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The Generalized Taylor Expansion Method for Solving Some Types of Fractional Non-local Problems

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Abstract

The aim of this paper is to prove the existence and the uniqueness of the solution for some types of fractional non-local problems, namely the non-linear non-local initial value problems for fractional Fredholm-Volterra integro-differential equations. Also, the generalized Taylor expansion method is used to solve the non-local initial value problem that consists of the linear fractional Fredholm-Volterraintegro-differential equation together with the linear non-local initial condition with some illustrative examples.

Keywords**:** Non-Local Problems, Taylor Expansion Method, Fractional Fredholm-Voltera Integro-Differential Equations.

Introduction

The nonlocal conditions for the initial value problems appear when values of the function on the interval are connected to values inside the domain. Such problems are known as nonlocal problems, [١]. Many researchers studied the nonlocal problems, say [١] discussed the existence and uniqueness for the solutions of the nonlocal initial value problems for the non-linear ordinary differential equations, [٤] used the finite difference method to solve special types of nonlocal problems for partial differential equations, [٧] used the homotopy perturbation method to solve some types of the non-local initial value problems of fractional differential and integro-differential equations.

The fractional nonlocal Problems have been studied by several researchers such as [١١] discussed the Nonlocal Cauchy problem for fractional evolution equations, [٣] discussed the Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, [١٠] discussed the nonlocal problems for fractional differential equations in Banach space, [٢] discussed the Nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions.

Existence and Uniquenees of the Solutions for the Non-Local Initial Value Problems for Non-Linear Fractional Fredholm-Volterra Integro-Differential Equations

Recall that if u is an absolutely continuous function on [a,b], the left and the right hand Caputo fractional derivative of u of order $\alpha > ٠$, can be defined as:

$$^{C}D^{\alpha}_{a+}u(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{u^{(n)}(y)}{(x-y)^{\alpha-n+1}}dy,$$

$$a \le x \le b$$

and

$$^{C}D^{\alpha}_{b-}u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{u^{(n)}(y)}{(x-y)^{\alpha-n+1}}dy,$$

$$a \le x \le b$$

respectively, where $n-١ < \alpha \le n$, n is a non-negative integer, [٨].

In this section we shall discuss the existence of the unique solution for the non-linear non-local initial value problem that consists of the non-linear fractional Fredholm-Volterra integro-differential equation of order α:

$$^{C}D^{\alpha}_{a+}u(x) = f(x,u(x)) + \int_a^b k(x,y,u(y))dy + \int_a^x l(x,y,u(y))dy \quad ....(1.1)$$

Together with the non-linear non-local initial condition:

$$u(a)=\int_a^b w(x,u(y))dy\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

where $u \in C[a, b]$, $k: [a, b] \times [a, b] \times R \longrightarrow R$ and $l : [a, b] \times [a, b] \times R \longrightarrow R$ are continuous functions, $f: [a,b] \times R \longrightarrow R$, $w: [a, b] \times R \longrightarrow R$ are continuous functions and ${}^{C}D^{\alpha}_{a^+}$ is the left hand Caputo fractional derivative of u of order α, $0<\alpha\le 1$. To do this we shall give the following theorem.

Theorem (٣,١):

Consider the non-linear non-local initial value problem given by equations (٢,١)-(٢,٢). If the following conditions are satisfied:

(١) f and w satisfy the Lipschitz condition with respect to the second argument with Lipschitz constants F and W respectively.

(٢) k and ℓ satisfy Lipschitz condition with respect to the third argument with Lipschitz constants K and L respectively.

(٣) $W(b-a)+\dfrac{F(b-a)^{\alpha}+\left(K+L\right)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}<1.$

Then the non-linear non-local initial value problem given by equations (٢,١)-(٢,٢) has a unique solution.

Proof:

It is known that C[a,b] is a Banach space with respect to the following norm:

$$\|u\|_{C[a,b]}=\sup_{a\le x\le b}|u(x)|.$$

It is easy to check that the non-local initial value problem given by equations (١.١)-(١.٢) is equivalent to the non-linear integral equation:

$$u(x)=\int_a^b w(y,u(y))dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y,u(y))dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s,u(s))ds\right]dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y l(y,s,u(s))ds\right]dy$$

.... (١.٣)

Let A be an operator that is defined by

$$Au(x)=\int_a^b w(y,u(y))dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y,u(y))dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s,u(s))ds\right]dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y l(y,s,u(s))ds\right]dy$$

$$|Au(x)-Av(x)|\le\int_a^b |w(y,u(y))-w(y,v(y))|dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|f(y,u(y))-f(y,v(y))|dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |k(y,s,u(s))-k(y,s,v(s))|ds\right]dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y |l(y,s,u(s))-l(y,s,v(s))|ds\right]dy$$

$$\le \int_a^b W\left|u(y)-v(y)\right|dy + \frac{F}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left|u(y)-v(y)\right|dy + \frac{K}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b \left|u(s)-v(s)\right|ds\right]dy + \frac{L}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \left|u(s)-v(s)\right|ds\right]dy$$

$$\le W\left\|u-v\right\|_{C[a,b]}(b-a) + \left(\frac{F}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy\right)\left\|u-v\right\|_{C[a,b]} + \left(\frac{K(b-a)}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy\right)\left\|u-v\right\|_{C[a,b]} + \left(\frac{L(b-a)}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy\right)\left\|u-v\right\|_{C[a,b]}$$

$$\le \left[W(b-a) + \frac{F(b-a)^{\alpha} + \left(K+L\right)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right].\left\|u-v\right\|_{C[a,b]}$$

Since

$$W(b-a) + \frac{F(b-a)^{\alpha} + \left(K+L\right)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} < 1$$

one can have A is a contraction operator. Therefore by using the Banach fixed point theorem, there exists a unique solution to the integral equation (١.٣) which is the unique solution for the equations (١.١)-(١.٢)

Some Basic Concepts of Generalized Taylor Formula

In this section we shall give some basic concepts for the generalized Taylor formula.

***Theorem (٣,٢):*, (Generalized Taylor Formula), [٩]**

Suppose that

$$\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u \in C[a,b],\ i=0,1,\ldots,N+1$$

where ٠ < α ≤ ١, then

$$u(x) = \sum_{i=0}^{N} \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha} + \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1} u\right)(c)}{\Gamma\left((N+1)\alpha+1\right)}(x-a)^{(N+1)\alpha}$$

where $a \le c \le x\ \ \forall\ x \in (a, b]$.

Remarks, [٩]:

(١) For α = ١, theorem (٣.٢) reduces to the classical Taylor formula.

(٢) The generalized Taylor series for u ∈ C [a, b] takes the form:

$$\sum_{i=0}^{\infty} \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha}$$

(٣) Suppose that

$$\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u \in C[a,b],\ i=0,1,\ldots,N+1$$

and ٠< α ≤ ١, then

$$u(x) \cong u_N(x) = \sum_{i=0}^{N} \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha}$$

Furthermore, the error term $R_N(x)$ has the form:

$$R_N(x) = \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1} u\right)(a)}{\Gamma\left((N+1)\alpha+1\right)}(x-a)^{(N+1)\alpha}$$

where $a \le c \le x\ \ \forall\ x \in (a, b]$.

The Generalized Taylor Expansion Method for Solving Linear Fractional Fredholm-Volterra Integro-Differential Equations with Non-local Initial Condition

In this section we will use the generalized Taylor expansion method to solve the non-local initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation of order α of the second kind:

$$ {}^{C}D_{a+}^{\alpha}u(x) = g(x) + \lambda_1 \int_a^b k(x,y)u(y)dy + \lambda_2 \int_a^x l(x,y)u(y)dy, $$

$0<\alpha \le 1$ ....(1.4)

together with the linear non-local initial condition:

$$ u(a) = \mu_1 \int_a^b u(y)dy + \mu_2 \text{ ....(1.5)} $$

where g, u∈C[a, b],

$k:[a,b]\times[a,b] \longrightarrow R,\ \ell:[a,b]\times[a,b] \longrightarrow R$ are continuous functions, ${}^{C}D_{a+}^{\alpha}u(x)$ is the left hand Caputo derivative of u of order α, $\mu_1, \mu_2, \lambda_1, \lambda_2$ are known constants.

To do this, we assume that the solution u of the non-local initial value problem given by equations (١.٤)-(١.٥) can be approximated as a generalized Taylor's formula:

$$ u(x) \cong u_N(x) = \sum_{i=0}^{N} \frac{\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha}, $$

$a \le x \le b$

.... (١.٦)

By substituting equation (١.٦) into equations (١.٤)-(١.٥), one can have:

$$ {}^{C}D_{a+}^{\alpha}u(x) = g(x) + \sum_{i=0}^{N} \frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a)\left[\lambda_1 \int_a^b k(x,y)(y-a)^{i\alpha}dy + \lambda_2 \int_a^x \ell(x,y)(y-a)^{i\alpha}dy\right] $$

and

$$ u(a) = \mu_1 \sum_{i=0}^{N} \frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a)\int_a^b (y-a)^{i\alpha}dy + \mu_2 $$

$$ = \mu_1 \sum_{i=0}^{N} \frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a) + \mu_2 $$

So

$$ {}^{C}D_{a+}^{\alpha}u(a) = g(a) + \sum_{i=0}^{N} \frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a).\left[\lambda_1 \int_a^b k(a,y)(y-a)^{i\alpha}dy\right] $$

.... (١.٧)

and

$$ \left[1-\mu_1(b-a)\right]u(a) - \mu_1 \sum_{i=1}^{N} \frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a) = \mu_2 $$

.... (١.٨)

Let

$$ a_{i,0} = \frac{-\lambda_1}{\Gamma(i\alpha+1)}\int_a^b k(a,y)(y-a)^{i\alpha}dy, $$

$i=0,1,\dots,N,\ f_0 = 1-\mu_1(b-a)$

and $f_i = -\mu_1 \dfrac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}$, i=1,2,...,N.

Then equations (١.٧)-(١.٨) become:

$$ \sum_{\substack{i=0\\ i\neq 1}}^{N} a_{i,0}\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a) + \text{ .... (١.٩)} $$

$$ (1+a_{1,0})\,{}^{C}D_{a+}^{\alpha}u(a) = g(a) $$

and

$$ \sum_{i=0}^{N} f_i\left(\left({}^{C}D_{a+}^{\alpha}\right)^i u\right)(a) = \mu_2 \text{ ....(1.10)} $$

Let

$$ m_i(x) = \int_a^b k(x,y)(y-a)^{i\alpha}dy,\ p_i(x) = \int_a^x l(x,y)(y-a)^{i\alpha}dy,\ i=0,1,\dots,N $$

Then

$$\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j+1}u\right)(x)=\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}g\right)(x)+\sum_{i=0}^{N}\left[\frac{\lambda_1\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}m_i\right)(x)+\lambda_2\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}p_i\right)(x)}{\Gamma(i\alpha+1)}\right].\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)$$

where $j=1,2,...,N-1$.

So,

$$\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j+1}u\right)(a)=\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}g\right)(a)+\sum_{i=0}^{N}\left[\frac{\lambda_1\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}m_i\right)(a)+\lambda_2\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}p_i\right)(a)}{\Gamma(i\alpha+1)}\right].\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a) \qquad \text{………………………(١.١١)}$$

where $j=1,2,...,N-1$.

Let

$$a_{i,j}=-\frac{\lambda_1\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}m_i\right)(a)+\lambda_2\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}p_i\right)(a)}{\Gamma(i\alpha+1)},$$

$i=0,1,...,N,\ j=1,2,...,N-1.$

Then equation (١.١١) becomes

$$\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j+1}u\right)(a)+\sum_{i=0}^{N}a_{i,j}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)=\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{j}g\right)(a),\ j=1,2,...,N-1 \qquad \text{………………………(١.١٢)}$$

Thus, by evaluating equation (١.٢٤) at each j=١,٢,…,N–١ and by using equations (١.٩)-(١.١٠), one can have the following linear system of N+١ equations with (N+١) unknowns $\left\{\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)\right\}_{i=0}^{N}$:

AU=B ……………………………………………(١.١٣)

where

$$A=\begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdots & f_{N-1} & f_N \\ a_{0,0} & 1+a_{1,0} & a_{2,0} & a_{3,0} & \cdots & a_{N-1,0} & a_{N,0} \\ a_{0,1} & a_{1,1} & 1+a_{2,1} & a_{3,1} & \cdots & a_{N-1,1} & a_{N,1} \\ a_{0,2} & a_{1,2} & a_{2,2} & 1+a_{3,2} & \cdots & a_{N-1,2} & a_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{0,N-2} & a_{1,N-2} & a_{2,N-2} & a_{3,N-2} & \cdots & 1+a_{N-1,N-2} & a_{N,N-2} \\ a_{0,N-1} & a_{1,N-1} & a_{2,N-1} & a_{3,N-1} & \cdots & a_{N-1,N-1} & 1+a_{N,N-1} \end{bmatrix},$$

$$U=\begin{bmatrix} u(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{2}u\right)(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{3}u\right)(a) \\ \vdots \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)^{(N-1)}(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)^{(N)}(a) \end{bmatrix}$$

and

$$B=\begin{bmatrix} \mu_2 \\ g(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)g\right)(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{2}g\right)(a) \\ \vdots \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)g\right)^{(N-2)}(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)g\right)^{(N-1)}(a) \end{bmatrix}.$$

By solving the above linear system of equations, one can get the values of

$$\left\{\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)\right\}_{i=0}^{N}.$$

These values are substituted into equation (١.٦) to get the approximated solution of the non-local initial value problem given by equations (١.٤)-(١.٥).

To illustrate this method, consider the following example:

Example:

Consider the nonlocal initial value problem that consists of the fractional linear Fredholm-Volterraintegro-differential equation of order $\frac{1}{2}$:

$$ {}^{C}D_{0^+}^{1/2}u(x) = -\frac{3}{2} - \frac{25}{4}x^2 + \frac{16}{\sqrt{\pi}}x^{\frac{5}{2}} - \frac{23}{4}x^5 + \int_0^1 (x^2+y)u(y)dy + \int_0^x (3x+2y)u(y)dy \quad \text{............................ (١.١٤)} $$

together with the nonlocal linear initial condition:

$$ u(0) = 2\int_0^1 u(y)dy - \frac{7}{2} \quad \text{............................ (١.١٥)} $$

We use the generalized Taylor expansion method to solve this fractional linear nonlocal initial value problem. To do this, let N =١, then equation (١.٦) takes the form:

$$ u(x) \cong u_1(x) = u(0) + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}\sqrt{x}, $$

$0 \le x \le 1$

$$ \text{........................... (1.16)} $$

Then the system given by equation (١.١٣) takes the form:

$$ \begin{pmatrix} -1 & \frac{-8}{3\sqrt{\pi}} \\ -\frac{1}{2} & \frac{5\sqrt{\pi}-4}{5\sqrt{\pi}} \end{pmatrix} \begin{pmatrix} u(0) \\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0) \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{3}{2} \end{pmatrix} $$

which has the solution:

$$ u(0) = \frac{3(35\sqrt{\pi}+12)}{2(15\sqrt{\pi}+8)} \cong 3.21087 $$

and

$$ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0) = \frac{1}{4} - \frac{2}{15\sqrt{\pi}+8} \cong 0.192174. $$

By substituting these values into equation (١.١٦) one can have:

$$ u(x) \cong u_1(x) = 3.21087 + 0.216846\sqrt{x}, $$

$$ 0 \le x \le 1 $$

By substituting this approximated solution into equation (١.١٤) one can have:

$$ {}^{C}D_{0^+}^{1/2}u_1(x) + \frac{3}{2} + \frac{25}{4}x^2 - \frac{16}{\sqrt{\pi}}x^{\frac{5}{2}} + \frac{23}{4}x^5 - \int_0^1 (x^2+y)u_1(y)dy - \int_0^x (3x+2y)u_1(y)dy $$

$$ \cong 5.52233 - 5.625x^2 - 92.625x^{\frac{5}{2}} + 5.75x^5 $$

Since the right hand side of the above equation does not equal zero, so we must increase the value of N. Therefore, let N=٢, then equation (٤,٦) takes the form:

$$ u(x) \cong u_2(x) = u(0) + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^2 u\right)(0)}{\Gamma(2)}x, \quad 0 \le x \le 1 $$

$$ \text{............................ (١.١٧)} $$

Thus the system given by equation (١.١٣) takes the form:

$$ \begin{pmatrix} -1 & \frac{-8}{3\sqrt{\pi}} & -1 \\ \frac{-1}{2} & \frac{5\sqrt{\pi}-4}{5\sqrt{\pi}} & \frac{-1}{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(0) \\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0) \\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^2 u\right)(0) \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{3}{2} \\ 0 \end{pmatrix} $$

which has the solution:

$$ u(0) = \frac{3(35\sqrt{\pi}+12)}{2(15\sqrt{\pi}+8)} \cong 3.21087 $$

$$\left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0)=\frac{1}{4}-\frac{2}{15\sqrt{\pi}+8}\cong 0.192174$$

and $\left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{2}u\right)(0)=0.$

By substituting these values into equation (١.١٧) one can have:

$$u(x)\cong u_2(x)=3.21087+0.216846\sqrt{x},$$

$$0\le x\le 1.$$

Since $u_2(x)=u_1(x)$, so we must increase the value of N. By continuing in this manner one can get for N=٦, equation (١،٦) takes the form:

$$\begin{aligned}u(x)\cong u_6(x)&=u(0)+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{2}u\right)(0)}{\Gamma(2)}x\\&+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{3}u\right)(0)}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{4}u\right)(0)}{\Gamma(3)}x^{2}\\&+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{5}u\right)(0)}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{6}u\right)(0)}{\Gamma(4)}x^{3},\end{aligned}$$

$$0\le x\le 1$$

............................(١.١٨)

Then the system given by equation (١.١٣) takes the form:

$$\begin{pmatrix}-1 & \frac{-8}{3\sqrt{\pi}} & -1 & \frac{-16}{15\sqrt{\pi}} & \frac{-1}{3} & \frac{-32}{105\sqrt{\pi}} & \frac{-1}{12}\\ \frac{-1}{2} & 1-\frac{4}{5\sqrt{\pi}} & \frac{-1}{3} & \frac{-8}{21\sqrt{\pi}} & \frac{-1}{8} & \frac{-16}{135\sqrt{\pi}} & \frac{-1}{30}\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ -10 & \frac{-8}{3\sqrt{\pi}} & -1 & \frac{-16}{15\sqrt{\pi}} & \frac{-1}{3} & 1-\frac{32}{105\sqrt{\pi}} & \frac{-1}{12}\\ 0&0&0&0&0&0&1\end{pmatrix}.$$

$$\begin{pmatrix}u(0)\\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0)\\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{2}u\right)(0)\\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{3}u\right)(0)\\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{4}u\right)(0)\\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{5}u\right)(0)\\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{6}u\right)(0)\end{pmatrix}=\begin{pmatrix}-\frac{7}{2}\\ -\frac{3}{2}\\ 0\\ 0\\ 0\\ -\frac{25}{2}\\ 30\end{pmatrix}$$

which has the solution:

$$u(0)=1,$$

$$\left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{i}u\right)(0)=0,\ \ i=1,2,3,4,5$$

and

$$\left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^{6}u\right)(0)=30.$$

By substituting these values into equation (١.١٨) one can have:

$$u(x)\cong u_6(x)=1+5x^{3},\quad 0\le x\le 1$$

By substituting this approximated solution into equations (١.٩)-(١.١٠) one can get:

$$
{}^{C}D_{0^+}^{1/2}u_6(x) + \frac{3}{2} + \frac{25}{4}x^2 - \frac{16}{\sqrt{\pi}}x^{\frac{5}{2}} + \frac{23}{4}x^5
$$

$$
-\int_0^1 (x^2 + y)u_6(y)dy - \int_0^x (3x + 2y)u_6(y)dy = 0
$$

and

$$
u_6(0) = 2\int_0^1 u_6(y)dy - \frac{7}{2}.
$$

Therefore u_6 is the exact solution of the linear nonlocal prolom given by equations (١.١٤)-(١.١٥).

Conclusions

From this work, one can concluded that the following apects:

(١) The existence and the uniqueness of the solution for the non-linear non-local initial value problem is a generalization of the existence and the uniqueness of the solution for the linear local initial value problem.

(٢) The generalized Taylor expansion method like the classical Taylor expansion method gave more accurate results as N increases.

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الخلاصة

الهدف من هذا البحث هو لأثبات وجود ووحدانية الحل بعض الانواع من المسائل اللامحلية الكسورية.

هذه الانواع هي مسائل القيم الابتدائية اللامحلية الكسورية التي تتضمن معادلات فريدهولم- فولتيرا التكاملية التفاضلية الكسورية اللاخطية مع الشروط الابتدائية اللامحلية اللاخطية.

وكذلك تم تقديم طريقة توسيع معمم تيلر لحل مسائل القيمة الابتدائية اللامحلية التي تتضمن معادلة فريدهولم- فولتيرا التكاملية التفاضلية الكسورية الخطية مع الشرط الابتدائي اللامحلي, مع بعض الامثلة التوضيحية.