z_i + w$, where $z_i \in A\cap B, w \in J(M)\cap[(A\cap B)\cap IM]$, then$f(x) = \sum r_i f(z_i) + f(w)$,hence$y = x + f(x) = \sum_{i=1}^{n} r_i(z_i + f(z_i)) + w + f(w) \in IT + J(M)\cap(T\cap IM)$.Thus $T\cap IM = IT + J(M)\cap(T\cap IM)$ and T is N – pure in M.
We claim that: $\ker f = (A\cap B)\cap T$:
Let $x \in \ker f$, then $x \in (A\cap B)\cap T$, let $x \in (A\cap B)\cap T$, then $x = w + f(w), x \in A \cap B$, then $x - w = f(w) \in A\cap Imf = \{0\}$. Thus $f(x) = f(w) = 0$ and $x \in \ker f$.Since M has the NPIP, then $(A\cap B)\cap T = \ker f$ is N – pure in M.

By the same argument one can prove.

Theorem (2.6):

Let M be an R- module. If M has the NPIP, then for every N - pure submodules A and B of M and for every R homomorphism $f: A \cap B \to C$, where C is a submodule of M such that $A \cap C = \{0\}$ and $A + C$ is pure in M, $\ker f$ is N - pure in M.

The following corollaries follow immediately from theorem (2.6).

Corollary (2.7):

Let M be an R–module with the NPIP. Let A and B be N - pure submodules of M such that $A \cap B = \{0\}$ and A +B is N - pure in M. Then for each R–homomorphism $f: A \to B$, $\ker f$ is N-pure in M.

Corollary(2.8):

Let M be an R module with the NPIP. Then for every decomposition $M = A \oplus B$ and for every R-homomorphism $f: A \to B$, $\ker f$ is N - pure in M.

Proof:

Since $A \cap B = \{0\}$ and $A + B = M$ is N- pure in M and $A = A \cap M$, then by Theorem (2.6), $\ker f$ is N - pure in M.

Proposition(2.9):

Let M be an R module, if $M \oplus \frac{M}{N}$ has the NPIP for every submodule N of M, then M is N - regular.

Proof:

Let N be a submodule of M. There is a natural map $\pi: M \to \frac{M}{N}$. Since $M \oplus \frac{M}{N}$ has the NPIP, then by corollary.(2.8) $\ker \pi = N$ is N - pure in M. Hence M is N – regular.

By the same argument of the previous proof and by [3], we get:

Corollary(2.10):

Let M be an R module, if $M \oplus \frac{M}{Rm}$ has the NPIP, for every $0 \neq m \in M$, then every cyclic submodule of M is N – pure. In fact if M is projective R - module, then M is N – regular.

Recall that a submodule N of an R - module M is called *Fully invariant* if for every endomorphism $f: M \to M$, $f(N) \subseteq N$, [4].

Proposition(2.11):

Let $M = \oplus_{i \in I} M_i$ where M_i is a submodule of M. If M has the NPIP, then each M_i has the NPIP. The converse is true if each N - pure submodule of M is fully invariant and $J(M) \cap M_i = J(M_i)$ for each $i \in I$.

Proof:

Suppose that M has the NPIP. Since M_i is summand of M, then M_i is N - pure in M, and hence M_i has the NPIP.

For the converse, let A be an N-pure submodule of M and $\pi_i: M \to M_i$ be the natural epimorphism on M_i for each $i \in I$. Let $a \in A$, then $a = \sum_{i \in I} m_i$, $m_i \in M$. $\pi(a) = m_i$. Since A is N-pure in M, then A is fully invariant and hence $\pi_i(A) \subseteq A \cap M_i$. Thus $\pi_i(a) = m_i \in A \cap M_i$. Thus $a \in \oplus_{i \in I}(A \cap M_i)$. Therefore $A \subseteq (A \cap M_i)$. But $\oplus_{i \in I}(A \cap M_i) \subseteq A$. Thus $A = \oplus_{i \in I}(A \cap M_i)$.

Now, if A and B are N - pure submodules of M, then

$$A \cap B = \left(\oplus_{i \in I}(A \cap M_i)\right) \cap \left(\left(\oplus_{i \in I}(B \cap M_i)\right)\right)$$
$$= \oplus_{i \in I}\left((A \cap M_i) \cap (B \cap M_i)\right)$$

Since A is N-pure in M, $A \subseteq M_i$ and $J(M) \cap M_i = J(M_i)$ for each $i \in I$. Then by proposition (1.2) A is N pure in M_i. But M_i has the NPIP, thus $A \cap M_i$ is N - pure in M_i, also $B \cap M_i$ is N-pure in M_i, hence $(A \cap M_i) \cap (A \cap M_i)$ is N-pure in M_i. By proposition (1.5), $\oplus_{i \in I}\left((A \cap M_i) \cap (B \cap M_i)\right)$ is N-pure in $\oplus_{i \in I} M_i = M$. Therefore $A \cap B$ is N-pure in M.

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد وليكن M مقاسا على R يسمى المقاس الجزئي N نقي تقريبا في M، اذا كان: $N \cap IM = IN + J(M) \cap (N \cap IM)$ لكل مثالي I في R الهدف الرئيسي من هذا البحث هو دراسة مفهوم المقاسات التي تمتلك خاصية التقاطع النقي تقريبا (NPIP)، وبينا اذا كان $M = A \oplus B$ تمتلك خاصية التقاطع النقي تقريبا و $f: A \to B$ تشاكل حلقي حيث A و B مقاسين جزئيين نقيين تقريبا فان نواة f نقي تقريبا في M.