Banach Fixed Point Theorem in Fuzzy Metric Spaces

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Abstract

In this paper, we study certain type of fuzzy metric spaces and then prove the completeness of this metric space, this metric space will be denoted by (F,ρ) where F is the family of all fuzzy subsets of the set of fuzzy numbers and ρ is the distance function. Then the statement and the proof of Banach fixed point theorem in the fuzzy metric space (F,ρ) is given as a main result.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh in 1965. Since then, this concept is used in topology and some branches of analysis, many authors have expensively developed the theory of fuzzy sets and application.

I. Kramosil and J. Michalek [5] in 1975 introduced the concept of fuzzy metric spaces, which opens an avenue for further development of analysis in such spaces. Consequently, in the same time some metric fixed point results were generalized to fuzzy metric spaces and given by A. George and P. Veeramani [6] in 1994 and M. Grabiec [7] in 1998 and others.

Several approaches are proposed to study fuzzy metric spaces depending on the definition of the distance function either using the α-level sets, or using the membership function, or using fuzzy numbers, etc.

Therefore, the study of some well know results on fuzzy metric spaces will depend on the structure of the fuzziness, such as the completeness of such spaces, fixed point theorem, etc.

In this paper, we prove the proposed fuzzy metric space is complete and the fixed point theorem is valued.

2. Preliminaries

In this section, some fundamental and primitive concepts related to fuzzy set theory, in general, and fuzzy metric space, in particular are given.

Definition 2.1, [2]:

If X is a collection of objects with generic element x, then a fuzzy subset Ã of X is characterized by a membership function; $\mu_{\tilde{A}}: X \longrightarrow I$, where I = [0,1], then we write a fuzzy set Ã by the set of points:

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$.

Remark 2.2, [2]:

The set of all fuzzy subsets of a set X is denoted by I^X that is

$I^X = \{\tilde{A} : \tilde{A} \text{ is fuzzy subset of } X\}$.

Definition 2.3, [2]:

The support of a fuzzy set $\tilde{A}$ is the crisp set (or nonfuzzy set) of all x ∈ X, such that $\mu_{\tilde{A}}(x) > 0$.

Definition 2.4, [3], [10]:

The height of a fuzzy set $\tilde{A}$ (denoted by hgt ($\tilde{A}$)) is the supremum of $\mu_{\tilde{A}}(x)$ over all x∈X. If hgt ($\tilde{A}$) = 1, then $\tilde{A}$ is normal, otherwise it is subnormal, and a fuzzy set may be always normalized by defining the scaled membership function:

$$\mu^*_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{\text{hgt}(\tilde{A})}, \forall x \in X$$

Definition 2.5, [3], [4]:

A fuzzy point x_r in a set X is a fuzzy set with membership function:

$$\mu_{x_r}(w) = \begin{cases} r, & \text{for } w = x \\ 0, & \text{for } w \ne x \end{cases}$$

where x ∈ X and 0 < r ≤ 1, w is called the support of x_r and r the value of x_r.

Two fuzzy points x_r and y_s are said to be distinct if and only if x ≠ y, i.e., their support sets are different from each other.

Definition 2.6, [3], [4]:

A fuzzy point x_r is said to be belong to a fuzzy subset Ã in X, denoted by $x_r \in \tilde{A}$ if and only if $r \le \mu_{\tilde{A}}(x)$.

Definition 2.7, [9]:

Let $\tilde{A}$ be a fuzzy subset of X, for any $\alpha \in (0,1]$, the set of all elements $x \in X$, such that $\mu_{\tilde{A}}(x) \geq \alpha$ is called the α-level (or α-cut) set of $\tilde{A}$ and is denoted by:

$\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$.

The next definition depends on the extension principle, which extends the definition of a function to fuzzy function:

Definition 2.8, [2]:

Let f be a function from the universal set X to the universal set Y. Let $\tilde{A}$ be a fuzzy subset in X with membership function $\mu_{\tilde{A}}(x)$. The image of $\tilde{A}$, written as $f(\tilde{A})$, is a fuzzy subset in Y whose membership function is given by:

$$\mu_{f(\tilde{A})}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_{\tilde{A}}(x)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.9, [2] :

A fuzzy subset $\tilde{A}$ of □ is said to be convex if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$$

for all $x_1, x_2 \in$ □ , and all $\lambda \in [0, 1]$.

Definition 2.10, [10]:

If a fuzzy set is convex and normalized, and its membership function is defined in ℝ and piecewise continuous such that there exist a unique $x_0 \in \mathbb{R}$, with $\mu_{\tilde{M}}(x_0) = 1$, then it is called a fuzzy number.

Remark 2.11, [10]:

Fuzzy number is expressed as a fuzzy interval in the real line ℝ represented by the peak point b and two end points a and c, which is abbreviated as [a,b,c], such that the height of the fuzzy number at b equals 1.

Now, we can put (Remark 2.9) as in the following definition in order to make the definition of fuzzy number more reliable.

Definition 2.12, [4],[8]:

A fuzzy number $\tilde{M}$ with the membership function:

$$\mu_{\tilde{M}}(x) = \begin{cases} f_L(x); & \text{if } a \leq x \leq b \\ f_R(x); & \text{if } b \leq x \leq c \\ 0 & \text{other wise} \end{cases}$$

where $f_L(x)$ is a continuous increasing function in [a,b], $f_R(x)$ is a continuous decreasing function in [b,c] and $f_L(a) = f_R(c) = 0, f_L(b) = f_R(b) = 1$, is called general fuzzy number.

The family of the above kind of fuzzy numbers will be denoted by:

$F_G = \{[a, b, c]: \forall\, a \leq b \leq c;\ a, b, c \in \mathbb{R}\}$.

Definition 2.13, [4],[8]:

Let $\tilde{M}$ be any general fuzzy number, then if $f_L(x) = (x-a)/(b-a)$, $a \leq x \leq b$ and $f_R(x) = (c-x)\,/\,(c-b)$, $b \leq x \leq c$, then we call this fuzzy number as triangular fuzzy number which is denoted by $\tilde{M} = (a, b, c)$ with $a < b < c$.

The family of all triangular fuzzy numbers is denoted by F_T, where:

$F_T = \{(a,b,c): \forall\, a < b < c,\ a, b, c \in \mathbb{R}\}$.

Remark 2.14, [4],[8]:

All λ-levels of a fuzzy point a_λ, form a family given by:

$F_P(\lambda) = \{a_\lambda : a \in \mathbb{R}\}$

and if $\lambda = 1$, then this generates a fuzzy number (a, b, c), with $a = b = c$, which is denoted by $\tilde{a} = (a, a, a)$.

i.e., $F_P(1) = \{a_1 : a \in \mathbb{R}\} = \{\tilde{a} = (a, a, a) \mid a \in \mathbb{R}\}$ and let $F_P = \bigcup_{0<\lambda\leq 1} F_P(\lambda)$.

Remark 2.15, [4], [8]:

Let F be the family of all fuzzy sets in general fuzzy numbers F_G and fuzzy points F_P , such that $F_G \cap F_P \subset F$ and $F_G \cup F_P \subset F$.

Notation 2.16, [8]:

For each λ, $0 < \lambda \leq 1$, there is a one-one and onto mapping between $F_P(\lambda)$ and ℝ, which maps $a_\lambda \in F$ onto $a \in \mathbb{R}$.

The next definition is a modified approach for the distance function between two fuzzy numbers, which will be used later in defining the fuzzy metric space.

Definition 2.17, [8]:

Let $\tilde{A}, \tilde{B} \in F$ where $\tilde{A} = [a_1, a_2, a_3], \tilde{B} = [b_1, b_2, b_3]$, then the distance between $\tilde{A}$ and $\tilde{B}$ is defined as:

$$\rho(\tilde{A}, \tilde{B}) = \tfrac{1}{2}\int_0^1 |x_{\tilde{A}L}(\alpha) - x_{\tilde{B}L}(\alpha)|\,d\alpha + \tfrac{1}{2}\int_0^1 |x_{\tilde{A}R}(\alpha) - x_{\tilde{B}R}(\alpha)|\,d\alpha$$

where $\alpha \in (0,1]$, and

$x_{\tilde{A}L}(\alpha) = a_1 + (a_2 - a_1)\alpha$ is α-cuts of $\tilde{A}$ from the left-hand side.

$x_{\widetilde{B}L}(\alpha) = b_1 + (b_2 - b_1)\alpha$ is α-cuts of $\widetilde{B}$ from the left-hand side.

$x_{\widetilde{A}R}(\alpha) = a_3 - (a_3 - a_2)\alpha$ is α-cuts of $\widetilde{A}$ from the right-hand side.

$x_{\widetilde{B}R}(\alpha) = b_3 - (b_3 - b_2)\alpha$ is α-cuts of $\widetilde{B}$ from the right-hand side.

Proposition 2.18, [4], [8]:

1- If $a_\lambda, b_\beta \in F_P, 0 < \lambda < \beta \leq 1$, then:

$$\rho(a_\lambda, b_\beta) = \int_0^\lambda |a-b|d\alpha + \int_\lambda^\beta |b|d\alpha = \lambda|a-b| + (\beta-\lambda)|b|.$$

2- If $\tilde{a} = (a,a,a), \tilde{b} = (b,b,b) \in F_P(1)$, then $\rho(\tilde{a},\tilde{b}) = |b-a|$.

As an illustration, consider the following example:

Example 2.19:

Let $\widetilde{A}$ = (1,3,5) and $\widetilde{B}$ = (2,4,6) be two fuzzy numbers with membership functions:

$$\mu_{\widetilde{A}}(x) = \begin{cases} f_L(x) = \dfrac{x-1}{2}; & \text{if } 1 \leq x \leq 3 \\ f_R(x) = \dfrac{5-x}{2}; & \text{if } 3 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\widetilde{B}}(x) = \begin{cases} f_L(x) = \dfrac{x-2}{2}; & \text{if } 2 \leq x \leq 4 \\ f_R(x) = \dfrac{6-x}{2}; & \text{if } 4 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

then in order to find the distance between $\widetilde{A}$ and $\widetilde{B}$, we must find the α-cuts of $\widetilde{A}$ and $\widetilde{B}$, where α∈(0,1], as follows:

For $\widetilde{A}$ the left-hand side is $x_{\widetilde{A}L}(\alpha) = a_1 + (a_2 - a_1)\alpha = 1 + 2\alpha$, and the right-hand side is $x_{\widetilde{A}R}(\alpha) = a_3 - (a_3 - a_2)\alpha = 5 - 2\alpha$.

Similarly, for $\widetilde{B}$ the left-hand side is $x_{\widetilde{B}L}(\alpha) = b_1 + (b_2 - b_1)\alpha = 2 + 2\alpha$, and the right-hand side is $x_{\widetilde{B}R}(\alpha) = b_3 - (b_3 - b_2)\alpha = 6 - 2\alpha$.

Then the distance between $\widetilde{A}$ and $\widetilde{B}$ is:

$$\begin{aligned}\rho(\widetilde{A},\widetilde{B}) &= \tfrac{1}{2}\int_0^1 |x_{\widetilde{A}L}(\alpha) - x_{\widetilde{B}L}(\alpha)|d\alpha + \tfrac{1}{2}\int_0^1 |x_{\widetilde{A}R}(\alpha) - x_{\widetilde{B}R}(\alpha)|d\alpha \\ &= \tfrac{1}{2}\int_0^1 |1 + 2\alpha - 2 - 2\alpha|d\alpha + \tfrac{1}{2}\int_0^1 |5-2\alpha-6+2\alpha|\, d\alpha \\ &= \tfrac{1}{2}\int_0^1 1d\alpha + \tfrac{1}{2}\int_0^1 1d\alpha = 1\end{aligned}$$

Theorem 2.20, [8]:

Let $\rho: F \times F \to \mathbb{R}^+$, then the distance ρ on F satisfies the following three axioms of the distance function:

(a) $\rho(\widetilde{A},\widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$.
(b) $\rho(\widetilde{A},\widetilde{B}) = \rho(\widetilde{B},\widetilde{A})$.
(c) $\rho(\widetilde{A},\widetilde{B}) \leq \rho(\widetilde{A},\widetilde{C}) + \rho(\widetilde{C},\widetilde{B})$, ∀ $\widetilde{A},\widetilde{B},\widetilde{C} \in F$.

so, (F, ρ) is a fuzzy metric space.

Remark 2.21, [8]:

Two fuzzy metric spaces $(F_P(1), \rho)$ and $(\mathbb{R}, \rho')$ are said to be isometric, which is denoted by $(F_P(1), \rho) \cong (\mathbb{R}, \rho')$, if there is a function $f : (F_P(1), \rho) \to (\mathbb{R}, \rho')$, such that $f(\tilde{a}) = a$, ∀ $\tilde{a} \in F_P(1)$, which is also a one-one and onto mapping and:

$$\rho(\tilde{a},\tilde{b}) = |b-a| = \rho'(a,b) = \rho'\left(f(\tilde{a}), f(\tilde{b})\right).$$

3. The Completeness of the Fuzzy Metric Space and the Banach Fixed Point Theorem

In order to investigate fixed point theorem in the fuzzy metric space (F, ρ), we must first show that the fuzzy metric space (F, ρ) is complete. So some additional basic definitions are introduced. For simplicity, the fuzzy point xα will be denoted by $\widetilde{P}$.

Definition 3.1:

Let $\{\widetilde{P}_n\}$, n ∈ ℕ be a sequence of fuzzy points in F, then $\{\widetilde{P}_n\}$ is said to be converge to $\widetilde{P}_0$ (written as $\widetilde{P}_n \to \widetilde{P}_0$) if and only if for each ε > 0, ∃ k ∈ ℕ such that $\rho(\widetilde{P}_n, \widetilde{P}_0) < \varepsilon, \forall n > k$.

Definition 3.2:

The sequence $\{\widetilde{P}_n\}$ is called Cauchy sequence in (F,ρ) if and only if for each $\varepsilon > 0, \exists k \in \mathbb{N}$ such that $\rho(\widetilde{P}_n, \widetilde{P}_m) < \varepsilon$, ∀ n, m > k.

An important characterization result which may be considered as the main result of this chapter is the next theorem, which relates between the convergence of a sequence of fuzzy points with the convergence of two sequences in ordinary sense. This theorem is of great importance, which will be used later on and seems to be new, to the best of our knowledge.

Theorem 3.3:

A sequence of fuzzy points $\{\widetilde{P}_n\}$, $n \in \mathbb{N}$ is converge to $\widetilde{P}$ if and only if there exists two nonfuzzy sequences, namely the sequence of supports $\{x_n\} \subset X$ and monotonic sequence of levels $\{\lambda_n\} \subseteq (0,1]$, $n \in \mathbb{N}$, such that $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$, $x \in X$, $\lambda \in (0, 1]$.

Proof:

$\Rightarrow$ If $\{\widetilde{P}_n\}$ is converge to $\widetilde{P}$, so for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$, such that $\rho(\widetilde{P}_n, \widetilde{P}) < \varepsilon$, for all $n \geq k$. Hence:

$\rho(\widetilde{P}_n,\widetilde{P}) = \lambda_n|x_n - x| + (\lambda_n - \lambda)|x| < \varepsilon$

and since $\lambda_n|x_n - x| \geq 0$ and $(\lambda_n - \lambda)|x| \geq 0$; and from the properties of the positive real numbers, there exists $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that $k = \max\{k_1, k_2\}$, and:

$|x_n - x| < \varepsilon_1$, $|\lambda_n - \lambda| < \varepsilon_2$, for all $n \geq k$

Then, $x_n \longrightarrow x$, and $\lambda_n \longrightarrow \lambda$.

$\Leftarrow$ If $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$

Hence, for all $\varepsilon > 0$, there exist $k_1, k_2 \in \mathbb{N}$, such that:

$|x_n - x| < \dfrac{\varepsilon}{2c_1}$, $|\lambda_n - \lambda| < \dfrac{\varepsilon}{2c_2}$, where $n \geq k = \max\{k_1, k_2\}$ and λ_n and $|x|$ are bounded by two non zero positive real numbers c_1 and c_2.

Therefore, for all $\varepsilon > 0$, there exist $k = \max\{k_1, k_2\} \in \mathbb{N}$, such that:

$$\begin{aligned}\rho(\widetilde{P}_n,\widetilde{P}) &= \lambda_n|x_n - x| + (\lambda_n - \lambda)|x| \\ &\leq \lambda_n \frac{\varepsilon}{2c_1} + |x|\frac{\varepsilon}{2c_2} \\ &\leq c_1 \frac{\varepsilon}{2c_1} + c_2\frac{\varepsilon}{2c_2} \\ &= \varepsilon, \ \forall\, n \geq k\end{aligned}$$

Hence, $\{\widetilde{P}_n\}$ is converge sequence of fuzzy points. ■

Similarly, as in the proof of theorem (3.3), we can prove state and prove the next proposition:

Proposition 3.4:

A sequence of fuzzy points $\{\widetilde{P}_n\}$, $n \in \mathbb{N}$ is Cauchy sequence if and only if there exists two nonfuzzy Cauchy sequences, namely the Cauchy sequence of supports $\{x_n\} \subset X$ and monotonic Cauchy sequence of images $\{\lambda_n\} \subseteq (0, 1]$, $n \in \mathbb{N}$.

Theorem 3.5:

Every convergent sequence of fuzzy points $\{\widetilde{P}_n\}$, $n \in \mathbb{N}$ in a fuzzy metric space (F,ρ) is a Cauchy sequence.

Proof:

Let $\{\widetilde{P}_n\}$, $n \in \mathbb{N}$ be a Cauchy sequence in (F,ρ).

Then using theorem (3.3), there exist sequence of supports $\{x_n\} \subset X$ and monotonic sequence of images $\{\lambda_n\} \subseteq (0, 1]$, $n \in \mathbb{N}$, such that

$x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$, $x \in X$, $\lambda \in (0, 1]$.

Since $\{x_n\}$ is convergent nonfuzzy sequence, hence it is a Cauchy sequence in X

Also, since $\{\lambda_n\}$ is convergent sequence of images in $(0, 1] \subset \mathbb{R}$

Now, using proposition (3.4), the sequence of fuzzy points $\{\widetilde{P}_n\}$, $n \in \mathbb{N}$ is a Cauchy sequence. ■

Definition 3.6:

A fuzzy metric space (F, ρ) is said to be complete if and only if every Cauchy sequence in (F, ρ) is converge.

Theorem 3.7:

The fuzzy metric space (F, ρ) is complete.

Proof:

To prove (F, ρ) is complete, we must prove any Cauchy sequence in F is convergent.

Let $\{\tilde{p}_n\}$ be any Cauchy sequence of fuzzy points, such that $\tilde{p}_1 = x_{1_{\lambda_1}}, \tilde{p}_2 = x_{2_{\lambda_2}}, \ldots, \tilde{p}_n = x_{n_{\lambda_n}}, \ldots$

Consider the following three cases:

1- If $\{\tilde{p}_n\} \in F_P(1)$, for all $n \in \mathbb{N}$.

Then $\tilde{p}_1 = x_1, \tilde{p}_2 = x_2, \ldots, \tilde{p}_n = x_n, \ldots$

Since $\{\tilde{p}_n\}$ is a Cauchy sequence, then we have $\forall\, \varepsilon > 0, \exists\, k \in \mathbb{N}$, such that:

$\rho(\tilde{p}_n, \tilde{p}_m) < \varepsilon, \ \forall\, n, m > k$.

And since $\{\tilde{p}_n\} \in F_P(1)$ for all $n \in \mathbb{N}$, implies from (proposition (2.18)(2))

$\rho(\tilde{p}_n, \tilde{p}_m) = |x_n - x_m| < \varepsilon, \forall\, n, m > k$.

Thus $\{x_n\}$ is Cauchy sequence in $\mathbb{R}$, but $\mathbb{R}$ is complete metric space, then $\{x_n\}$ is a convergent sequence.

2- If $\{\tilde{p}_n\} \in F_P(\lambda)$; $0 < \lambda < 1$

Then $\tilde{p}_1 = x_{1_\lambda}, \tilde{p}_2 = x_{2_\lambda}, \ldots, \tilde{p}_n = x_{n_\lambda}, \ldots$

Since $\{\tilde{p}_n\}$ is a Cauchy sequence, and $\lambda > 0$, then from Archimedean property for any $\lambda\varepsilon > 0, \exists\, k \in \mathbb{N}$ such that $\rho(\tilde{p}_n, \tilde{p}_m) < \lambda\varepsilon, \forall n, m > k$.

For $\tilde{p}_n = x_{n_\lambda}, \tilde{p}_m = x_{m_\lambda} \in F_P(\lambda)$; $0 < \lambda < 1$, their α-cuts, $0 < \alpha \le 1$, and from the definition of the fuzzy points, are:

$x_{\tilde{p}_n L}(\alpha) = x_{\tilde{p}_n R}(\alpha) = x_n$, if $0 < \alpha \le \lambda$

$x_{\tilde{p}_n L}(\alpha) = x_{\tilde{p}_n R}(\alpha) = 0$, if $\lambda < \alpha \le 1$

$x_{\tilde{p}_m L}(\alpha) = x_{\tilde{p}_m R}(\alpha) = x_m$, if $0 < \alpha \le \lambda$

$x_{\tilde{p}_m L}(\alpha) = x_{\tilde{p}_m R}(\alpha) = 0$, if $\lambda < \alpha \le 1$

Therefore, by (proposition (2.18)(1)) the distance between $\tilde{p}_n, \tilde{p}_m$ is:

$$\rho(\tilde{p}_n, \tilde{p}_m) = \int_0^\lambda |x_n - x_m| d\alpha + \int_\lambda^1 |0| d\alpha = \lambda |x_n - x_m|$$

Since $\rho(\tilde{p}_n, \tilde{p}_m) < \lambda\varepsilon, \forall n, m > k$, implies $\rho(\tilde{p}_n, \tilde{p}_m) = \lambda|x_n - x_m| < \lambda\varepsilon, \forall n, m > k$, $0 < \lambda < 1$

Therefore $|x_n - x_m| < \varepsilon, \forall\, n, m > k$

And hence $\{x_n\}$ is Cauchy sequence in $\mathbb{R}$, but $\mathbb{R}$ is a complete metric space, then $\{x_n\}$ is a convergent sequence.

3- If $\{\tilde{p}_n\} \in F_P$, then $\tilde{p}_1 = x_{1_{\lambda_1}}, \tilde{p}_2 = x_{2_{\lambda_2}}, \ldots, \tilde{p}_n = x_{n_{\lambda_n}}, \tilde{p}_m = x_{m_{\lambda_m}}, \ldots$; such that $0 < \lambda_i \le 1$, $\forall\, i = 1,2,3, \ldots$, where from proposition 2.18(1); it is supposed that $\{\lambda_i\}$ is a monotonic increasing sequence of levels, then for $m > n$, we have $\lambda_m > \lambda_n$ or $\lambda_m - \lambda_n > 0$. Since $\{\tilde{p}_n\}$ is a Cauchy sequence in F, and suppose that:

$\lambda_n \varepsilon + (\lambda_m - \lambda_n)|x_m| > 0$, then $\exists\, k \in \mathbb{N}$, such that:

$\rho(\tilde{p}_n, \tilde{p}_m) < \lambda_n \varepsilon + (\lambda_m - \lambda_n)|x_m|$, $\forall n, m > k$, by (proposition (2.18)(1)), we have:

$$\begin{aligned} \rho(\tilde{p}_n, \tilde{p}_m) &= \lambda_n |x_n - x_m| + (\lambda_m - \lambda_n)|x_m| \\ &< \lambda_n \varepsilon + (\lambda_m - \lambda_n)|x_m|, \ \forall\, n, m > k \\ &= \lambda_n |x_n - x_m| < \lambda_n \varepsilon, \forall\, n, m > k \\ &= |x_n - x_m| < \varepsilon, \forall\, n, m > k \\ &= \rho'(x_n, x_m) < \varepsilon, \ \forall\, n, m > k \end{aligned}$$

Implies $\{x_n\}$ is Cauchy sequence in $\mathbb{R}$, but $\mathbb{R}$ is a complete metric, therefore $\{x_n\}$ is a convergent sequence.

Thus in all cases $\{\tilde{p}_n\}$ is a convergent sequence, and therefore (F, ρ) is a complete fuzzy metric space. ■

Now, we are ready to investigate the Banach fixed point theorem in the fuzzy metric space (F, ρ).

<u>Theorem 3.8:</u>

Let (F, ρ) be a complete fuzzy metric space, and f: $(F, \rho) \to (F, \rho)$ satisfy $\rho(f(\tilde{p}), f(\tilde{q})) \le r\rho(\tilde{p}, \tilde{q})$, where $0 \le r < 1$. Then f has a unique fixed point $\tilde{p}$.

Proof:

For any fuzzy point $\tilde{p}_0 \in F$. Let $\tilde{p}_1 = f(\tilde{p}_0)$, $\tilde{p}_2 = f(\tilde{p}_1) = f\left(f(\tilde{p}_0)\right) = f^2(\tilde{p}_0)$, …, $\tilde{p}_n = f^n(\tilde{p}_0)$

So, we have a sequence of fuzzy points $\{\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \ldots\}$.

Now, we show that for each $\tilde{p}_0$, $\{\tilde{p}_n\}$ is Cauchy sequence.

If $m > n$,

$$\begin{aligned} \rho(\tilde{p}_m, \tilde{p}_n) &= \rho(f^m(\tilde{p}_0), f^n(\tilde{p}_0)) \\ &\le r^n \rho(f^{m-n}(\tilde{p}_0), \tilde{p}_0) \\ &= r^n \rho(\tilde{p}_{m-n}, \tilde{p}_0) \\ &\le r^n\{\rho(\tilde{p}_0, \tilde{p}_1) + \rho(\tilde{p}_1, \tilde{p}_2) + \ldots + \rho(\tilde{p}_{m-n-1}, \tilde{p}_{m-n})\} \\ &\le r^n \rho(\tilde{p}_0, \tilde{p}_1)\{1 + r + r^2 + \cdots + r^{m-n-1}\} \\ &\le \frac{r^n \rho(\tilde{p}_0, \tilde{p}_1)}{1 - r} \end{aligned}$$

Since $r < 1$, $\forall\, \varepsilon > 0$, $\exists n \in \mathbb{N}$ and as $n \to \infty$, then $\frac{r^n \rho(\tilde{p}_0, \tilde{p}_1)}{1-r} < \varepsilon$

Then $\rho(\tilde{p}_m, \tilde{p}_n) < \frac{r^n \rho(\tilde{p}_0, \tilde{p}_1)}{1-r} < \varepsilon, \forall\, n, m > k$

Thus $\{\tilde{p}_n\}$ is Cauchy sequence. Since (F, ρ) is complete fuzzy metric space, there exists $\tilde{p}$ such that $\{\tilde{p}_n\} \to \tilde{p}$.

Similarly if $n > m$.

Now, we show that $\tilde{p}$ is fixed point in F, then we must prove $f(\tilde{p}) = \tilde{p}$.

$$\begin{aligned} \rho(f(\tilde{p}), \tilde{p}) &\le \rho(f(\tilde{p}), \tilde{p}_n) + \rho(\tilde{p}_n, \tilde{p}) \\ &= \rho\left(f(\tilde{p}), f(\tilde{p}_{n-1})\right) + \rho(\tilde{p}_n, \tilde{p}) \\ &\le r\,\rho(\tilde{p}, \tilde{p}_{n-1}) + \rho(\tilde{p}_n, \tilde{p}) \end{aligned}$$

Since $\{\tilde{p}_n\} \to \tilde{p}$, thus $\rho(f(\tilde{p}), \tilde{p}) \le r\varepsilon + \varepsilon = r(1 + \varepsilon)$. Therefore $\rho(f(\tilde{p}), \tilde{p}) = 0$, implies $f(\tilde{p}) = \tilde{p}$.

Next, we must show that $\tilde{p}$ is the unique fixed point of f, suppose there exists another fixed point $\tilde{q} \in F$, such that $f(\tilde{q}) = \tilde{q}$.

$\rho(\tilde{p}, \tilde{q}) = \rho(f(\tilde{p}), f(\tilde{q})) \le r\rho(\tilde{p}, \tilde{q})$

Thus $\rho(\tilde{p}, \tilde{q}) \le r\rho(\tilde{p}, \tilde{q})$

Therefore $\rho(\tilde{p}, \tilde{q}) = 0$ implies $\tilde{p} = \tilde{q}$.

Thus, there is a unique fixed point $\tilde{p}$ of f in F. ■

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