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Minimal and Maximal Beta Open Sets

Qays Rashid Shakir
Operations Management Techniques Department-Technical College of Management Baghdad, Baghdad-Iraq.
E-mail: qays.shakir@gmail.com.

Abstract

The present paper deals and discusses new types of sets all of these concepts completely depended on the concept of Beta open set. The importance concepts which introduced in this paper are minimal β-open and maximal β-open sets. Besides, new types of topological spaces introduced which called $T_{\beta min}$ and $T_{\beta max}$ spaces. Also we present new two maps of continuity which called minimal β-continuous and maximal β-continuous. Additionally we investigated some fundamental properties of the concepts which presented in this paper.

Keywords: minimal β-open set, maximal β-open set, minimal β-continuous and maximal β-continuous

Introduction

Minimal and maximal sets play an important role in the researches of generalized topological spaces, Nakaoka and Oda introduced these concepts in [1] and [2] and they used them to investigate many topological properties. In this paper we introduced the notion of minimal β-open and maximal β-open and their complements.

Definition (1) [1]:

A proper nonempty open subset O of a topological space X is said to be minimal open set if any open set which is contained in O is ϕ or O.

Definition (2) [2]:

A proper nonempty open subset O of a topological space X is said to be maximal open set if any open set which is contains O is O or X.

Definition (3) [3]:

A proper nonempty closed subset O of a topological space X is said to be minimal closed set if any closed set which is contained in O is ϕ or O.

Definition (4) [3]:

A proper nonempty closed subset O of a topological space X is said to be maximal closed set if any closed set which is contains O is O or X.

Definition (5) [4]:

Let A be a subset of a topological space X, then the union of all open subset of X which contained in A is called the interior of A and denoted by A^o and the intersection of all closed subset of X which contain A is called the closure of A and denoted by $\overline{A}$.

Definition (6) [5]:

A subset A of a space X is called a β-open set if $A \subseteq \overline{\overline{A}^o}$.The complement of a β-open set is defined to be a β-closed set.

Definition (7) [5]: Let X and Y be topological spaces and f:X→Y is a map then f is called a β-continuous function if f^{-1}(A) is a β-open set in X for every open set A in Y.

Minimal and Maximal β-open sets

Definition (8):

A proper β-open subset B of a topological space X is said to be a minimal β-open set if any β-open set which is contained in B is ϕ or B.

Definition (9):

A proper nonempty β-open subset B of a topological space X is said to be a maximal β-open set if any β-open set which contains B is X or B.

Definition (10):

A proper nonempty β-closed subset F of a topological space X is said to be a minimal β-closed set if any β-closed set which is contained in F is ϕ or F.

Definition (11):

A proper nonempty β-closed subset F of a topological space X is said to be a maximal β-closed set if any β-closed set which contains F is X or F.

Remarks (12):

(1) The family of all minimal β-open (resp. minimal β-closed) sets of a topological space X is denoted by $M_i\beta O(X)$ (resp. $M_i\beta C(X)$).

(2) The family of all maximal β-open (resp. maximal β-closed) sets of a topological space X is denoted by $M_a\beta O(X)$ (resp. $M_a\beta C(X)$).

Remark (13):

The concept of minimal β-open, maximal β-open, minimal β-closed and maximal f-closed are independent of each other as in the following example. Example (14): let X = {a, b, c} and

$\tau = \{\phi, \{a\}, \{a, b\}, X\}$ so $\beta O(X) = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$, $M_i\beta O(X) = \{\{a\}\}$, $M_i\beta C(X) = \{\{c\}, \{b\}\}$, $M_a\beta O(X) = \{\{a,b\}, \{a,c\}\}$, $M_a\beta C(X) = \{\{b,c\}\}$

So the following table show that the new sets are independent each other.

Table (1).

	{a}	{b}	{a,b}	{b,c}
Minimal β-open	Yes	No	No	No
Minimal β-closed	No	Yes	No	No
Maximal β-open	No	No	Yes	No
Maximal β-closed	No	No	No	Yes

Theorem (15):

Let F be a subset of a topological space X, then F is a minimal β-closed if and only if X-F is maximal β-open set.

Proof:

$\Rightarrow$let F be a minimal β-closed, so X-F is β-open. We have to show that X-F is maximal β-open suppose not, so there is a β-open subset D of X such that $X - F \subset D$ hence $X - D \subset F$ and this contradict being F is minimal β-closed.

$\Leftarrow$let F be a β-closed subset of X, suppose that there is a β-closed $K \neq \phi$ such that $K \subset F$ thus $X - F \subset X - K$ but X-K is proper β-open set. Contradiction to the assumption of being X-F is maximal β-open.■

Theorem (16):

Let U and V be maximal β-open subsets of a Topological space X, then $U \cup V = X$ or U=V.

Proof:

If $U \cup V = X$ then the proof is complete.

If not, i.e. $U \cup V \neq X$ so we have to show that U=V.

Since $U \cup V \neq X$ so $U \subset U \cup V$ and $V \subset U \cup V$.

But U is maximal β-open set, so $U \cup V = X$ or $U \cup V = U$

Thus $U \cup V = U$ and so $V \subset U$.

Now since $V \subset U \cup V$ and V is maximal f-open set, so $U \cup V = X$ or $U \cup V = V$,but $U \cup V \neq X$ so $U \cup V = V$ and hence $U \subset V$

Therefore U=V.■

Theorem (17):

Let U be a maximal β-open and V be a β-open subsets of a Topological space X then $U \cup V = X$ or $V \subset U$.

Proof:

If $U \cup V = X$ then the proof is complete.

If $U \cup V \neq X$ so $U \subset U \cup V$ and $V \subset U \cup V$.

Since U is maximal β-open and $U \subset U \cup V$ so by definition of Maximal β-open we have that $U \cup V = X$ or $U \cup V = U$ but $U \cup V \neq X$ so $U \cup V = U$ and hence $V \subset U$.■

Theorem (18):

Let U be a maximal β-open subset of a Topological space X with $x \in X/U$ then $X/U \subset V$ for any β-open subset of X with $x \in V$.

Proof:

Let $x \in X/U$ and $x \in V$, so $V \not\subset U$, thus by (17) we have that $U \cup V = X \Rightarrow (X \setminus U) \cap (X \setminus V) = \phi$ $\Rightarrow X \setminus U \subset V$.■

Theorem (19):

Let F be a minimal β-closed and K be a β-closed subsets of a topological space X then $F\cap K=\phi$ or $\mathbf{F\subset K}$.

Proof:

If $F\cap K=\phi$ then the proof is complete.

If $F\cap K\neq\phi$ then we have to show that $\mathbf{F\subset K}$.

Since $F\cap K\neq\phi$ then $F\cap K\subset F$ and $F\cap K\subset K$.

But F is minimal β-closed, so we have $F\cap K=F$ or $F\cap K=\phi$.

Thus $F\cap K=F$

So $\mathbf{F\subset K}$.■

Theorem (20):

Let F and K be minimal β-closed subsets of a topological space X then $F\cap K=\phi$ or $\mathbf{F=K}$.

Proof:

If $F\cap K=\phi$ then the proof is complete.

If $F\cap K\neq\phi$ then we have to show that $\mathbf{F=K}$.

Since $F\cap K\neq\phi$ so $F\cap K\subset F$ or $F\cap K\subset K$.

Since F is minimal β-closed so we have $F\cap K=F$ or $F\cap K=\phi$. But $F\cap K\neq\phi$ hence $F\cap K=F$ which means $\mathbf{F\subset K}$.

Now since K is minimal β-closed so we have $F\cap K=K$ or $F\cap K=\phi$. But $F\cap K\neq\phi$ hence $F\cap K=K$ which means $\mathbf{K\subset F}$. Therefore F=K. ■

Theorem (21):

Let U, V and W be maximal β-open subsets of a Topological space X such that $\mathbf{U\neq V}$, if $U\cap V\subset W$, then either U=W or V=W.

Proof:

Suppose that $U\cap V\subset W$, if U=W then the proof is complete.

If $\mathbf{U\neq W}$ we have to show that V=W

$V\cap W=V\cap(X\cap W)$ *Set Theory*

$=V\cap[W\cap(U\cup V)]$ *by* (2.9)

$=V\cap[(W\cap U)\cup(W\cap V)]$ *Set Theory*

$=(V\cap W\cap U)\cup(V\cap W\cap V)$ *Set Theory*

$=(U\cap V)\cup(V\cap W)$ since $U\cap V\subset W$

$=(U\cup W)\cap V$ *Set Theory*

$=X\cap V$ since $U\cup W=X$ Thus $V\cap W=V$

$=V$

implies $\mathbf{V\subset W}$ but V is maximal β-open therefore V=W or $V\cup W=X$ but $V\cup W\neq X$ so V=W. ■

Theorem (22):

U, V and W be maximal β-open subsets of a Topological space X which are different from each other, then $U\cap V\not\subset U\cap W$

Proof:

Let $U\cap V\subset U\cap W$

$\Rightarrow(U\cap V)\cup(W\cap V)\subset(U\cap W)\cup(W\cap V)$

$\Rightarrow(U\cap W)\cup V\subset(U\cap V)\cup W$

$\Rightarrow X\cup V\subset X\cup W$

$\Rightarrow V\subset W$

But V is maximal β-open and W is a proper subset of X so V=U, this result contradicts the fact that U, V and W are different from each other. Hence $\mathbf{U\cap V\not\subset U\cap W}$ ■

Theorem (23):

Let F be a minimal β-closed subset of a Topological space X , if $\mathbf{x\in F}$ then $\mathbf{F\subset K}$ for any β-closed subset K of X containing x.

Proof:

Suppose $\mathbf{x\in K}$ and $\mathbf{F\not\subset K}$ so $F\cap K\subset F$ and $F\cap K\neq\phi$ since $x\in F\cap K$

But F is minimal β-closed so $F\cap K=F$. or $F\cap K=\phi$.

hence $F\cap K=F$ which contract the relation $F\cap K\subset F$. Therefore $\mathbf{F\subset K}$.■

Theorem:

Let F and $\mathbf{F_\alpha(\alpha\in A)}$ be minimal β-closed sets if $F\subset\bigcup_{\alpha\in A}F_\alpha$ then there exists $\alpha_o\in\mathbf{A}$ such that $\mathbf{F=F_{\alpha_o}}$.

Proof:

First we have to show that $F \cap F_{\alpha_0} \neq \phi$, suppose that $F \cap F_{\alpha_0} = \phi$ then $F_{\alpha_0} \subset X \backslash F$ and so $F \subset \bigcup_{\alpha \in A} F_\alpha \subset X \backslash F$ which is a contradiction. So $F \cap F_{\alpha_0} \neq \phi$. and hence $F \cap F_{\alpha_0} \subset F$ and $F \cap F_{\alpha_0} \subset F_{\alpha_0}$
since $F \cap F_{\alpha_0} \subset F$ and F is minimal β-closed then $F \cap F_{\alpha_0} = F$ or $F \cap F_{\alpha_0} = \phi$
thus $F \cap F_{\alpha_0} = F$ and hence $F_{\alpha_0} \subset F$. Now since $F \cap F_{\alpha_0} \subset F_{\alpha_0}$ and F_{α_0} is minimal β-closed then $F \cap F_{\alpha_0} = F_{\alpha_0}$ or $F \cap F_{\alpha_0} = \phi$. Thus $F \cap F_{\alpha_0} = F_{\alpha_0}$ and hence $F \subset F_{\alpha_0}$. Therefore $F = F_{\alpha_0}$. ■

$T_{\beta min}$ and $T_{\beta max}$ space

Definition (24):

A topological space X is said to be $T_{\beta min}$ space if every nonempty proper β-open subset of X is minimal β-open set.

Definition (25):

A topological space X is said to be $T_{\beta max}$ space if every nonempty proper β-open subset of X is maximal β-open set.

Example (26):

Let X={a, b, c} and $\tau = \{\phi, \{a,b\}, \{c\}, X\}$ thus $\beta O(X) = \tau$, it is clear that {a, b} and {c} are maximal and minimal β-open sets thus the space X is both $T_{\beta min}$ and $T_{\beta max}$.

Remark (27):

$T_{\beta min}$ and $T_{\beta max}$ spaces are identical.

Theorem (28):

A space X is $T_{\beta min}$ if and only if it is $T_{\beta max}$.

Proof:

$\Rightarrow$ Let X is $T_{\beta min}$ space. Suppose that X is not $T_{\beta max}$, so there is a proper β-open subset K of X which is not maximal, this mean there exist a β-open subset of X with $K \subset H \neq \phi$. Thus we get that H is not minimal which is contradict of being X is $T_{\beta max}$.

$\Leftarrow$ Let X is $T_{\beta max}$ space. Suppose that X is not $T_{\beta min}$, so there is a proper β-open subset K of X which is not minimal, this mean there exist an β-open subset of X with $\phi \neq H \subset K$. Thus we get that H is not maximal which is contradict of being X is $T_{\beta max}$. ■

Theorem (29):

A topological space X is $T_{\beta min}$ space if and only if every nonempty proper β-closed subset of X is maximal β-closed set in X.

Proof:

$\Rightarrow$ let F be a proper β-closed subset of X and suppose F is not maximal.
So there exists an β-closed subset K of X with $K \neq X$ such that $F \subset K$.
Thus $X - K \subset X - F$. Hence X-F is a proper β-open which is not minimal and this contradicts of being X is $T_{\beta min}$ space.

$\Leftarrow$ Suppose U is a proper β-open subset of X. thus X-U is a proper β-closed subset of X, so X-U is maximal β-closed subset of X. and by (15) U is minimal β-open. thus X is $T_{\beta min}$ space. ■

Theorem (30):

A topological space X is $T_{\beta max}$ space if and only if every nonempty proper β-closed subset of X is minimal β-closed set in X.

Proof:

$\Rightarrow$ let F be a proper β-closed subset of X, suppose F is not minimal β-closed in X, so there is a proper β-closed subset of X such that $K \subset F$
Thus $X - F \subset X - K$ but X-K is proper β-open in X so X-F is not maximal in X. Contradiction to the fact X-F is maximal β-open.

$\Leftarrow$ let U be a proper β-open subset of X, then X-U is a proper β-closed subset of X and so it is minimal β-closed set. By (15) we get that U is maximal β-open. ■

Theorem (31):

Every pair of different minimal β-open sets of $T_{\beta min}$ are disjoint.

Proof: Let U and V be minimal f-open subsets of $T_{\beta min}$ space X such that $U \neq V$ to show that $U \cap V = \phi$ suppose not i.e. $U \cap V \neq \phi$.
So $U \cap V \subset U$ and $U \cap V \subset V$. Since $U \cap V \subset U$ and U is minimal β-open then $U \cap V = U$ or $U \cap V = \phi$ thus $U \cap V = U$.
Now since $U \cap V \subset V$ and V is minimal β-open then $U \cap V = V$ or $U \cap V = \phi$ thus $U \cap V = V$.
Hence we get that U=V this result contradicts the fact that U and V are different. Therefore $U \cap V = \phi$.■

Theorem (32):

Union of every pair of different maximal β-open sets in $T_{\beta max}$ space X is X.

Proof: Let U and V be maximal β-open subsets of $T_{\beta max}$ space X such that $U \neq V$ to show that $U \cup V = X$ suppose not i.e. $U \cup V \neq X$.
So $U \subset U \cup V$ and $V \subset U \cup V$.
Since $U \subset U \cup V$ and U is maximal β-open then $U \cup V = U$ or $U \cup V = X$.
Thus $U \cup V = U$ … (1).
Now since $V \subset U \cup V$ and V is maximal β-open then $U \cup V = V$ or $U \cap V = X$
Thus $U \cup V = V$ … (2)
Hence from (1) and (2) we get that U=V this result contradicts the fact that U and V are different. Therefore $U \cap V = X$.■

Continuity with Minimal and Maximal β-open Sets

Definition (33):

Let X and Y be topological spaces, a map $f: X \to Y$ is called minimal β-continuous if $f^{-1}(U)$ is minimal β-open in X for any open subset U of Y.

Example (34):

Let X = Y = {a, b, c} and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, where τ={ ϕ, {a}, {a, b}, X} and σ={ ϕ, {a}, Y} then f is minimal β-continuous since the only proper open subset of Y is {a} and $f^{-1}(\{a\}) = \{a\}$ is minimal β-open in X.

Definition (35):

Let X and Y be topological spaces, a map $f: X \to Y$ isn called maximal β-continuous if $f^{-1}(U)$ is maximal β-open in X for any open subset U of Y.

Example (36):

Let X = Y = {a, b, c} and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, where τ={ ϕ, {a}, {b},{a, b}, X} and σ={ ϕ, {a, c}, Y} then f is maximal β-continuous since the only proper open subset of Y is {a, c} and $f^{-1}(\{a, c\}) = \{a, c\}$ is maximal β-open in X.

Theorem (37):

Every minimal β-continuous map is β-continuous.

Proof:

Let $f: X \to Y$ be a minimal β-continuous map and U be open subset of Y. then $f^{-1}(U)$ is minimal β-open in X and so $f^{-1}(U)$ is β-open subset of X.■

Remark 38:

The converse is not true in general as in the following example.

Example (39):

Let X = Y = {a, b, c} and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, where τ={ ϕ, {a}, {c}, {a, c}, X} and σ={ ϕ, {a, c}, Y} then f is β-continuous but f is not minimal β-continuous since $f^{-1}(\{a,c\}) = \{a,c\}$ is not minimal β-open since $\{a\} \in \beta O(X)$ and $\phi \neq \{a\} \subset \{a,c\}$.

Theorem (40):

Let X and Y be topological spaces, if $f: X \to Y$ is an β-continuous onto map and X is $T_{\beta min}$ space then f is minimal β-continuous.

Proof:

It is clear that the inverse image of ϕ and Y are f-open subsets of X. So let U be a proper open subset of Y. Since f is f-continuous so

$f^{-1}(U)$ is proper f-open subset of X, but X is $T_{\beta mir}$ so $f^{-1}(U)$ minimal f-open.■

Remark (41):

The converse is not true in general as in the following example.

Example (42):

In (34) f is minimal f-continuous but X is not $T_{\beta mir}$.

Theorem (43):

Let X and Y be topological spaces, if $f: X \to Y$ is a β-continuous onto map and X $T_{\beta mir}$ space then f is maximal β-continuous.

Proof:

It is clear that the inverse image of ϕ and Y are β-open subsets of X. So let U be a proper open subset of Y. Since f is β-continuous so $f^{-1}(U)$ is a proper β-open subset of X but X is $T_{\beta mir}$ so $f^{-1}(U)$ is maximal β-open.■

Remark (44):

The converse is not true in general as in the following example.

Example (45): In (36) f is maximal β-continuous but X is not $T_{\beta mir}$ space.

Theorem (46):

Every maximal β-continuous map is β-continuous.

Proof:

Let $f: X \to Y$ be a maximal β-continuous map and U be open subset of Y. then $f^{-1}(U)$ is maximal β-open in X and so $f^{-1}(U)$ is β-open subset of X.■

Remark (47):

The Converse is not true in general as in the following example.

Example (48):

Let X = Y = {a, b, c} and $f: (X, \tau) \to (Y, \sigma)$ is the identity map, then where τ={ ϕ, {a}, {a, c}, X} and σ={ ϕ, {a}, Y} then f is β -continuous but f is not maximal β -continuous since $f^{-1}(\{a\}) = \{a\}$ is not maximal β -open since $\phi \neq \{a,c\} \supset \{a\}$.

Remark (49):

Minimal β -continuous and maximal β-continuous maps are independent of each other and the following examples show that.

Example (50):

In (36) f is maximal β-continuous since $f^{-1}(\{a,c\}) = \{a,c\}$ is β-open but f is not minimal β-continuous.

Example (51):

In (34) f is minimal β-continuous but it is not maximal β-continuous since $f^{-1}(\{b\}) = \{b\}$ is not maximal β-open in X.

Theorem (52):

Let $f: X \to Y$ be a map and X and Y be topological spaces, then f is maximal (resp. minimal) β-continuous if and only if $f^{-1}(F)$ is minimal (resp. maximal) β-closed subset of X for each closed subset F of Y.

Proof: $\Rightarrow$ let F be a closed set in Y. thus Y-F is open and so $f^{-1}(Y-F)$ is maximal (resp. maximal) β-open. but $f^{-1}(Y-F) = X - f^{-1}(F)$ so $f^{-1}(F)$ is minimal (resp. maximal) β-closed.

Theorem (53):

Let X,Y and Z be topological spaces, if $f: X \to Y$ is a minimal (respect. maximal) β-continuous map and $g: Y \to Z$ is a continuous map then $g \circ f: X \to Z$ is a minimal (resp. maximal) β-continuous map.

Proof: Let U be an open subset of Z, since g is continuous so $g^{-1}(U)$ is an open subset of Y. But f is minimal (respect. maximal) β-continuous thus $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}$ is a minimal (respect. maximal) β-open subset of X.■

Conclusion

In this paper we get some theorems presented to reveal many various properties of the minimal β-open and maximal β-open and their complements and we defined two types of topological spaces and finally we defined continuity over the new sets which produced here.

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الخلاصة

يناقش البحث الحالي أنواعا جديدة من المجموعات ويتعامل معها. وكل هذه المفاهيم تعتمد على مفهوم المجموعة المفتوحة من النمط بيتا وأهم هذه المفاهيم التي قدمت في هذا البحث هي المجموعة المفتوحة بيتا الاصغرية و المجموعة المفتوحة بيتا الاكبرية. كما تم تقديم نوعين جديدين من الفضاءات التبولوجيا سميت الفضاء من النمط $T_{\beta min}$ و الفضاء من النمط $T_{\beta max}$. كذلك قمنا بتعريف نمطين جديدين من الدوال المستمرة هما الدالة المستمرة الاصغرية من النمط بيتا و الدالة المستمرة الاكبرية من النمط بيتا، بالاضافة الى ذلك تم دراسة بعض الخواص الاساسية للمفاهيم المطروحة في هذا البحث.