# Fully Cancellation and Naturally Cancellation Modules

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**Abstract**

In this paper, we introduce two types of cancellation modules namely fully cancellation and naturally cancellation. Some characterizations of these concepts are given and some properties of these concepts in the class of multiplication modules are presented. Also the direct sum of fully cancellation module and the behaviour of fully cancellation (naturally cancellation) are discussed.

Keywords**:** Fully cancellation modules, Naturally cancellation modules, Weak-cancellation modules.

## 1. Introduction

Let M be an R-module, where R is a commutative ring with unity. Gilmer in [7] introduced the concept of cancellation ideal where, an ideal I of a ring R is said to be cancellation if whenever J and K are ideals of R, J.I = K.I, then J = K. Also, D.D. Anderson and D.F. Anderson in [2], studied the concept of cancellation ideals. In 1992, A.S. Mijbass in [11], give the generalization of this concept namely cancellation module (weakly cancellation module), where an R-module is called cancellation (weakly cancellation) if whenever I and J are ideals of R, I.M = J.M implies I = J ($I+ann_RM = J+ ann_RM$).

In 2000, B.N. Shihab [14], introduce and studied restricted (and weakly restricted) cancellation module, if whenever I and J are two ideals of R, with IM = JM and IM≠0, then I = J ($I+ann_RM = J+ ann_RM$).

Next, in [12], L.M, Selman, B.N. Shihab and T. Rashed introduced and studied cancellation, Relatively cancellation module, where an R-module is called an R-module M is called Relatively cancellation whenever IM = KM, with I is a prime ideal of R and K is any ideal of R, implies I = K.

In this paper, we introduce two types of cancellation modules namely fully cancellation and naturally cancellation, where M is called a fully cancellation module if for every submodule A, B of M, IA = IB implies A = B. naturally cancellation module is introduce by using the naturally product of submodules which is introduced in [3], where for each submodules A and B of M, the naturally product of A and B (denoted by A.B) is define by $(A:_R M)(B:_R M)M$. We that M is a naturally cancellation module if for each submodules A, B, C of M, A.B = A.C implies B = C.

In $2 0f this paper, some characterizations related with these concepts are given. Also some relationships between these concepts and cancellation modules are presented.

In $3, we discuss the direct sum of fully cancellation modules.

In $4 we study the behavior of fully (naturally) cancellation modules under localization.

## 2. Fully (Naturally) Cancellation Modules

In this section, we will introduce a new concept (to the best of our knowledgment) namely fully cancellation and naturally cancellation.

We give some basic results and characterizations of these concepts and some relationships between them.

***Definition 2.1:***

Let M be an R-module .The module M is called fully cancellation if for every non zero ideal I of R and for every submodules N and W of M such that IN=IW then N=W**.**

***Definition 2.2:***

Let M be an R-module .M is called naturally cancellation if whenever N, $W_1$ and $W_2$ are submodule of M such that N. $W_1$ = N. $W_2$ then $W_1$= $W_2$.

***Remarks and Examples 2.3:***

**1-** Z- as Z-module is a fully cancellation module, since if we take I=nZ, N=< $m_1$> and W=<$m_2$ >, where $m_1$, $m_2$ ∈Z. Now, assume that IN=IW then n $m_1$ Z= m $m_2$ Z, which implies n $m_1$ = n $m_2$ a and n $m_2$ = n $m_1$ b (for some a, b∈Z) .Hence n $m_1$ = n $m_1$b a, then 1=b a and consequently either, a=b=1 or a=b=-1. In each case we get n $m_1$ = n $m_2$ which leads to $m_1$ = $m_2$. Hence N=W.

**2-** The Z-module $Z_4$ is not fully cancellation, since if we take I=4Z, N = $(\bar{2})$ and W=( $\bar{Z}_4$ ) then (4Z)($\bar{2}$)=(4) ($Z_4$) but $(\bar{2}) \neq Z_4$ .

**3-**Let M be a fully cancellation R-module which is not simple, then M is faithful. To prove this let r ∈ ann M. Supposes r ≠ 0, then rM = 0, and let N be proper submodule of M. Hence rN=0, thus rM = rN and this implies M = N, which is contradiction.

**4-** Any submodule N of a fully cancellation R-module M is fully cancellation.
To prove this let I be a non-zero ideal of a ring R. For any two submodules $M_1$, $M_2$ of N, if $IM_1$ = $IM_2$ and since $M_1$ and $M_2$ are submodules of M which is fully cancellation module, then $M_1$ =$M_2$.Thus N is fully cancellation.

**5-** The homomorphic image of the fully cancellation module is not necessary be a fully cancellation module. For example:

Let $\pi : Z \to \dfrac{Z}{<4>} \cong Z_4$, Z is fully cancellation but $Z_4$ is not fully cancellation by (Rem & Ex 2.3 (1) & (3)).

**1-**If $M_1 \cong M_2$ then M is a fully cancellation module if $M_2$ is a fully cancellation module.

**2-**Not every fully cancellation is naturally cancellation module as the following example shows.

Consider Q as a Z-module. Q is not a naturally cancellation module, since if

we take $A = \frac{1}{4}Z$, $B = \frac{1}{2}Z$ and $C = Z$

which are submodules of Q. Then

$A.B = (\frac{1}{4}Z:Q)\ (\frac{1}{2}Z:Q)Q = 0$ and

$A.C = (\frac{1}{4}Z:Q)(\frac{1}{2}Z:Q)Q = 0$ but B≠C. Also, one can easily show that Q as a Z-module is fully cancellation. Let I = (n) where n∈Z. If IA = IB where A, B ≤ Q then for every x∈A, nx∈IA=IB, thus nx = ny (for some y∈B). Hence x = y∈B. Therefore A⊆B and similarly B⊆A, hence A = B. Thus Q as a Z-module is fully cancellation but is not naturally cancellation.

Recall that an R-module is called multiplication if for each $N \leq M$, there exists an ideal I of R such that N = IM. Equivalently, M is a multiplication R-module if for each $N \leq M$, $N = (N :_R M)M$, where $(N :_R M) = \{r \in R : rM \subseteq N\}$,[4][5].

The next theorem shows that the two concepts are equivalent if the module M is multiplication.

***Theorem 2.4:***

Let M be a multiplication R-module, then M is naturally cancellation if and only if M is fully cancellation.

***Proof*** **(⇐)** Let N,$W_1$ and $W_2$ are submodules of M such that N. $W_1$ = N. $W_2$ ,then $(N:_RM)(W_1:_R M)M = (N:_R M)(W_2:_R M)M$ but $(W_1:_R M)M = W_1$ and $(W_2:_R M)M = W_2$ ,then $(N:_R M)W_1=(N:_R M)W_2$ ,hence $W_1$= $W_2$, since M is a fully cancellation module.

**(⇒)** Let I be a non zero ideal of R and N, W be two submodules of M such that IN = IW. Now, since $IN=I(N:_R M)M = (N:_R M)IM = (N:_R M)(IM:_R M)M$ = N.IM and similarly IW = W.IM thus N.IM = W.IM. But M is naturally cancellation, so N = W .Therefore M is fully cancellation.□

The following examples illustrate the above theorem.

***Example 2.5:***

**1-** Z as Z-module is fully cancellation and since Z is multiplication Z-module then by Th. (2.4) Z is a naturally cancellation module.

**2-** Consider $Z_{p^\infty}$ as Z-module. We know that $Z_{p^\infty}$ as Z-module is not multiplication, also it is not fully cancellation, since $(P^2)(\frac{1}{P^2}+Z)=(P^2)(\frac{1}{P}+Z)$ but notice that $(\frac{1}{P^2}+Z)\neq(\frac{1}{P}+Z)$. Also, $Z_{p\infty}$ as Z-module is not naturally cancellation, since for every A, B, And C submodules of $Z_{p\infty}$ such that B≠C, we have (B: $Z_{p\infty}$) $Z_{p\infty}$ =0 and A.C = (A: $Z_{p\infty}$) (C: $Z_{p\infty}$) $Z_{p\infty}$=0 .This means that A.B = A.C but B≠C .

The following is a characterization of fully cancellation modules. Compare this result with [11, Th.1.9]

***Theorem 2.6:***

Let M be an R-module. Let N and W are two submodules of M, let I be a non zero ideal of R .Then following statements are equivalent

**1-** M is fully cancellation module.

**2-** If IN⊆ IW then N⊆W.

**3-** If I<a>⊆IW then a∈W, where a∈M.

**4-** $(IN:_R IW)=(N:_R W)$.

***Proof*:**

**(1) ⇒(2)**

If IN⊆IW, then IW = IN+IW = I.(N+W)[9,Prop 2.1(4)]. Since M is a fully cancellation module then W = N+W and this means that N⊆W.

**(2) ⇒(3)** Clear

**(3) ⇒(1)**

If IN = IW, to prove N = W. Let a∈N, then I<a>⊆IN⊆IW, by (3) a∈W. Thus N⊆W. Similarly W⊆N. Hence N =**W**

**(1) ⇒(4)**

Let r ∈(IN:$_R$ IW), then rIW⊆IN. So, IrW⊆IN and since (1) implies (2), we have rW ⊆N, therefore $r\in(N:_R W)$. Hence $(IN:_R IW)\subseteq(N:_R W)$. The reverse conclusion is clear. Thus $(IN:_R IW)=(N:_R W)$.

**(4) ⇒(1)**

Let IN = IW, then by (4) (IN:$_R$ IW) = (N:$_R$ W). But (IN:$_R$ IW) = R (since IN = IW). Thus R = (N:$_R$ W) and so W⊆N. Similarly, (IW:$_R$ IN) = (W:$_R$ N), thus R = (W:$_R$ N) and hence N⊆W. Thus N =W. □

The following proposition gives a new characterization a bout naturally cancellation module when it is multiplication R-module.

***Theorem 2.7:***

Let M be multiplication R-module. A, B and C are submodules of M and a∈M. Then the following statements are equivalent

**1-** M is a naturally cancellation R- module

**2-** M is a fully cancellation R-module

**3-** If A.B ⊆ A.C, where A,B and C are submodules of M .Then B ⊆ C.

**4-** If A.(a) ⊆ A.B then a∈B

**5-** (A.B:$_R$A.C) = (B:$_R$C)

***Proof:***

**(1)□(2):** (see Th2.4)

**(2)⇒(3):**

Let A.B ⊆ A.C where A,B,C ≤M . then (A:$_R$ M)(B:$_R$M)M⊆(A:$_R$ M)(C:$_R$M)M. Since M is multiplication, (A:$_R$ M)B⊆(A:$_R$ M)C. But M is fully cancellation by (2), so B⊆C.

**(3)⇒(4):** It is clear

**(2)⇒(5):**

Let a∈(B:$_R$ C) . Then aC⊆B, hence a(C:M)M⊆(B:M)M, since M is multiplication. It follows that a.(A:$_R$ M) (C:$_R$ M)M⊆(A:$_R$ M) (B:$_R$ M)M that is a.(A.C) ⊆ A.B. Thus a∈(A.B:$_R$ A.C). Now, if a∈(A.B:$_R$ A.C) then a(A.C) ⊆ A.B, hence a.(A:$_R$ M)C⊆ (A:$_R$ M)B, since M is multiplication. By (2), M is fully cancellation, so by Th 2.6 aC⊆ B. Thus a∈(B:C).

**(5)⇒(2):**

Let A.B = A.C, for A,B,C ≤M .Thus (A.B:$_R$A.C) = R. By (5) , we get R=(B:C) and C⊆ B. Similarly (A.B:$_R$A.C) = R=(C:$_R$ B) and hence B⊆C. Thus B = C.

**(4)⇒(1):**

Let A.B = A.C. Then R = (A.B:$_R$ A.C) By Cond (4), (A.B:$_R$ A.C) = B:$_R$ C). Thus R = (B:$_R$ C) and hence C⊆ B. Similarly A.C = A.B implies R = (A.C:$_R$ A.B) = (C:$_R$ B). Thus B⊆C. Therefore B = C.□

The following proposition shows that every multiplication submodule of fully cancellation module is naturally cancellation.

***Proposition 2.8:***

Let M fully cancellation R-module and let K be multiplication submodule of M then K is naturally cancellation module.

***Proof*:**

Since K≤M and M is a fully cancellation R-module, K is a fully cancellation (Rem & Ex 2.3(4)). But K is a multiplication R-module, so K is naturally cancellation, by Th. 2.4. □

An ideal I of a ring R is called cancellation ideal if AI = BI then A = B, where A and B are two ideals of R [7].

Next, we end this section by some relationships between fully (naturally) cancellation and cancellation modules.

***Proposition 2.9:***

Let M be a multiplication and cancellation R-module. If every ideal of R is cancellation then M is naturally cancellation R-module.

***Proof*:**

Let A, B and C are submodules of M such that A.B = A.C. Then $(A:_R M)(B:_R M)M = (A:_R M)(C:_R M)M$. Since M is cancellation module then we get $(A:_R M)(B:_R M) = (A:_R M)(C:_R M)$. By assumption $(A:_R M)$ is a cancellation ideal, thus $(B:_R M) = (C:_R M)$. This implies $(B:_R M)M = (C:_R M)M$ a multiplication module so that B = C and hence M is M is a naturally cancellation R-module. □

***Corollary 2.10:***

Let M be a finitely generated faithfull multiplication R-module if every ideal of R is a cancellation module, then M is naturally cancellation ideal.

***Proof*:**

Since M is a finitely generated faith. mult, M is a cancellation R-module by [5.Th3.1]. Hence the result follows by prop 2.8.

***Proposition 2.11:***

Let M be a fully cancellation R-module. If M is a cancellation module, then every non zero ideal of R is a non zero cancellation ideal.

***Proof*:**

Let I be a non zero ideal of R such that IJ = IK where J, K are any two ideals of R. To prove that J = K. We have IJM =IKM, but M is fully cancellation R-module implies JM = KM. Also, since M is cancellation module, then J = K. Thus I is cancellation ideal of R. □

***Corollary 2.12:***

Let M be a multiplication cancellation R-module. Then M is fully cancellation if and only if every non zero ideal of R is cancellation ideal.

***Proof*: (⇒)**

It follows directly by proposition 2.11

**(⇐)**

By Prop.2.9, M is a naturally cancellation module. And by Th. 2.4, M is a fully cancellation R-module. □

An element x in an R-module A is called a torsion element if rx = 0 for some non zero divisor element r∈R [9].

In the following proposition we introduce a necessary condition for a module to be fully cancellation modules.

***Proposition 2.13:***

Let M be module over principal ideal ring R such that every element in M is non torsion. Then M is a fully cancellation module.

***Proof*:**

Let I be a non zero ideal of R and A, B are submodules of M such that IA = IB. By assumption I = (x), for some x≠0, x∈R. Hence (x)A = (x)B. Now, for any a∈A we have xa∈ (x)B. So, xa = xb, for some b∈B, thus x (a-b) = 0. If a-b≠0 then x≠0 since a-b is non torsion which is a contradiction. Thus (a-b) = 0, that is a = b and so A = B. Therefore M is a fully cancellation module. □

The following lemma will be used in our work later on.

***Lemma 2.14:***

Let M be a module over an integral domain R. if M = <m>, for some non torsion element m∈M. then every non zero element of M is non torsion.

***Proof*:**

Let x∈M, x≠0, and suppose that x is torsion element. So, there exists r∈R,r≠0, such that rx = 0. But x∈<m>, so x = tm, for some t∈R. Thus rx = rtm = 0. But m is not torsion element thus rt = 0. Also, since R is an integral domain and $r \neq 0$ then $t = 0$. Thus $tm = x = 0$ which is a contradiction. □

***Corollary 2.15:***

Let M be a module over a PID (principal ideal domain) R. If M=<m> where m is non torsion element then M is a fully cancellation module.

***Proof*:**

By lemma 2.14, every non zero element of M is non torsion; hence by Prop. 2.12 M is a fully cancellation module.□

***Example 2.16:***

Consider $Z_2$ as Z-module. We know that Z is a PID domain principal. Also, $Z_2 = <\bar{1}>$ , but $\bar{1}$ is torsion element (Since 2. $\bar{1} = \bar{0}$). And $Z_2$ as Z-module is not fully cancellation since $(2)(\bar{1}) = (2)(\bar{0})$ but $(\bar{1}) \neq (\bar{0})$. □

**3. Direct Sum of Fully Cancellation Modules**

In this section we discuss the direct sum of fully cancellation modules.

***Proposition 3.1:***

Let $M = M_1 \oplus M_2$ be an R-module, where $M_1$, $M_2$ are two submodules of M such that ann $M_1$ +ann $M_2$ =R.Then $M_1$ and $M_2$ are fully cancellation R-modules if and only if M is fully cancellation.

***Proof* (⇒)**

To prove M is fully cancellation. Let I be a non zero ideal of R and A,B are submodules of M such that IA = IB. Since ann $M_1$+ann $M_2$=R then by [1,Th 4.2 P.28] we get $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ for some $A_1$, $B_1$ are submodules of $M_1$ and $A_2$, $B_2$ are submodules of $M_2$.Thus $I(A_1 \oplus A_2) = I(B_1 \oplus B_2)$. Hence $IA_1 \oplus IA_2 = IB_1 \oplus I.B_2$.This implies that $IA_1 = IB_1$ and $IA_2 = IB_2$ .But $M_1$ and $M_2$ are fully cancellation R-modules then $A_1 = B_1$ and $A_2 = B_2$. Hence A = B.

**(⇐)** It is clear by (Rem &Ex 2.3(3)). □

Recall that a submodule N of an R-module M is called fully invariant if for each $f \in End(M)$, and for each $N \leq M$, $f(N) \subseteq N$,[6].

The following proposition also shows that the direct sum of fully cancellation modules is also fully cancellation, under another condition.

***Proposition 3.2:***

Let $M = M_1 \oplus M_2$ be an R-module, where $M_1$, $M_2$ are two submodules of M such that $M_1$, $M_2$ are fully invariant submodules. Then $M_1$, $M_2$ are fully cancellation R-modules if and only if M is a fully cancellation R-module.

***Proof*: (⇒)**

Let A, B are submodules of M and let I be a non zero ideal of R. Suppose that IA = IB. Since $M_1$, $M_2$ are fully invariant submodules, then $A = (A \cap M_1) \oplus (A \cap M_2)$ and $B = (B \cap M_1) \oplus (B \cap M_2)$ [1]. Therefore $I(A \cap M_1) \oplus (A \cap M_2) = I(B \cap M_1) \oplus (B \cap M_2)$. So $I(A \cap M_1) = I(B \cap M_1)$ and $I(A \cap M_2) = I(B \cap M_2)$. Hence $A \cap M_1 = B \cap M_1$ and $A \cap M_2 = B \cap M_2$, since $M_1$, $M_2$ are fully cancellation. Thus A = B.

**(⇐)** It follows by (Rem & Ex.2.3 (2)) □

**4. Localization of Fully Cancellation (naturally Cancellation) Modules**

In this section, we study the localization of fully (naturally) cancellation modules, also we study the behaviour of a module if its localization is fully.

This section start with the following lemma which is a generalization of Prop.4.13,P 70 in [10].

***Lemma 4.1:***

Let M be an R-module, and let A, B are submodules of M. Then A = B if and only if $A_p = B_p$ ,for every maximal ideal P of R.

***Proof*: (⇒) Clear**

**(⇐)** To prove that A = B, let $a \in A$, then $\frac{a}{1} \in A_p \subseteq B_p$ for every maximal ideal P of R. thus $\frac{a}{1} = \frac{b}{t_p}$ for some $t_p \notin P, b \in B$. Hence there exists $c_p \notin P$ such that $c_p t_p a = b t_p \in B$, put $c_p t_p = r_p \notin P$ .For every maximal ideal P of R, there exist $r_p \notin P, r_p a \in B$. Now, let E be the ideal by$\{r_p$ :P is maximal ideal of R}, then E = R because if E≠R,∃ J, a maximal ideal of R such that $C \subseteq J$. So $r_q \in J$ this is contradiction. Hence C=R and so $1 = a_1 r_{p_1} + a_2 r_{p_2} + ... + a_k r_{p_k}$ for some $k \in Z$, therefore $a = a_1 r_{p_1} + a_2 r_{p_2} + ... + a_k r_{p_k}$,

$a \in B$.Thus A⊆B, Similarly B⊆A, and hence A=B. □

The following proposition shows that a finitely generated R-module is naturally cancellation if $M_p$ is a naturally cancellation $R_p$ –module. Compare with [11,Th.3.3].

***Proposition 4.2:***

Let M be a finitely generated R-module, and let $M_p$ be a naturally cancellation $R_p$–module (for every maximal ideal P of R), then M is a naturally cancellation R –module.

***Proof*:**

Let A,B,C are submodules of M. Assume A.B = A.C then $(A.B)_p = (A.C)_p$ for every maximal ideal P of R; that is

$[(A:M)(B:M)M]_p=[(A:M)(C:M)M]_p$. Then by [13, P.172 Exc 9.11(i)] $(A:M)_p$ $(B:M)_p$ $M_p$ = $(A:M)_p$ $(C:M)_p$ $M_p$. But M finitely generated so by [11, Lemma 9.12(ii) P.172] $(A_p :_{R_p} M_p)(B_p :_{R_p} M_p) M_p = (A_p :_{R_p} M_p)(C_p :_{R_p} M_p) M_p$. This means $A_p . B_p = A_p . C_p$, for every maximal ideal P of R. Hence $B_p = C_p$. Thus by lemma (4.1), B = C. □

The converse of the last proposition will be given through the next proposition.

***Proposition 4.3:***

Let M be a finitely generated naturally cancellation R–module. Then $M_p$ is a naturally cancellation $R_p$ –module provided that if $U_p = V_p$, then U = V, for every submodules U,V of M.

***Proof*:**

Let $U_p$, $V_p$, $A_p$, $B_p$ are submodules of $M_p$ and assume that $U_p .A_p = U_p .B_p$. Hence $(U_p :_{R_p} M_p)(A_p :_{R_p} M_p) M_p = (U_p :_{R_p} M_p)(B_p :_{R_p} M_p) M_p$. Since M is finitely generated, then $(U:_R M)_p (A:_R M)_p M_p = (U:_R M)_p (B:_R M)_p M_p$. Hence $[(U:_R M)(A:_R M) M]_p = [(U:_R M)(B:_R M) M]_p$ .Thus $(U.A)_p = (U.B)_p$ [2].Then by hypothesis U.A = U.B and M is naturally cancellation then A = B. Thus $A_p = B_p$. □

Now, we will study this property on the fully cancellation module. Compare this result with Prop.4.2 ,Prop.4.3.

***Proposition 4.4:***

Let M be R-module, then $M_p$ is fully cancellation (for every maximal ideal P of R) iff M is a fully cancellation R-module.

***Proof*: (⇒)**

Suppose that IA = IB, where I is an ideal of R and A, B are submodules of M. Then $(IA)_p = (IB)_p$ ,for every maximal ideal P of R. Then $I_p A_p = I_p B_p$ [11, Exc 9.11(i) P.172]. But $M_p$ is fully cancellation, so $A_p = B_p$, for every maximal ideal P of R. Thus by lemma (4.1) we have A = B.

**(⇐)** Let P be any maximal ideal, I be an ideal of R and let A be submodule of M, we have $I_p \frac{a}{s} \in I_p B_p$,where $I_p$ is an ideal of the ring $R_p$, $A_p$, $B_p$ are submodules of $R_p$ –module $M_p$ and $\frac{a}{s} \in A_p$ . Thus for any x∈I, we have $\frac{x}{1} \in I_p$ and $\frac{x}{1}.\frac{a}{s} \in I_p .B_p$ and then $\frac{xa}{s} = \sum_{i=1}^{n} \frac{k_i}{s_i} \frac{b_i}{t_i}$, where $k_i \in I, b_i \in B, s_i \notin P$, $t_i \notin P$ .Thus $\frac{xa}{s} = \sum_{i=1}^{n} \frac{k_i b_i}{s_i^{/}}$ where $s_i^{/} = s_i.t_i$ . Therefore

$\frac{xa}{s} = \frac{k_1 b_1 u_1 + k_2 b_2 u_2 + ... + k_n b_n u_n}{v}$ where $v = s_1^{/}.s_2^{/}...s_n^{/}$,

$u_1 = s_1^{/}.s_3^{/}...s_n^{/}, ... u_n = s_1^{/}.s_2^{/}...s_{n-1}^{/}$ .Thus there exists $k \notin P$ such that $k\,x\,a\,v = (k_1 b_1 u_1 + k_2 b_2 u_2 + ... + k_n b_n u_n) sk$, but $k\,x\,a\,v \in$ IA, $(k_1 b_1 u_1 + k_2 b_2 u_2 + ... + k_n b_n u_n) sk \in IB$. But M is fully cancellation so by Th 2.6 [6] we have $a \in B$. Thus $\frac{a}{s} \in B_p$. Therefore $M_p$ is fully cancellation $R_p$ –module. □

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**الخلاصة**

في هذا البحث قدمنا نوعين من المقاسات وهما مقاس الحذف التام ومقاس الحذف الطبيعي. واعطيت بعض التشخيصات حول هذه الافكار وبعض خصائص هذه الافكار في صنف مقاسات الضرب قد درست. وأخيرا الجمع المباشر للمقاسات التامة وسلوك المقاسات التامة (المقاسات الحذف الطبيعي) قد تمت مناقشتها.