Solution of Nonlinear Ordinary Delay Differential Equations Using Variational Approach

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Abstract

The main objective of this paper is to study the variational formulation of nonlinear problems, which may be given in general as a nonlinear ordinary differential equation, nonlinear partial differential equation, nonlinear integral equation , etc. and then introducing the variational formulation of nonlinear ordinary delay differential equations with its solution using the proposed approach given by Tonti.

Keyword: Nonlinear Ordinary Delay Differential Equations,Variational Approach.

Introduction

Many problems in science and engineering may be formulated as a variational problem of ordinary delay differential equations where certain functional whose domain consists of a certain vector space of functions of an independent and dependent variables with its derivatives up to certain order and the delay terms may occur in one or more of these variables, is required to be minimized, [1].

The variational calculus gives a method for finding the maximal and minimal values of functionals. Problems that consist of finding the maxima or the minima of a functional are called variational problems, [2]. As an example, the solution of any problem (such as partial differential equations, ordinary differential equations, integral equations, etc.) is equivalent to the problem of minimizing a functional that corresponding to this problem, [3].

The basic analysis of the subject of calculus of variation depends mainly on minimizing some functional with a suitable condition to be satisfied. However, the problem of evaluating this functional has some difficulties and therefore, much attention on Tonti's approach is given for every nonlinear operator equation of the form N(u) = 0, [4].

In addition, delay differential equations occur as in the works of L. Euler (in the second half of the eighteenth century), but systematically the study of such equations was first considered in the twentieth century, to meet the demands of applied science, in particular of control theory. The significance of these equations lies in their ability to describe processes with after effect. The importance of these equations in various branches of technology, economics, biology and medical sciences have been recognized recently and has caused mathematicians to study with increasing interest, [5].

Preliminary Concepts in Calculus of Variation

Following are some of the fundamental concepts related to this paper considering the subject of calculus of variation and its inverse problem for evaluating the related functional of certain problem. Throughout this work, U and V are assumed to be vector spaces and L : D (L) $\subseteq$ U $\longrightarrow$ R (L) $\subseteq$ V be a linear operator, where D(L) is the domain of L and R(L) is the range.

The following definitions seem to be necessary in this paper.

Definition (1), [3]:

Let $<.,.>: \mathrm{U} \times \mathrm{V} \rightarrow \mathbb{R}$ be a bilinear form, a linear operator L: D (L) $\subseteq$ U $\longrightarrow$ R(L) $\subseteq$ V is said to be symmetric with respect to the chosen bilinear form <. , .> if L satisfie: $< \mathrm{u}_2, \mathrm{Lu}_1 >=< \mathrm{u}_1, \mathrm{Lu}_2 >, \forall \mathrm{u}_1, \mathrm{u}_2 \in \mathrm{D(L)}$

Definition (2), [6]:

An operator L: D (L) $\subseteq$ U $\longrightarrow$ R (L) $\subseteq$ V is said to be positive definite if the following two conditions are satisfied:

a-$< u, Lu >> 0, \forall u \in D(\mathrm{L})$ and u $\neq 0$

b-$< u, Lu >= 0 \; and \; only \; if \; u = 0$

Definition (3), [4]:

Let N: D (N) $\subseteq$ U $\longrightarrow$ R(N) $\subseteq$ V be a nonlinear operator. Then the Gateaux derivative of N at u is an operator and is denoted by $\acute{N}_u$ defined by:

$\acute{N}_u(\phi)=\frac{d}{d\varepsilon}[N(u+\varepsilon\phi)]_{\varepsilon=0},\forall\ u \in D(N),\varepsilon\in \mathbb{R}$

where ϕ is chosen such that $u+\varepsilon\phi$ belongs to D(N) for every u.

Ordinary Delay Differential Equations, [7]

An n-th order ordinary delay differential equation may take the following form:

$$F(t; y(t-\tau_1(t)),\dots,y(t-\tau_k(t)),y'(t-y_1(t)),\dots,y'(t-\tau_k(t)),\dots,y^{(n)}(t),\dots,y^n(t-\tau_k(t)) = g(t) \quad \dots\dots (1)$$

where F is a given function and $\tau_1(t)$, $\tau_2(t)$, …, $\tau_k(t)$ are given real valued functions called "time delays" .

For simplicity, one can write the first order delay differential equation with constant coefficients and with one delay as follows:

$$a_0y'(t) + a_1y'(t-\tau(t)) + b_0y(t) + b_1y(t-\tau(t)) = g(t) \quad \dots\dots (2)$$

Equation (3) can be classified into three kinds: the first kind (retarded) occurs when ($a_0 \neq 0$ and $a_1 = 0$), i.e., the delay comes in y only, and the differential equation takes the form:

$$a_0y'(t) + b_0y(t) + b_1y(t-\tau(t)) = g(t) \quad \dots\dots (3)$$

The second kind (neutral) is obtained when ($a_1 \neq 0$ and $b_1 = 0$), i.e., the delay comes in y′, and the differential equation takes the form:

$$a_0y'(t) + a_1y'(t-\tau(t)) + b_0y'(t) = g(t) \quad \dots\dots (4)$$

The third kind (mixed) is obtained when ($a_1 \neq 0$ and $b_1 \neq 0$), i.e., the delay comes in y and y′, and the differential equation also takes the form:

$$a_0y'(t) + a_1y'(t-\tau(t)) + b_0y(t) + b_1y(t-\tau(t)) = g(t) \quad \dots\dots (5)$$

Tonti's Approach, [4]

Tonti in 1984 gave the variational formulation for every nonlinear equation, with ordinary or partial derivatives, of any order (odd or even). This equation may be of an integral or integro-differential type, or it may even be a system of differential or integral equations, etc.

Now, in order to find the variational formulation corresponding to Tonti's approach, we follow the following steps:

Step (1)

Find an integral operator K of the form $Ku(x) = \int_a^b k(x,t)u(t)dt$ that transform the given problem into another problem, with the following two conditions must be satisfied:

(1) The integral operator must be invertible to ensure that the new problem has the same solution as the original one.

(2) The operator of the new problem must be a gradient of a functional with respect to the bilinear form.

Step (2)

Find a functional F, such that the operator of the new problem is the gradient of F.

Now the procedure that Tonti had used to find the variational formulation for every nonlinear problem is given in the following theorem:

Theorem (1), [4]:

Consider the nonlinear problem:

$$N(u) = 0$$

where N: D (N) $\subseteq$ U $\longrightarrow$ R (N) $\subseteq$ V is a nonlinear operator, such that:

1- The solution of this problem exists and unique.

2- D(N) is simply connected.

3- $N'_u(\varphi)$ exists, where φ is an arbitrary element in D(N).

4-$D(N'_u)$ is dense in U.

5-$N'^*_u(\varphi)$ is invertible for every $u\in D(N)$, where * stands for the adjoint operator.

Then for every operator K that satisfy the following conditions:

6- K is linear with D (K)$\supseteq$R(N) and R(K)$\subseteq$(N'^*_u).

7- K is invertible.

8- K is symmetric.

Then the operator $\overline{N}$ defined by:

$$\overline{N}=N'^*_u(KN(u)) \quad \dots\dots (7)$$

has the following properties.

a- Its domain coincides with that of N.

b-The problems $N(u) = 0$ and $\bar{N}(u) = 0$ have the same solution;

c- It is a gradient of a certain functional.

From properties (b) and (c) it follows that the solution of problem (6) is the critical point of the functional:

$$\bar{F}[u] = \frac{1}{2} < N(u), KN(u) >$$

Whose gradient the operator $\bar{N}$.The functional vanishes when the solution is reached.

Moreover if:

9- K is positive definite, then $\bar{F}[u]$ has its minimum value at the critical point.

The Variational Formulation of Nonlinear Ordinary Delay Differential Equations

In this section, we use Tontie's approach to find the variational formulation of nonlinear ordinary delay differential equations with single constant delay τ:

$\frac{d}{dt}u(t) = F(t, u(t), u(t-\tau))$, $t \geq 0$, $u \in C^1[0, T]$. ....(8)

with initial condition

$u(t) = g(t)$, $-\tau \leq t \leq 0$

where F is a nonlinear function g is a continuous function, and to solve this problem for all $t \geq 0$ using the method of steps in connection with Tontie's approach given by theorem (1).

A first step we must check if the related operator of eq.(8) satisfies the conditions of theorem (1). Now, let:

$N(u) = \frac{d}{dt}u(t) - F(t, u(t), u(t-\tau))$, where $u(t) = g(t), -\tau \leq t \leq 0$,

$u(t_0) = g(t_0)$, $u \in C^1[0, T]$.

$= \frac{d}{dt}u(t) - F(t, u(t), g(t-\tau))$, $u(t_0) = g(t_0)$, $u \in C^1[0,T]$

by using the method of steps ,since $g(t-\tau)$ is a function of t then

$N(u) = \frac{d}{dt}u(t) - F_1(t, u(t))$, $t \geq 0$,

$u \in C^1[0,T]$ .... (9)

$u(t_0) = g(t_0)$, $-\tau \leq t \leq 0$

then, equation (9) we will get an ordinary differential equation which is indeed the Cauchy problem and then existed and the uniqueness of the solution is assured,[8] therefore condition (1) of theorem(1) is satisfied .

Now, if u_1 , $u_2 \in D(N)$ then $\eta(t)=\lambda u_1(t)+(1-\lambda)u_2(t) \in D(N)$, $\lambda \in [0,1]$ since the addition of continuously differentiable functions is a continuously differentiable function hence the D(N) is convex set and hence is simply connected since every convex set is simply connected,[9] therefore condition (2) of theorem (1) is satisfied .

Now, $N(u + \varepsilon\phi) = \frac{d}{dt}(u + \varepsilon\phi)(t) - F_1(t, (u + \varepsilon\phi)(t)), \phi \in D(N)$.........................(10)

$$N'_u(\phi) = \frac{d}{d\varepsilon}[N(u + \varepsilon\phi)]_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}[\frac{d}{dt}(u + \varepsilon\phi)(t) - F_1(t, (u + \varepsilon\phi))(t)]_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}[\frac{d}{dt}(u + \varepsilon\phi)(t)]_{\varepsilon=0} - \frac{d}{d\varepsilon}[F_1(t, (u + \varepsilon\phi))(t)]_{\varepsilon=0}$$

$$= \frac{d}{dt}\phi(t) - \frac{\partial F_1}{\partial u}\phi(t)$$

Hence the Gateaux derivative is:

$N'_u(\phi) = \frac{d}{dt}\phi(t) - \frac{\partial F_1}{\partial u}\phi(t)$, $\phi(0) = 0$, $\phi \in D(N)$

Therefore condition (3) of theorem (1) is satisfied .

In this case $D(N'_u)$ is formed of class $C^1[0,T]$ .Is form a dense subset of U with respect to the topology induced by norm $||u|| = \sqrt{\int_0^T u^2(t)dt}$, therefore condition (4) of theorem (1) is satisfied.

Now, in order to find the adjoint of Gateaux derivative, we must satisfy:

$$\int_0^T \psi N'_u(\phi)dt = \int_0^T \phi N'^*_u(\psi)dt, \psi, \phi \in C^1[0, T] \quad \text{................................(11)}$$

Take the left hand side, we get:

$$\int_0^T \psi N'_u(\phi)dt = \int_0^T \psi\left[\frac{d}{dt}\phi(t) - \frac{\partial F_1}{\partial u}\phi(t)\right]dt$$

$$= \int_0^T \psi(t)\frac{d}{dt}\phi(t)dt - \int_0^T \psi(t)\frac{\partial F_1}{\partial u}\phi(t)dt$$

$$\int_0^T \psi N'_u(\phi)dt = \int_0^T \psi(t)\frac{d}{dt}\phi(t)dt - \int_0^T \phi(t)\frac{\partial F_1}{\partial u}\psi(t)dt$$

and using the method of integrating by parts to the $\int_0^T \psi(t)\frac{d}{dt}\phi(t)dt$,we get:

$$\int_0^T \psi N_u' (\phi)dt = \psi(t)\phi(t)|_0^T - \int_0^T \phi(t)\frac{d\psi(t)}{dt}dt - \int_0^T \phi(t)\frac{\partial F_1}{\partial u}\psi(t)dt$$

Since $\psi(0) = \psi(T) = \phi(0) = \phi(T) = 0$, hence:

$$\int_0^T \psi N_u' (\phi)dt = -\int_0^T \phi(t)\frac{d\psi(t)}{dt}dt - \int_0^T \phi(t)\frac{\partial F_1}{\partial u}\psi(t)dt$$

$$=\int_0^T \phi[-\frac{d\psi(t)}{dt} - \frac{\partial F_1}{\partial u}\psi(t)]dt$$

Substituting in eq. (11), we get

$$\int_0^T \phi[-\frac{d\psi(t)}{dt} - \frac{\partial F_1}{\partial u}\psi(t)]dt = \int_0^T \phi\, N_u'^*(\psi)dt$$, hence:

$$N_u'^*(\psi) = -\frac{d\psi(t)}{dt} - \frac{\partial F_1}{\partial u}\psi(t), \psi(T) = 0, \psi \in C^1[0,T]$$

Then, the adjoint homogeneous problem is given by

$$-\frac{d\psi(t)}{dt} - \frac{\partial F_1}{\partial u}\psi(t) = 0, \psi(T) = 0, \psi \in U$$

which is a linear equitation with variable coefficients and can be solved to have the null solution, [8], [2], and thus the operator $N_u'^*$ is invertible since (a linear operator L is invertible if and only if homogeneous equation LU=0 has the null solution) [10]. Therefore conditions (5) of theorem (1) is satisfied.

Also, choose the integral operator to be

$$Kv = \int_0^T e^{st}\phi(t)\phi(s)v(s)ds$$

where ϕ satisfy the homogeneous initial conditions for $D(N_u'^*)$, which is $\phi(T) = 0$ and $\phi(0) = 0$. Therefore, letting $\phi(t) = t^2 - Tt$, yields:

$$Kv = \int_0^T e^{st}(t^2 - Tt)(s^2 - Tt)v(s)ds$$

and hence from theorem (4.1), we have the functional:

$\bar{F}(u) = \frac{1}{2}\langle N(u), KN(u)\rangle$, where the bilinear form $< u, v >$ is taking to be

$< u, v > = \int_0^T u(t)v(t)dt$,

$u \in C^1[0,T]$, $v \in C[0,T]$.

$$\bar{F}(u) = \frac{1}{2}\int_0^T \left(\frac{du}{dt} - F(t,u(t))\right)\left(\int_0^T e^{st}(t^2 - Tt)(s^2 - Ts)\left(\frac{du}{ds} - F(s,u(s))\right)\right)dsdt \quad \text{...............................(12)}$$

which can be minimized using the direct Ritz method to find the critical points of eq.(12) which are equivalent to the solution of the nonlinear ordinary delay differential equation given by eq.(8) with initial condition.

As an illustration consider the following example:

Example:

Consider the nonlinear delay differential equation:

$$\frac{du}{dt} = u^2(t) + u(t-1) - t^4 + t + 1 , t \geq 0$$

with initial condition:

$u(t) = g(t) = t$, $t \in [-1,0]$

which has for compares the exact solution $u(t) = t^2$ .This example can be solved for the first time step [0, 1] and by using the method of steps, then we have:

$$\frac{du(t)}{dt} = u^2(t) + u(t-1) - t^4 + t + 1$$
$$= u^2(t) + g(t-1) - t^4 + t + 1$$
$$= u^2(t) + 2t - t^4$$
$$= F(t,u(t))$$

Hence, using the variational formulation (12), we have the functional:

$$\bar{F}(u) = \frac{1}{2}\int_0^1 \left(\frac{du}{dt} - F(t,u(t))\right)\left(\int_0^1 e^{st}(t^2 - t)(s^2 - s)\left(\frac{du}{ds} - F(s,u(s))\right)\right)dsdt$$

$$= \frac{1}{2}\int_0^1 \left(\frac{du}{dt} - u^2(t) - 2t + t^4\right)\left(\int_0^1 e^{st}(t^2 - t)(s^2 - s)\left(\frac{du}{ds} - u^2(s) - 2s + s^4\right)\right)dsdt \quad \text{...............................(13)}$$

and by using the direct Ritz method, let:

$u(t) = a_1 + a_2 t + a_3 t^2$

and since $u(0)=0$, then $a_1 = 0$ and hence:

$u(t) = a_2 t + a_3 t^2$ ....(14)

Thus, substitute (14) in (13), yields to:

$$\bar{F}(a_2,a_3) = \frac{1}{2}\int_0^1 [a_2 + 2a_3t - (a_2t + a_3t^2)^2 - 2t + t^4]\int_0^1 [e^{ts}(t^2-t)(s^2-s)(a_2 + 2a_3s - (a_2s + a_3s^2)^2 - 2s + s^4]dsdt \quad \ldots\ldots (15)$$

which may be minimized using MATHCAD computer program and find the critical points a_2^*, a_3^* for the function which are found to be:

$$a_2^* = 2.354 \times 10^{-6} \quad , \quad a_3^* = 0.999$$

Therefore:

$$u(t) = 2.354 \times 10^{-6}t + 0.999t^2$$

which is the approximate solution of the problem, and the absolute error for $t \in [0,1]$ with step size 0.1,is given in Table (1).

Table (1)

Exact and Approximation Results of Example.

t	*Exact solution*	*Approximation solution*	*Absolute error*
0.1	0.01	0.01	2.30×10^{-7}
0.2	0.04	0.04	4.50×10^{-7}
0.3	0.09	0.090001	6.59×10^{-7}
0.4	0.16	0.160001	8.58×10^{-7}
0.5	0.25	0.250001	1.05×10^{-6}
0.6	0.36	0.360001	1.23×10^{-6}
0.7	0.49	0.490001	1.39×10^{-6}
0.8	0.64	0.640002	1.55×10^{-6}
0.9	0.81	0.810002	1.70×10^{-6}
1	1	1.000002	1.83×10^{-6}

Conclusion and Recommendations for Future Work

From the present study of this paper one may conclude that Variational approach gives a very good result in comparison with exact solution. Also, we may recommend to study and solve problem including differential equations with multi delay.

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الخلاصة

الهدف الرئيس من هذا البحث هو دراسة الصياغة التغايرية **(**Variational Formulation) للمعادلات الغير خطية والتي قد تعطى بصورة عامة بشكل معادلة تفاضلية أعتيادية غير خطية، معادلة تفاضلية جزئية غير خطية، معادلة تكاملية غير خطية، الخ ومن ثم اعطاء الصياغة التغايرية للمعادلات التفاضلية التباطؤية الغير خطية (Nonlinear Delay Differential Equations) وإيجاد حلها باستخدام الاسلوب المقترح من قبل تونتي (Tonti).