The Effect of Two Infectious Diseases and Harvesting in the Eco-Epidemiological Model

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Abstract

In this paper, an Eco-Epidemiological model with different infectious diseases in prey population and the optimal harvesting in predator population is proposed and studied. Linear type used to describe the functional response. Sufficient and necessary conditions for existence each equilibrium points of the system are established, the bounded and stability analysis of all possible equilibrium points are studied. The effect of harvest on the stability of this system is investigated. Finally, the dynamical behavior of system is discussed using Numerical simulation.

Keywords: SIS epidemics disease, Eco-Epidemiological model, Harvest management; stability analysis.

Introduction

Diseases in a prey-predator system have received significant interest in recent years. In fact, any given environment may contain many or hundreds of species. Since any species has at least the potential to interact with any other species in its environment, the possibility of spread of the diseases in a community rapidly becomes astronomical as the number of infected species in the environment increases. Therefore, it is more of biological significance to study the effect of disease on the dynamical behavior of interacting species. The first model described a prey–predator model involving disease in prey species formulated by Anderson and May [1], Later on many researchers, especially in the last two decades, have proposed and studied different predator–prey models in presence of disease in one of the species see for example [2-11] and the references there in. The effect of constant-rate harvesting on the dynamics of predator-prey systems has been investigated by many authors, see, for example [12-16] and the references there in. On the other hand, many researchers proposed and study eco-epidemical models containing two disease strains in the same population. see for example [17,19] and the references there in. On contrast to all the above studies, in this paper a prey-predator model involving, in additional to harvest in predator species, two different SIS infectious diseases in prey species is proposed and analyzed. It is assumed that both the diseases spread within prey population by contact, between susceptible individuals and infected individuals. Furthermore, in this model, using linear functional response and linear incidence rate to describe spread both of diseases first and second.

Mathematical Model

To describe the model for an Eco-Epidemiological system, we consider the following notation:

1. Let $N(t)$ and $p(t)$ be the population densities of the prey species and predator species at time t, respectively.
2. The prey grows logistically with intrinsic growth rate $h_1 > 0$ and carrying capacity $h_2 > 0$.
3. There are two different SIS epidemic diseases spread among the prey population and it transmitted between the prey individuals (but not the predator) by contact, according to linear incidence rate with first and second infection rate constants $h_3 > 0$ and $h_4 > 0$, respectively. Therefore, the total prey population is divided into three classes: susceptible that is denoted by $x(t)$, infected by first disease that is denoted by $y(t)$ infected by second disease that is denoted by $z(t)$. Hence at any time t the total prey population is $N(t) = x(t) + y(t) + z(t)$.
4. The predator preys upon prey according to linear functional response with maximum attack rates $h_5 > 0$, $h_9 > 0$ and $h_{11} > 0$,

respectively. Furthermore it is assumed that $e>0$ represent the conversion rate constant

5. Both of the infected prey can be recovered and become susceptible again with recovery rate constant $h_6>0$ and $h_7>0$, respectively.
6. Furthermore it is assumed that there is disease induced mortality rate represented by $h_8>0$ and $h_{10}>0$, respectively.
7. The predator grows logistically with intrinsic growth rate $h_{12}>0$ and carrying capacity $h_{13}>0$.
8. Finally, $q>0$ is the catch ability co-efficient of the predator, $E>0$ is the harvesting effort and qEp is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis.

Consequently, the model with the above assumptions can be written in the following form:

$$\frac{dx}{dt}=x\left[h_1(1-\frac{x+y+z}{h_2})-h_3y-h_4z-h_5p\right]+h_6y+h_7z$$
$$\frac{dy}{dt}=y(h_3x-h_6-h_8-h_9p)$$
$$\frac{dz}{dt}=z(h_4x-h_7-h_{10}-h_{11}p)$$
$$\frac{dp}{dt}=p\left[h_{12}\left(1-\frac{p}{h_{13}}\right)+eh_5x+eh_9y+eh_{11}z-qE\right]$$
.................................(1)

The system (1) has the following domain $\Re_+^4=\{(x,y,z,p),\ x\geq 0, y\geq 0, z\geq 0\ and\, p\geq 0\}$.
Moreover, the above four nonlinear differential equations are continuously differentiable on int. $\Re_+^4$ and hence they are Lipschizian on $\Re_+^4$. Thus, for each set of initial conditions, say $x(0)\geq 0, y(0)\geq 0, z(0)\geq 0$ and $p(0)\geq 0$, system (1) has a unique solution. Therefore, the domain $\Re_+^4$ is an invariant for the system (1). Further in the following theorem the sufficient condition for uniformly bounded of the solution of the system (1) is established.

Theorem (1):

All the trajectories of system (1), which initiate in $\Re_+^4$ are uniformly bounded.

Proof:

From the first equation of system (1) we in the absence of diseases and predator obtain that $\frac{dx}{dt}\leq h_1x\left(1-\frac{x}{h_2}\right)$

From the four equation in the absence of prey we have $\frac{dp}{dt}\leq h_{12}p\left(1-\frac{p}{h_{13}}\right)$

Clearly by solving the above differential inequalities we get $\limsup_{t\to\infty} x(t)\leq h_2$ and $\limsup_{t\to\infty} p(t)\leq h_{13}$

Define the function $\mathrm{M}(t)=x(t)+y(t)+z(t)+\frac{1}{e}p(t)$ and then take its time derivative along the solution of system (1), gives

$$\frac{d\mathrm{M}}{dt}\leq h_1x+p(e)^{-1}(h_{12}-qE)-\phi y-\phi z$$
$$\leq \pi-\phi\mathrm{M}\quad \text{where}\quad \phi=\min\{h_8, h_{10}\}$$

and $\pi=(h_1+\phi)h_2+h_{13}(e)^{-1}((h_{12}-qE)+\phi)$

Now, by using Gronwall lemma [20], it obtains that $0<\mathrm{M}(t)\leq \mathrm{M}(0)e^{-\phi t}+\pi(\phi)^{-1}(1-e^{-\phi t})$ which yields $\limsup_{t\to\infty}\mathrm{M}(t)\leq \pi(\phi)^{-1}$ that is independent of the initial conditions.

For later purposes, it is necessary to have the Jacobian of system(1) at hand, it is reported below.

$$J_k=\left(\beta_{ij}^{[k]}\right)=\begin{bmatrix}\beta_{11}^{[k]} & \beta_{12}^{[k]} & \beta_{13}^{[k]} & \beta_{14}^{[k]}\\ \beta_{21}^{[k]} & \beta_{22}^{[k]} & 0 & \beta_{24}^{[k]}\\ \beta_{31}^{[k]} & 0 & \beta_{33}^{[k]} & \beta_{34}^{[k]}\\ \beta_{41}^{[k]} & \beta_{42}^{[k]} & \beta_{43}^{[k]} & \beta_{44}^{[k]}\end{bmatrix}$$

Where: $i=1,2,3,4$; $j=1,2,3,4$; $k=0,1,\ldots,9$ and

$\beta_{11}^{[k]}=h_1\left(1-\frac{2x}{h_2}\right)-\left(h_3+\frac{h_1}{h_2}\right)y-\left(h_4+\frac{h_1}{h_2}\right)z-h_5p$;

$\beta_{12}^{[k]}=\frac{-h_1}{h_2}x-h_3x+h_6$; $\beta_{13}^{[k]}=\frac{-h_1}{h_2}x-h_4x+h_7$;

$\beta_{14}^{[k]}=-h_5x$; $\beta_{21}^{[k]}=h_3y$; $\beta_{22}^{[k]}=h_3x-h_6-h_8-h_9p$;

$\beta_{23}^{[k]}=0$; $\beta_{24}^{[k]}=-h_9y$; $\beta_{31}^{[k]}=h_4z$; $\beta_{32}^{[k]}=0$;

$\beta_{42}^{[k]}=eh_9p$; $\beta_{33}^{[k]}=h_4x-h_7-h_{10}-h_{11}p$;

$\beta_{34}^{[k]}=-h_{11}z$; $\beta_{41}^{[k]}=eh_5p$; $\beta_{43}^{[k]}=eh_{11}p$;

$\beta_{44}^{[k]}=h_{12}\left(1-\frac{2p}{h_{13}}\right)+eh_5x+eh_9y+eh_{11}z-qE$

Equilibrium points

System (1) has the following equilibrium:

1. The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exists.
2. The free diseases and predator and diseases equilibrium point $E_1 = (x_1, 0,0,0)$ where $x_1 = h_2$, E_1 always exists.
3. The free prey equilibrium point $E_2 = (0,0,0,p_2)$ where: $p_2 = h_{13}(h_{12})^{-1}[h_{12} - qE]$ exists if and only if $h_{12} > qE$.
4. The first disease equilibrium point $E_3 = (x_3, y_3, 0,0)$ where: $x_3 = \frac{h_6 + h_8}{h_3}$ and $y_3 = \frac{h_1 x_3 (h_2 - x_3)}{(h_2 h_8 + h_1 x_3)}$, exists uniquely in the interior of the first quadrant of $xy-$plane under the following necessary and sufficient condition $x_3 < h_2$.
5. The second disease equilibrium point $E_4 = (x_4, 0, z_3, 0)$ where: $x_4 = (h_7 + h_{10})(h_4)^{-1}$ and $z_4 = \frac{h_1 x_4 (h_2 - x_4)}{(h_2 h_{10} + h_1 x_4)}$ exists uniquely in the interior of the first quadrant of $xz-$plane under the following necessary and sufficient condition $x_4 < h_2$
6. The susceptible prey-predator equilibrium point $E_5 = (x_5, 0, 0, p_5)$ where: $x_5 = \frac{h_2[h_5 h_{13} qE + h_1 h_{12} - h_5 h_{12} h_{13}]}{[h_1 h_{12} + e h_2 h_{13} (h_5)^2]}$ and $p_5 = \frac{h_1(h_2 - x_5)}{h_2 h_5}$, exists uniquely in the interior of the first quadrant of $xp-$plane if and only if $x_5 < h_2$ and $h_5 h_{13} qE + h_1 h_{12} > h_5 h_{12} h_{13}$
7. The free first disease equilibrium point $E_6 = (x_6, 0, z_6, p_6)$ where $p_6 = (h_4 x_6 - h_7 - h_{10})(h_{11})^{-1}$ and $z_6 = \frac{x_6[h_1(h_2 - x_6) - h_2 h_5 p_6]}{[h_1 x_6 + h_2 h_4 x_6 - h_2 h_7]}$ while x_6 represents a positive root of the equation $A_1 x^2 + A_2 x + A_3 = 0$ where:
$$A_1 = e h_1 h_5 h_{11} h_{13} - h_1 h_4 h_{12} - h_2 h_{12} (h_4)^2 - e h_1 h_{13} (h_{11})^2$$
$$A_2 = h_1 h_{11} h_{12} h_{13} + h_1 h_7 h_{12} + h_1 h_{10} h_{12} + h_2 h_4 h_{10} h_{12} + h_2 h_4 h_{11} h_{12} h_{13} + 2 h_2 h_4 h_7 h_{12} + e h_1 h_2 h_{13} (h_{11})^2 + e h_2 h_5 h_{10} h_{11} h_{13} - qE h_1 h_{11} h_{13} - qE h_2 h_4 h_{11} h_{13}$$
$$A_3 = qE h_2 h_7 h_{11} h_{13} - h_2 h_7 h_{10} h_{12} - h_2 h_{12} (h_7)^2 - h_2 h_7 h_{11} h_{12} h_{13}$$

Obviously, E_6 exists uniquely in the interior of the first octant of $xzp-$ space if and only if the following conditions are hold.

$A_1 > 0 \,;\, A_3 < 0$ .... (2.a)

or

$A_1 < 0 \,;\, A_3 > 0$ .... (2.b)

with $h_7 + h_{10} < h_4 x_6 \,;\, h_2 h_5 p_6 + h_1 x_6 < h_1 h_2$ and $h_2 h_7 < h_1 x_6 + h_2 h_4 x_6$

8. The free second disease equilibrium point $E_7 = (x_7, y_7, 0, p_7)$ where: $y_7 = \frac{\lfloor qE h_{13} - h_{12} h_{13} + h_{12} p_7 - e h_5 h_{13} x_7 \rfloor}{e h_9 h_{13}}$ and $p_7 = \frac{(h_3 x_7 - h_6 - h_8)}{h_9}$ while x_7 represents a positive root of the equation $B_1 x^2 + B_2 x + B_3 = 0$ where:
$$B_1 = e h_1 h_5 h_9 h_{13} - h_1 h_3 h_{12} - h_2 h_{12} (h_3)^2 - e h_1 h_{13} (h_9)^2$$
$$B_2 = e h_1 h_2 h_{13} (h_9)^2 + e h_2 h_5 h_6 h_9 h_{13} + e h_2 h_5 h_8 h_9 h_{13} + h_2 h_3 h_6 h_{12} - e h_2 h_5 h_6 h_9 h_{13} - qE h_2 h_3 h_9 h_{13} + h_2 h_3 h_9 h_{12} h_{13} + h_2 h_3 h_6 h_{12} + h_2 h_3 h_8 h_{12} - qE h_1 h_9 h_{13} + h_1 h_9 h_{12} h_{13} + h_1 h_6 h_{12} + h_1 h_8 h_{12}$$
$$B_3 = qE h_2 h_6 h_9 h_{13} - h_2 h_6 h_9 h_{12} h_{13} - h_2 h_{12} (h_6)^2 - h_2 h_6 h_8 h_{12}$$

Obviously, E_7 exists uniquely in the interior of the first octant of $xyp-$ space if and only if the one set of following conditions are hold.

$B_1 > 0 \,;\, B_3 < 0$ .... (3.a)

or

$B_1 < 0 \,;\, B_3 > 0$ .... (3.b)

with $h_{13}(h_{12} + e h_5 x_7) < qE h_{13} + h_{12} p_7$ and $(h_6 + h_8) < h_3 x_7$

9. The free predator equilibrium point $E_8 = (x_8, y_8, z_8, 0)$ where $x_8 = \left\{\frac{h_6 + h_8}{h_3}, \frac{h_7 + h_{10}}{h_4}\right\}$ and

$$y_8 = \frac{h_1 x_8[h_2 - x_8] - h_3 z_8[h_2 h_{10} + h_1 x_8]}{[h_2 h_8 + h_1 x_8]}, \quad E_8$$

exists uniquely in the interior of the first octant of $xyz-$ space under the following necessary and sufficient conditions $x_8 < h_2$

$$\frac{(h_6 + h_8)}{h_3} = \frac{(h_7 + h_{10})}{h_4} \text{ and } z_8 < \frac{h_1 x_8[h_2 - x_8]}{[h_2 h_{10} + h_1 x_8]}$$

10. The coexistence equilibrium point $E_9 = (x_9, y_9, z_9, p_9)$ where

$$x_9 = \frac{h_{11}(h_6 + h_8) - h_9(h_7 + h_{10})}{(h_3 h_{11} - h_4 h_9)};$$

$$p_9 = \left\{ \frac{[h_3 x_9 - (h_6 + h_8)]}{h_9}, \frac{[h_4 x_9 - (h_7 + h_{10})]}{h_{11}} \right\}$$

$$y_9 = \frac{[h_{13}(qE - eh_5 x_9 - h_{12} - eh_{11} z_9) + h_{12} p_9]}{eh_9 h_{13}}$$

$$z_9 = \frac{\begin{bmatrix} eh_9 h_{13} x_9 [h_1(h_2 - x_9) - h_2 h_5 p_9] \\ + [qEh_{13} - eh_5 h_{13} x_9 - h_{12} h_{13} + h_{12} p_9] \\ \times [h_2 h_6 - (h_1 + h_2 h_3) x_9] \end{bmatrix}}{eh_{13} x_9 \left[h_2(h_6 h_{11} - h_7 h_9) - \begin{pmatrix} h_9(h_1 + h_2 h_4) \\ - h_{11}(h_1 + h_2 h_3) \end{pmatrix} \right]}$$

exists uniquely in the $\text{Int.}\,\Re^4_+$ under the following necessary and sufficient condition

$$h_4 h_9 < h_3 h_{11},\ h_9(h_7 + h_{10}) < h_{11}(h_6 + h_8),$$
$$eh_5 h_{13} x_9 + h_{12} h_{13} + eh_{11} h_{13} z_9 < qEh_{13} + h_{12} p_9$$
$$\max\left\{ \frac{(h_6 + h_8)}{h_3}, \frac{(h_7 + h_{10})}{h_4} \right\} < x_9$$
$$x_9 < \min\left\{ h_2, \frac{h_2 h_6}{(h_1 + h_2 h_3)} \right\}, p_9 < \frac{h_1(h_2 - x_9)}{h_2 h_5}$$

Stability analysis of the system

At the equilibrium point E_0 the eigenvalues are $h_1; -(h_6 + h_8); -(h_7 + h_{10}); (h_{12} - qE)$ showing that it is an unstable saddle.

Theorem (2):

The equilibrium point E_1 is locally asymptotically stable in $\Re^4_+$ if and only if

$$\left.\begin{matrix} h_2 < \min\{(h_6 + h_8)(h_3)^{-1};\ (h_7 + h_{10})(h_4)^{-1}\} \\ (h_{12} + eh_2 h_5) < qE \end{matrix}\right\} \text{.... (2)}$$

Proof:

The Jacobian matrix of the system (1) at E_1 is given by:

$$J_1 = \begin{pmatrix} \beta^{[1]}_{11} & \beta^{[1]}_{12} & \beta^{[1]}_{13} & \beta^{[1]}_{14} \\ 0 & \beta^{[1]}_{22} & 0 & 0 \\ 0 & 0 & \beta^{[1]}_{33} & 0 \\ 0 & 0 & 0 & \beta^{[1]}_{44} \end{pmatrix} \text{ where:}$$

$\beta^{[1]}_{11} = -h_1$, $\beta^{[1]}_{12} = (h_6 - h_1 - h_2 h_3)$, $\beta^{[1]}_{14} = -h_2 h_5$,
$\beta^{[1]}_{13} = (h_7 - h_1 - h_2 h_4)$, $\beta^{[1]}_{22} = (h_2 h_3 - h_6 - h_8)$,
$\beta^{[1]}_{33} = (h_2 h_4 - h_7 - h_{10})$, $\beta^{[1]}_{44} = (h_{12} + eh_2 h_5 - qE)$,

So, the characteristic equation of J_1 can be written by

$$0 = (-h_1 - \lambda^{[1]}_x)((h_2 h_3 - h_6 - h_8) - \lambda^{[1]}_y)$$
$$\times ((h_2 h_4 - h_7 - h_{10}) - \lambda^{[1]}_z)((h_{12} + eh_2 h_5 - qE) - \lambda^{[1]}_p)$$

from which, we obtain that:

$\lambda^{[1]}_y = h_2 h_3 - h_6 - h_8$, $\lambda^{[1]}_z = h_2 h_4 - h_7 - h_{10}$,

$\lambda^{[1]}_x = -h_1$ and $\lambda^{[1]}_p = h_{12} + eh_2 h_5 - qE$

Here $\lambda^{[1]}_x, \lambda^{[1]}_y, \lambda^{[1]}_z$ and $\lambda^{[1]}_p$ denote to the eigenvalues in the $x-$direction, $y-$direction, $z-$direction and $p-$direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if the condition (2) holds. Therefore, the equilibrium point E_1 is locally asymptotically stable in $\Re^4_+$. Similarly, the equilibrium point E_2 is locally asymptotically stable if and only if $p_2 > \frac{h_1}{h_5}$. E_3 is locally asymptotically stable if and only if $h_{12} + eh_5 x_3 + eh_9 y_3 < qE$ and $x_3 < \frac{(h_7 + h_{10})}{h_4}$. E_4 is locally asymptotically stable if and only if $2x_4 + z_4 < h_2; h_{12} + eh_5 x_4 + eh_{11} z_4 < qE$ $x_4 < \frac{(h_6 + h_8)}{h_3}$ and $h_1 < \frac{h_2 h_4 z_4}{(h_2 - 2x_4 - z_4)}$. E_5 is locally asymptotically stable if and only if $x_5 < \min\left\{ \frac{(h_6 + h_8 + h_5 p_5)}{h_3}, \frac{(h_7 + h_{10} + h_{11} p_5)}{h_4} \right\}$.

Theorem (3):

If the following conditions hold

$h_3 x_6 < h_6 + h_8 + h_9 p_6$ .... (3a)

$$\left.\begin{matrix} h_{11} h_{13} < \min\left\{ \frac{h_4 h_{12}}{eh_5}; \frac{h_5 h_{12} p_6}{h_4 z_6} \right\} \\ h_1 h_2 < 2h_1 x_6 + (h_2 h_4 + h_1) z_6 + h_2 h_5 p_6 \\ h_2 h_7 < (h_2 h_4 + h_1) x_6 \end{matrix}\right\} \text{....... (3b)}$$

then E_6 is a locally asymptotically stable.

Proof:

The Jacobian matrix of the system (1) at E_6 is given by:

$$J_6=\begin{pmatrix}\beta_{11}^{[6]} & \beta_{12}^{[6]} & \beta_{13}^{[6]} & \beta_{14}^{[6]}\\ 0 & \beta_{22}^{[6]} & 0 & 0\\ \beta_{31}^{[6]} & 0 & 0 & \beta_{34}^{[6]}\\ \beta_{41}^{[6]} & \beta_{42}^{[6]} & \beta_{43}^{[6]} & \beta_{44}^{[6]}\end{pmatrix}\text{ where:}$$

$$\beta_{11}^{[6]}=h_1\left(1-\frac{2}{h_2}x_6\right)-z_6\left(h_4+\frac{h_1}{h_2}\right)-h_5p_6\ ;$$
$$\beta_{12}^{[6]}=h_6-\left(\frac{h_1}{h_2}+h_3\right)x_6\ ;\ \beta_{13}^{[6]}=h_7-\left(\frac{h_1}{h_2}+h_4\right)x_6\ ;$$
$$\beta_{14}^{[6]}=-h_5x_6\ ;\beta_{22}^{[6]}=h_3x_6-h_6-h_8-h_9p_6\ ;$$
$$\beta_{31}^{[6]}=h_4z_6\ ;\ \beta_{34}^{[6]}=-h_{11}z_6\ ;\ \beta_{41}^{[6]}=eh_4p_6\ ;$$
$$\beta_{42}^{[6]}=eh_9p_6\ ;\beta_{43}^{[6]}=eh_{11}p_6\ ;\ \beta_{44}^{[6]}=\frac{-h_{12}}{h_{13}}p_6$$

So, the characteristic equation of J_6 can be written by

$$\left(\lambda_y^{[6]}-\beta_{22}^{[6]}\right)\left[\left(\lambda^{[6]}\right)^3+F_1\left(\lambda^{[6]}\right)^2+F_2\left(\lambda^{[6]}\right)+F_3\right]=0$$

with

$$F_1=-\left(\beta_{11}^{[6]}+\beta_{44}^{[6]}\right)\ ;$$
$$F_2=\beta_{11}^{[6]}\beta_{44}^{[6]}-\beta_{14}^{[6]}\beta_{41}^{[6]}-\beta_{13}^{[6]}\beta_{31}^{[6]}-\beta_{43}^{[6]}\beta_{34}^{[6]}\ ;$$
$$F_3=\beta_{43}^{[6]}\beta_{34}^{[6]}\beta_{11}^{[6]}+\beta_{13}^{[6]}\beta_{31}^{[6]}\beta_{44}^{[6]}-\beta_{14}^{[6]}\beta_{43}^{[6]}\beta_{31}^{[6]}-\beta_{13}^{[6]}\beta_{34}^{[6]}\beta_{41}^{[6]}$$

Here $\lambda_y^{[6]}$ denote to the eigenvalues in the $y-$direction. The Routh-Hurwitz conditions require $F_1>0$; $F_3>0$ and $\Delta=F_1F_2-F_3>0$, which follows from condition (3b) and in addition the negativity of the other eigenvalues, namely condition (3a). So, according to Routh-Hurwitz criterion E_6 is locally asymptotically stable.

<u>Theorem (4):</u>

If the following conditions hold

$h_4x_7<h_7+h_{10}+h_{11}p_7$ .... (4a)

$$\left.\begin{array}{l}h_9h_{13}<\min\{h_3h_{12}(eh_5)^{-1}\ ;h_5h_{12}p_7(h_3z_7)^{-1}\}\\ h_1h_2<2h_1x_7+(h_2h_3+h_1)y_7+h_2h_5p_7\\ h_2h_6<(h_2h_3+h_1)x_7\end{array}\right\}$$

....(4b)

then E_7 is a locally asymptotically stable.

Proof:

Due to the form of eigenvalues of J_7, the proof is follows directly as that given in theorem(3).

<u>Theorem (5):</u>

The equilibrium point E_8 is locally asymptotically stable in the subregion in $\Re_+^4$ if and only if:

$$\left.\begin{array}{l}x+y+z<h_2\ ;\ p<h_{13}\ ;\\ x<\min\left\{\frac{(h_6y+h_7z)}{(h_3y_8+h_4z_8)},x_8\right\}\\ eh_5x_8+eh_9y_8+eh_{11}z_8+h_{12}\left(1-\frac{p}{h_{13}}\right)<qE\end{array}\right\}\ ...(5)$$

Proof:

The Jacobian matrix of the system (1) at E_8 is given by:

$$J_8=\begin{pmatrix}\beta_{11}^{[8]} & \beta_{12}^{[8]} & \beta_{13}^{[8]} & \beta_{14}^{[8]}\\ \beta_{21}^{[8]} & 0 & 0 & \beta_{24}^{[8]}\\ \beta_{31}^{[8]} & 0 & 0 & \beta_{34}^{[8]}\\ 0 & 0 & 0 & \beta_{44}^{[8]}\end{pmatrix}\text{ where:}$$

$$\beta_{11}^{[8]}=h_1\left(1-\frac{2}{h_2}x_8\right)-y_8\left(h_3+\frac{h_1}{h_2}\right)-z_8\left(h_4+\frac{h_1}{h_2}\right);$$
$$\beta_{12}^{[8]}=-\left(\frac{h_1}{h_2}x_8+h_8\right);\beta_{13}^{[8]}=-\left(\frac{h_1}{h_2}x_8+h_{10}\right);$$
$$\beta_{14}^{[8]}=-h_5x_8\ ;\beta_{21}^{[8]}=h_3y_8\ ;\beta_{24}^{[8]}=-h_9y_8\ ;\beta_{31}^{[8]}=h_4z_8;$$
$$\beta_{34}^{[8]}=-h_{11}z_8;\beta_{44}^{[8]}=h_{12}+e(h_5x_8+h_9y_8+h_{11}z_8)-qE$$

So, the characteristic equation of J_8 can be written by

$$\lambda^{[8]}\left(\lambda_p^{[8]}-\beta_{44}^{[8]}\right)\left[\left(\lambda^{[8]}\right)^2-\beta_{11}^{[8]}\lambda^{[8]}-\left(\beta_{13}^{[8]}\beta_{31}^{[8]}+\beta_{12}^{[8]}\beta_{21}^{[8]}\right)\right]=0$$

Since $\lambda^{[8]}=0$ that mean E_8 is non-hyperbolic equilibrium point, then consider the function

$$V^{[8]}=\left(x-x_8-x_8\ln\frac{x}{x_8}\right)+\left(y-y_8-y_8\ln\frac{y}{y_8}\right)+\left(z-z_8-z_8\ln\frac{z}{z_8}\right)+\frac{p}{e}$$

Clearly, $V^{[8]}:\Re_+^4\to\Re$ and $V^{[8]}(E_8)=0$ with $V^{[8]}(E)\neq0\ \ \forall E\neq E_8\ ,E\in\Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[8]}$ with respect to the time t is given as follows.

$$\frac{dV^{[8]}}{dt}=\left[h_1\left(1-\frac{x+y+z}{h_2}\right)+\frac{(h_6y+h_7z)}{x}-h_3y_8-h_4z_8\right](x-x_8)+\frac{p}{e}\left[eh_5x_8+eh_9y_8+eh_{11}z_8+h_{12}\left(1-\frac{p}{h_{13}}\right)-qE\right]$$

In addition condition(5) guarantee that $\frac{dV^{[8]}}{dt} < 0$ on subregion of $\Re_+^4$, then $V^{[8]}$ is a Lyapunov function on that subregion. Therefore E_8 is a locally asymptotically stable but not globally.

Theorem (6):

The coexistence equilibrium point E_9 is locally asymptotically stable in $\Re_+^4$ if and only if

$$\frac{h_4 h_9 \beta_{11}^{[9]}}{h_3 \beta_{44}^{[9]}} < h_{11} < \min\left\{ \frac{h_9(h_2 h_{10} + h_1 x_9)}{(h_2 h_8 + h_1 x_9)} \left(1, \frac{\beta_{44}^{[9]}}{\beta_{11}^{[9]}}\right), \frac{h_4 h_9}{h_3}\left(1, \frac{\beta_{44}^{[9]}}{\beta_{11}^{[9]}}\right) \right\}$$

$$h_1 < \frac{2h_1}{h_2} x_9 + y_9\left(h_3 + \frac{h_1}{h_2}\right) + z_9\left(h_4 + \frac{h_1}{h_2}\right) + h_5 p_9;$$

$$e h_5 h_{13} < \min\left\{\frac{h_4 h_{12}}{h_{11}}, \frac{h_3 h_{12}}{h_9}\right\}$$

$$x_9 < \min\left\{\frac{h_9 h_{12} p_9}{h_3 h_5 h_{13}}, \frac{h_{11} h_{12} p_9}{h_4 h_5 h_{13}}\right\}$$

…………………………(6)

Proof:

The Jacobian matrix of the system (1) at E_9 is given by:

$$J_9 = \begin{pmatrix} \beta_{11}^{[9]} & \beta_{12}^{[9]} & \beta_{13}^{[9]} & \beta_{14}^{[9]} \\ \beta_{21}^{[9]} & 0 & 0 & \beta_{24}^{[9]} \\ \beta_{31}^{[9]} & 0 & 0 & \beta_{34}^{[9]} \\ \beta_{41}^{[9]} & \beta_{42}^{[9]} & \beta_{43}^{[9]} & \beta_{44}^{[9]} \end{pmatrix}$$ Where:

$$\beta_{11}^{[9]} = h_1\left(1 - \frac{2x_9}{h_2}\right) - y_9\left(h_3 + \frac{h_1}{h_2}\right) - z_9\left(h_4 + \frac{h_1}{h_2}\right) - h_5 p_9;$$

$$\beta_{12}^{[9]} = -\left(h_8 + h_9 p_9 + \frac{h_1}{h_2} x_9\right); \ \beta_{14}^{[9]} = -h_5 x_9;$$

$$\beta_{13}^{[9]} = -\left(h_{10} + h_{11} p_9 + \frac{h_1}{h_2} x_9\right); \beta_{21}^{[9]} = h_3 y_9; \beta_{24}^{[9]} = -h_9 y_9;$$

$$\beta_{31}^{[9]} = h_4 z_9; \beta_{34}^{[9]} = -h_{11} z_9; \beta_{41}^{[9]} = e h_5 p_9; \beta_{42}^{[9]} = e h_9 p_9;$$

$$\beta_{43}^{[9]} = e h_{11} p_9; \beta_{44}^{[9]} = \frac{-h_{12}}{h_{13}} p_9$$

So, the characteristic equation of J_9 can be written by

$$\left[(\lambda^{[9]})^4 + F_1(\lambda^{[9]})^3 + F_2(\lambda^{[9]})^2 + F_3(\lambda^{[9]}) + F_4\right] = 0$$

where:

$$F_1 = -\left(\beta_{11}^{[9]} + \beta_{44}^{[9]}\right); \ F_2 = \beta_{11}^{[9]}\beta_{44}^{[9]} - \beta_{13}^{[9]}\beta_{31}^{[9]} - \beta_{43}^{[9]}\beta_{34}^{[9]} - \beta_{14}^{[9]}\beta_{41}^{[9]} - \beta_{12}^{[9]}\beta_{21}^{[9]} - \beta_{24}^{[9]}\beta_{42}^{[9]};$$

$$F_3 = \beta_{13}^{[9]}\beta_{31}^{[9]}\beta_{44}^{[9]} - \beta_{13}^{[9]}\beta_{34}^{[9]}\beta_{41}^{[9]} - \beta_{14}^{[9]}\beta_{43}^{[9]}\beta_{31}^{[9]} + \beta_{43}^{[9]}\beta_{34}^{[9]}\beta_{11}^{[9]} + \beta_{12}^{[9]}\beta_{21}^{[9]}\beta_{44}^{[9]} - \beta_{14}^{[9]}\beta_{42}^{[9]}\beta_{21}^{[9]} - \beta_{12}^{[9]}\beta_{24}^{[9]}\beta_{41}^{[9]} + \beta_{24}^{[9]}\beta_{42}^{[9]}\beta_{11}^{[9]}$$

$$F_4 = \beta_{12}^{[9]}\beta_{21}^{[9]}\beta_{34}^{[9]}\beta_{43}^{[9]} - \beta_{13}^{[9]}\beta_{34}^{[9]}\beta_{42}^{[9]}\beta_{21}^{[9]} + \beta_{13}^{[9]}\beta_{31}^{[9]}\beta_{24}^{[9]}\beta_{42}^{[9]} - \beta_{12}^{[9]}\beta_{24}^{[9]}\beta_{43}^{[9]}\beta_{31}^{[9]}$$

Hence, the Routh-Hurwitz conditions require $F_1 > 0$; $F_3 > 0$; $F_4 > 0$; $\Delta = F_1 F_2 - F_3 > 0$ and $(F_1 F_2 - F_3)F_3 - (F_1)^2 F_4 > 0$, which follows from condition (6). So, according to Routh-Hurwitz criterion E_9 is locally asymptotically stable.

Numerical Simulations

The system (1) is solved numerically for different sets of parameters with different initial conditions, and then the time series for the trajectories of system (1) are drawn to confirm our obtained analytical results. Note that we will use the cont. line (—) for x, dash line(- -) for y, dot line(::) for z and dash-dot line(-.-) for p in the all of the following figures. Now, for the following parameters:

$$h_1 = 0.02, h_2 = 0.4, h_3 = 0.9, h_4 = 0.9, h_5 = 0.8,$$
$$h_6 = 0.05, h_7 = 0.05, h_8 = 0.01, h_9 = 0.29,$$
$$h_{10} = 0.01, h_{11} = 0.31, h_{12} = 0.03, h_{13} = 0.6, \quad \ldots(7)$$
$$q = 0.2, e = 0.8, E = 0.4$$

The time series of the trajectories of system (1) are drown in Fig.(1).

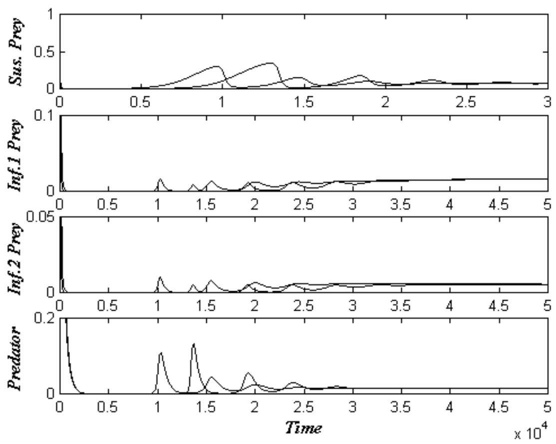

Fig.(1) The time series for the trajectories of system(1) starting from different initial points.

According to the above figure, system(1) approaches asymptotically to the stable coexistence equilibrium point $E_9 = (0.0712, 0.0165, 0.0051, 0.0138)$ starting from different initial points (0.5,0.5,0.5,0.5), (0.75,0.75,0.75,0.75), (1,1,1,1).

In order to investigate the effect of infection rate, due to existence of first disease, (i.e. parameter h_3) on the dynamics of system (1) in case of existence of E_9, the system is solved numerically for different values of h_3 with $h_3 = 0.8, 0.91$ while the rest of parameters as given in Eq. (7) and then the trajectories of system (1) are drown in the Fig.(2.a)-(2.b).

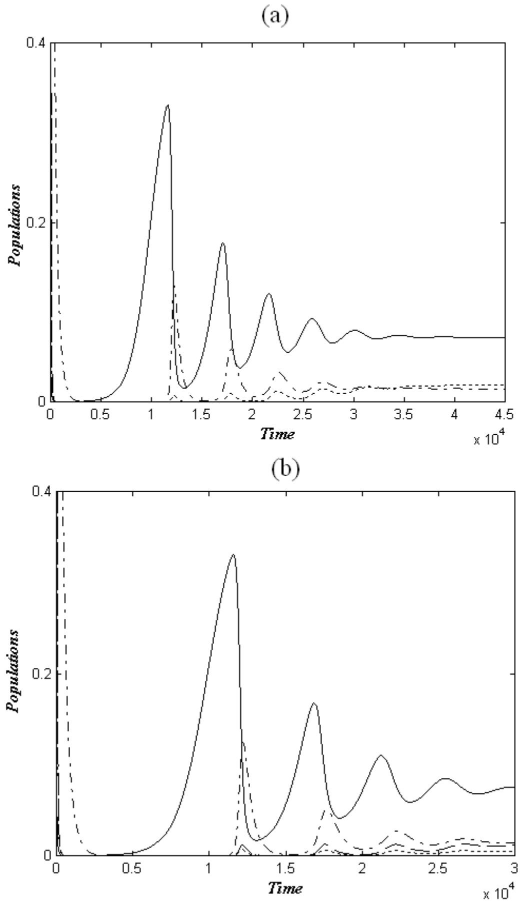

Fig.(2) Time series of trajectories for data given in Eq.(7) with different values of h_3 ***which shows that the trajectories approaches asymptotically to the stable point:(a)*** E_6 ***for*** $h_3 = 0.8$ ***(b)*** E_9 ***for*** $h_3 = 0.91$.

According to the above results, it is observed that the trajectory of system (1) approaches to stable point $E_6 = (0.0716, 0, 0.0195, 0.0144)$ for $h_3 < 0.88$ as shown in the typical Fig. (2.a), while it is approaches to asymptotically stable point $E_9 = (0.0712, 00165, 0.0051, 0.0138)$ for $0.88 \le h_3$ as shown in the typical Fig.(2.b). Similar observations have been obtained for increasing the recovery rate h_6 on the dynamical behavior of system (1).

Now the effect of infection rate, due to existence of second disease, on the dynamical behavior of the system (1) is studied numerically for parameters values given in Eq.(7) with $h_4 = 0.8, 0.92$ and the time series are drawn in Fig.(3.a)-(3.b).

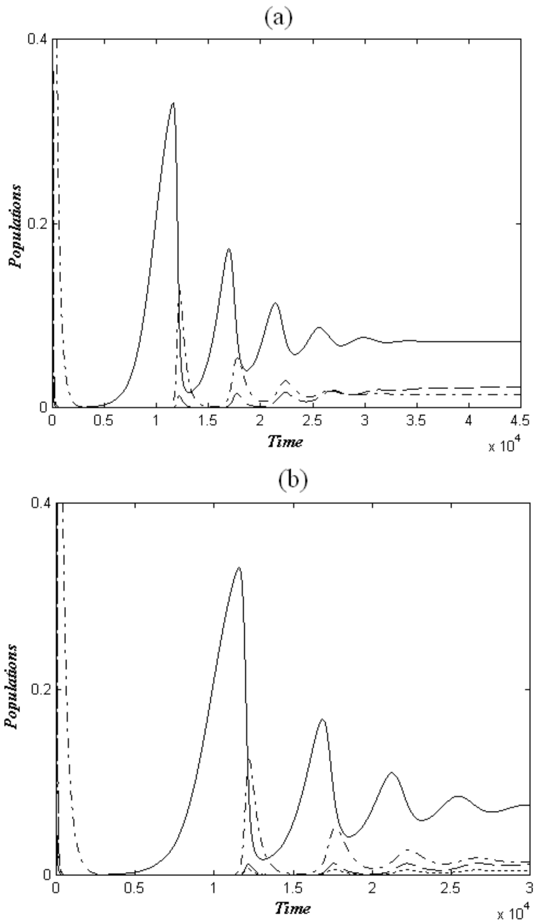

Fig.(3) Time series of the trajectories for data given in Eq.(7) with different values of h_4 ***which shows that the trajectories approaches asymptotically to the stable point: (a)*** E_7 ***for*** $h_4 = 0.8$ ***(b)*** E_9 ***for*** $h_4 = 0.92$.

Clearly, from the above figures, it is observed that decreasing the value of the infection rate causes decreasing in z and then the system (1) approaches to stable point $E_7 = (0.0711, 0.0223, 0, 0.0137)$ in $xyp-$plane that is means disappearing of the second disease. Similar observations have been obtained for increasing the recovery rate h_7 on the dynamical behavior of system (1).

Finally, we will investigate the effect of varying harvest rate qE, which is the catch ability co-efficient of the predator, on the dynamics of system (1). again the system(1) solved numerically for the values qE while the rest of parameters as given in Eq.(7) and then the trajectories of system(1) are drawn in the Figs.(4.a)-(4.c) for $qE = 0.03,\ 0.05,\ 0.1$.

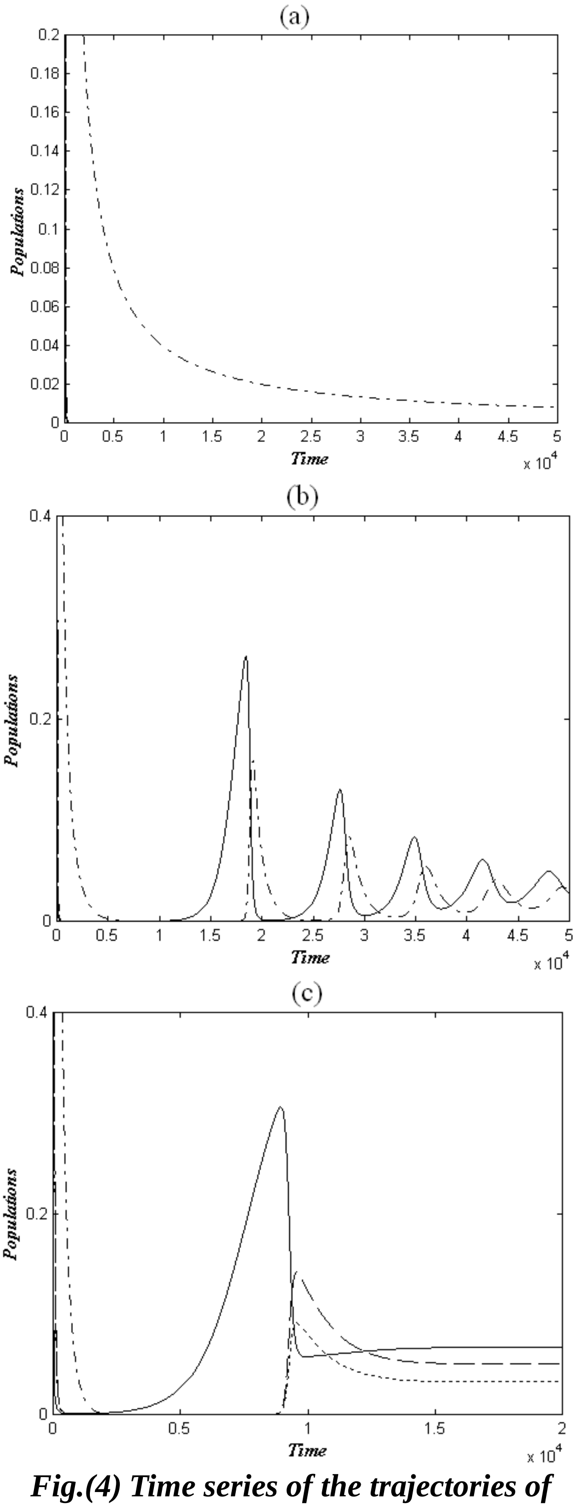

Fig.(4) Time series of the trajectories of system(1) for data given in Eq.(7) with different values of qE ***which shows that the trajectories approaches asymptotically to the stable point: (a)*** E_2 ***for*** $qE = 0.03$ ***(b)*** E_5 ***for*** $qE = 0.05$ ***(c)*** E_8 ***for*** $qE = 0.1$**.**

Obviously, from the above figures, as qE increases causes decreasing in the values of p species while the value of x, y, z species are increasing and the trajectory of system (1) approaches to the equilibrium point $E_2 = (0,0,0,0.008)$ as shown in Fig.(4.a). However the trajectory of system (1) approaches asymptotically to stable point $E_5 = (0.0264,0,0,0.031)$ as shown in Fig.(4.b), while it is approaches to $E_8 = (0.0667,0051,0.0327,0)$ as shown in Fig.(4.c).

Conclusions and Discussion

In this paper, an eco-epidemiological model has been proposed and analysed. In order to study the effect of two infection diseases and harvesting on the dynamical behavior of the prey-predator system, the dynamical behavior of system (1) has been investigated locally as well as globally. Now, we shall discuss the effects of changing the parameters on the dynamical behavior of system (1) according to the numerical results:

1. For the effect of the varying values of h_3, keeping other parameters fixed as in Eq.(7), when the infection rate by first disease in the rang $0.1 \le h_3 < 0.87$ then the trajectories of system (1) approaches to free first disease equilibrium point E_6, and if infection rate increasing $h_3 \ge 0.88$ that implies the first disease appears and the trajectories of system (1) approaches to coexistence equilibrium points E_9.similar effect have been obtained for increasing the recovery rate h_6 on the dynamical behavior of system (1).
2. If the infection rate by second disease in ($0.1 \le h_4 < 0.8$) the value of z disappear and then the system (1) approaches to stable point E_7, for $h_4 \ge 0.8$, the trajectory of system (1) approaches to E_9. Similar observations have been obtained for increasing the recovery rate h_7 on the dynamical behavior of system (1).
3. For small value of harvest rate say ($0.01 < qE \le 0.03$) the trajectory of system (1) approaches to E_2. As the harvest rate increases $0.05 < qE < 0.08$ the trajectory of

system (1) approaches to E_5. While for $0.08 \le qE < 0.1$, the trajectory of system (1) approaches to E_9. Finally, for $qE \ge 0.1$, the predator evanescent.

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الخلاصة

في هذا البحث، اقترحنا ودرسنا نموذج بيئي–وبائي مع وجود أمراض معدية مختلفة في مجتمع الفريسة وحصاد أمثل في مجتمع المفترس. أستخدمنا صيغة خطية لوصف استجابة وظيفي، أيجاد الشروط الكافية التي تضمن وجود كل نقاط التوازن الممكنة في هذا النظام، قمنا بدراسة تحليل الاستقرارية لجميع نقاط التوازن الممكنة، التحقق من تأثير الحصاد على استقرارية هذا النظام. أخيراً، السلوك الديناميكي لهذا النظام تمت مناقشته بأستخدام المحاكاة العددية.