Approximation to the Mean and Variance of Moments Method Estimate Due to Gamma Distribution

Akram M. Al-Abood and Yamama N. Mahmood
Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University.

Abstract

In this paper, we shall consider the approximation to the mean and variance of moments method estimators due to gamma distribution by using Taylor series expansion approach. This approach showed that the estimators are asymptotically unbiased with mean square error approach zero as the sample size approach infinity. The theoretical approach assessed practically by using monte-carlo simulation.

Keyword**:** Gamma distribution, moments, moments method estimation, Taylor method, Ratio approximation, monte-carlo simulation.

1. Introduction

The gamma distribution arise as a model from statistical studies of interval between events occurring in time or space, specifically when the interest in the waiting time from the occurrence of one event until r further events have occurred in a Poisson process with constant rate λ, [5]. This distribution sometimes referred to as a special Erlangian distribution after the Swedish scientist who used the distribution in early studies of queuing problem [1]. The gamma distribution has an important applications in the study of life time models, such as stops of a machine, failure or breakdowns of an equipment (e.g. electronic computer), air or road accidents [5], [8], coal mining disasters, telephone calls, daily rainfall [9], etc., are examples of such events that occur in a real time and have properties experted for gamma case [6]

2. Some fundamental concepts related to Gamma distribution are given for completeness: [3]

Definition: A random variable X is said to have a gamma distribution with parameters α and β, denoted by X~ $G(\alpha,\beta)$, if X has probability density function,

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1}e^{-x/\beta}, 0 < x < \infty$$
$$= 0 \qquad , e.w \quad .... (1)$$

Where α>0, β>0 and $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ is called a gamma function.

The following are some relations needed to our latter discussion:

(i) The r^{th} moment of random variable X~ $G(\alpha,\beta)$ is:

$$\mu_r{}' = E(X^r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\beta^r, r = 1,2,3,... \quad ...... (2)$$

(ii) The coefficient of kurtosis of random variable X~ $G(\alpha,\beta)$ is:

$$\frac{E[(X-\mu)^4]}{(\sigma^2)^2} - 3 = 6\alpha^{-1} \quad .... (3)$$

(iii) The r^{th} moment of the sample mean $\bar{X}$~$G(n\alpha,\frac{\beta}{n})$ is:

$$E(\bar{X}^r) = \frac{\Gamma(n\alpha+r)}{\Gamma(n\alpha)}(\frac{\beta}{n})^r, r = 1,2,3,... \quad .... (4)$$

(iv) If $X_1, X_2, ..., X_\alpha$ is a random sample of size α from exp (β), then the random variable Y=$\sum_{i=1}^{\alpha} X_i$~$G(\alpha,\beta)$

(v) The mean and variance of the sample variance S^2 are:

$$E(S^2) = \sigma^2 = \alpha \quad .... (6)$$

$$var(S^2) = \frac{1}{n}\left\{E\left[(X-\mu)^4 - \frac{n-3}{n-1}\sigma^4\right]\right\}$$
$$= \frac{\alpha\beta^4(6n-6-2n\alpha)}{n(n-1)} \quad .... (7)$$

3. Moments Method Estimation: [4]

It has been shown throughout the literature that the moments method estimators of the parameters α and β are:

$$\hat{\alpha} = \frac{n\bar{X}^2}{(n-1)S^2} \quad .... (8)$$

$$\hat{\beta} = \frac{(n-1)S^2}{n\bar{X}} \quad .... (9)$$

4. Expecctation of Quotient Function of Random Variables: [1]

In general, there is no simple exact formulae for the mean and variance of the quotient of two random variables in terms of the moments of the two random variables; however, there is an approximate formulae can be considered. One way of finding the approximate formula for $E(\frac{X}{Y})$ by considering Taylor series expansion of the function $g(x,y)=\frac{x}{y}$ expanded about the point $[E(X),E(Y)]$, where we drop all terms of order higher than 2, and then take the expectation of both sides. Furthermore the approximate formula for $var(\frac{X}{Y})$ is similarly obtained by expanding Taylor series and retaining only the second-order terms, we note that the function $g(x,y)=\frac{x}{y}, y\neq 0$ is analytic at point (μ_x,μ_y) and differentiable with respect to x and with respect to y up n and m times respectively.

The Taylor series expansion of the function $g(x,y)=\frac{x}{y}$ about the point (μ_x,μ_y) is

$$g(x,y)\approx g(\mu_x,\mu_y)+(x-\mu_x)\frac{\partial g(x,y)}{\partial x}\Big|_{\mu_x,\mu_y}+(y-\mu_y)\frac{\partial g(x,y)}{\partial y}\Big|_{\mu_x,\mu_y}+\frac{1}{2!}(x-\mu_x)^2\frac{\partial^2 g(x,y)}{\partial x^2}\Big|_{\mu_x,\mu_y}+\frac{1}{2!}(y-\mu_y)^2\frac{\partial^2 g(x,y)}{\partial y^2}\Big|_{\mu_x,\mu_y}+(x-\mu_x)(y-\mu_y)\frac{\partial^2 g(x,y)}{\partial x\partial y}\Big|_{\mu_x,\mu_y}+\cdots \quad (10)$$

Where $\mu_x=E(X)\neq 0$ and $\mu_y=E(Y)\neq 0$

$$\frac{\partial g(x,y)}{\partial x}=\frac{1}{y}\Rightarrow\frac{\partial g(x,y)}{\partial x}\Big|_{\mu_x,\mu_y}=\frac{1}{\mu_y}$$

$$\frac{\partial^2 g(x,y)}{\partial x^2}=0\Rightarrow\frac{\partial^2 g(x,y)}{\partial x^2}\Big|_{\mu_x,\mu_y}=0$$

$$\frac{\partial g(x,y)}{\partial y}=\frac{-x}{y^2}\Rightarrow\frac{\partial g(x,y)}{\partial y}\Big|_{\mu_x,\mu_y}=\frac{-\mu_x}{{\mu_y}^2}$$

$$\left.\begin{array}{l}\frac{\partial^2 g(x,y)}{\partial y^2}=\frac{2x}{y^3}\Rightarrow\frac{\partial^2 g(x,y)}{\partial y^2}\Big|_{\mu_x,\mu_y}=\frac{2\mu_x}{{\mu_y}^3}\\ \frac{\partial^2 g(x,y)}{\partial y\partial x}=\frac{\partial^2 g(x,y)}{\partial x\partial y}=\frac{-1}{y^2}\Rightarrow\\ \frac{\partial^2 g(x,y)}{\partial y\partial x}\Big|_{\mu_x,\mu_y}=\frac{-1}{{\mu_y}^2}\end{array}\right\}\quad (11)$$

Take the expectation of both sides of Eq.(10) and according to the results of Eq.(11), thus we have:

$$E\left(\frac{X}{Y}\right)\approx\frac{E(X)}{E(Y)}+\frac{E(X)}{\{E(Y)\}^3}var(Y)-\frac{1}{\{E(Y)\}^2}cov(X,Y)$$

$$=\frac{E(X)}{E(Y)}\left[1+\frac{var(y)}{\{E(Y)\}^2}-\frac{E(XY)-E(X)E(Y)}{E(X)E(Y)}\right]$$

$$=\frac{E(X)}{E(Y)}\left[2+\frac{var(y)}{\{E(Y)\}^2}-\frac{E(XY)}{E(X)E(Y)}\right] \quad (12)$$

Take the variance of both sides of Eq.(10) and according to the results of Eq.(11), thus we have:

$$var\left(\frac{X}{Y}\right)\approx\frac{var(X)}{\{E(Y)\}^2}+var(y)\frac{\{E(X)\}^2}{\{E(Y)\}^4}-2cov(X,Y)\frac{E(X)}{\{E(Y)\}^3}=\left[\frac{E(X)}{E(Y)}\right]^2\left[\frac{var(X)}{\{E(X)\}^2}+\frac{var(Y)}{\{E(Y)\}^2}-\frac{2cov(X,Y)}{E(X)E(Y)}\right]$$

$$=\left[\frac{E(X)}{E(Y)}\right]^2\left[\frac{var(X)}{\{E(X)\}^2}+\frac{var(Y)}{\{E(Y)\}^2}-\frac{2\{E(XY)-E(X)E(Y)\}}{E(X)E(Y)}\right]$$

$$=\left[\frac{E(X)}{E(Y)}\right]^2\left[2+\frac{var(X)}{\{E(X)\}^2}+\frac{var(Y)}{\{E(Y)\}^2}-\frac{2E(XY)}{E(X)E(Y)}\right] \quad (13)$$

5. Approximation to the Mean and Variance of Moments Method Estimators:

In this section, we shall consider the approximation to the mean and variance of moments method estimators by equations (12) and (13).

(5.1) Approximation to the mean of $\hat{\alpha}$

Consider the expectation of $\hat{\alpha}$ given by Eq.(8)

$$E(\hat{\alpha})=E\left[\frac{n\bar{X}^2}{(n-1)S^2}\right]=\left(\frac{n}{n-1}\right)E\left[\frac{\bar{X}^2}{S^2}\right]$$

Use of Eq.(12), with $X=\bar{X}^2$ and $Y=S^2$, we have:

$$E(\hat{\alpha})\approx\left(\frac{n}{n-1}\right)\left[\frac{E(\bar{X}^2)}{E(S^2)}\right]\left[2+\frac{var(S^2)}{\{E(S^2)\}^2}-\frac{E(\bar{X}^2S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

Consider

$$(\bar{X}^2 S^2) = E\left[\bar{X}^2 \frac{1}{(n-1)}\left\{\sum_{i=1}^{n} X_i^{\,2} - n\bar{X}^2\right\}\right] = \frac{1}{(n-1)} E\left[\bar{X}^2 \sum_{i=1}^{n} X_i^{\,2} - n\bar{X}^4\right]$$

$$= \frac{1}{n^2(n-1)} E\left[\left(\sum_{i=1}^{n} X_i\right)^2 \left(\sum_{i=1}^{n} X_i^{\,2}\right)\right] - \frac{n}{(n-1)} E(\bar{X}^4)$$
$$= A - B$$

Where

$A = \frac{1}{n^2(n-1)} E\left[\left(\sum_{i=1}^{n} X_i\right)^2 \left(\sum_{i=1}^{n} X_i^{\,2}\right)\right]$, and $B = \frac{n}{(n-1)} E(\bar{X}^4)$

Consider

$$A = \frac{1}{n^2(n-1)} E\left[\begin{matrix}\left\{\sum_{i=1}^{n} X_i^{\,2} + 2\sum_{i<j}\sum X_i X_j\right\} \\ \left(\sum_{i=1}^{n} X_i^{\,2}\right)\end{matrix}\right]$$
$$= \frac{1}{n^2(n-1)} E\left[\begin{matrix}\left(\sum_{i=1}^{n} X_i^{\,2}\right)^2 + 2\left(\sum_{i=1}^{n} X_i^{\,2}\right) \\ \left(\sum_{i<j}\sum X_i X_j\right)\end{matrix}\right]$$

$$= \frac{1}{n^2(n-1)} E\left[\sum_{i=1}^{n} X_i^{\,4} + 2\sum_{i<j}\sum X_i^{\,2} X_j^{\,2} + 2\sum_{i<j}\sum X_i^{\,3} X_j + 2\sum_{i<j<k}\sum\sum X_i^{\,2} X_j X_k\right]$$
$$= \frac{1}{n^2(n-1)}\left[\sum_{i=1}^{n} E\left(X_i^{\,4}\right) + 2\sum_{i<j}\sum E\left(X_i^{\,2}\right)E\left(X_j^{\,2}\right) + 2\sum_{i<j}\sum E\left(X_i^{\,3}\right)E\left(X_j\right) + 2\sum_{i<j<k}\sum\sum E\left(X_i^{\,2}\right)E\left(X_j\right)E\left(X_k\right)\right]$$

Use of Eq.(2), with r=1,2,3,4 , we have:

$$A = \frac{1}{n^2(n-1)}\left[\sum_{i=1}^{n} \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4 + 2\sum_{i<j}\sum \alpha^2(\alpha+1)^2\beta^4 + 2\sum_{i<j}\sum \alpha(\alpha+1)(\alpha+2)\beta^3\ \alpha\beta + 2\sum_{i<j<k}\sum\sum \alpha(\alpha+1)\beta^2(\alpha\beta)^2\right]$$

$$= \frac{1}{n^2(n-1)}\left[n\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4 + n(n-1)\alpha^2(\alpha+1)^2\beta^4 + 2n(n-1)\alpha^2(\alpha+1)(\alpha+2)\beta^4 + \frac{2\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)}{2}\alpha^3(\alpha+1)\beta^4\right]$$
$$= \frac{n\alpha(\alpha+1)\beta^4}{n^2(n-1)}\left[(n\alpha+2)(n\alpha+3)\right]$$

and

$$B = \frac{n}{(n-1)} E(\bar{X}^4)$$

Use of Eq.(4), with r=4, we have:

$$B = \frac{n}{(n-1)}\, n\alpha(n\alpha+1)(n\alpha+2)(n\alpha+3)\left(\frac{\beta}{n}\right)^4$$

$$= \frac{\alpha\beta^4}{n^2(n-1)}(n\alpha+1)(n\alpha+2)(n\alpha+3)$$

So

$$E(\bar{X}^2 S^2) = \frac{n\alpha(\alpha+1)\beta^4}{n^2(n-1)}\left[(n\alpha+2)(n\alpha+3)\right] - \frac{\alpha\beta^4}{n^2(n-1)}(n\alpha+1)(n\alpha+2)(n\alpha+3)$$

$$= \frac{\alpha\beta^4}{n^2}(n\alpha+2)(n\alpha+3)$$

Now

$$\frac{E(\bar{X}^2)}{E(S^2)} = \frac{n\alpha(n\alpha+1)\frac{\beta^2}{n^2}}{\alpha\beta^2} = \frac{n\alpha+1}{\mathrm{n}}$$
$$E(\bar{X}^2)\, E(S^2) = n\alpha(n\alpha+1)\frac{\beta^2}{n^2}\alpha\beta^2 = \frac{\alpha^2(n\alpha+1)\beta^4}{\mathrm{n}}$$
$$var(S^2) = \frac{\alpha\beta^4(6n-6+2n\alpha)}{\mathrm{n}(\mathrm{n}-1)}$$
$$\frac{var(S^2)}{[E(S^2)]^2} = \frac{6n-6+2n\alpha}{\mathrm{n}(\mathrm{n}-1)\alpha} = \frac{6}{n\alpha} + \frac{2}{n-1}$$

$$E(\hat{\alpha}) = \left(\frac{n}{n-1}\right)\left(\frac{n\alpha+1}{n}\right)\left[2 + \frac{6}{n\alpha} + \frac{2}{n-1} - \frac{\frac{\alpha\beta^4}{n^2}(n\alpha+2)(n\alpha+3)}{\frac{\alpha^2(n\alpha+1)\beta^4}{\mathrm{n}}}\right]$$

$$= \left(\frac{\alpha+\frac{1}{n}}{1-\frac{1}{n}}\right)\left(2 + \frac{6}{n\alpha} + \frac{2}{n-1} - \frac{\left(\alpha+\frac{2}{n}\right)\left(\alpha+\frac{3}{n}\right)}{\alpha\left(\alpha+\frac{1}{n}\right)}\right) \quad \text{....(14)}$$

By taking the limit as $n \to \infty$ to both sides of Eq.(14), we get

$$\lim_{n\to\infty} E(\hat{\alpha}) = \left(\frac{\alpha+0}{1}\right)\left(2+0+0-\frac{(\alpha+0)(\alpha+0)}{\alpha(\alpha+0)}\right) = \alpha$$

Therefore

$\hat{\alpha}$ is asymptotically unbiased estimator for α.

Where the bias of $\hat{\alpha}$ is

$b(\hat{\alpha}) = E(\hat{\alpha}) - \alpha$ ....(15)

(5.2) Approximation to the variance of $\hat{\alpha}$

Use of Eq.(13) with $X = \bar{X}^2$ *and* $Y = S^2$, we have :

$$var(\hat{\alpha}) = var\left[\frac{n\bar{X}^2}{(n-1)S^2}\right] = \left(\frac{n}{n-1}\right)^2 var\left(\frac{\bar{X}^2}{S^2}\right)$$

$$\approx \left(\frac{n}{n-1}\right)^2 \left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 \left[2 + \frac{var(\bar{X}^2)}{\{E(\bar{X}^2)\}^2} + \frac{var(S^2)}{\{E(S^2)\}^2} - \frac{2E(\bar{X}^2S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

$$= \left(\frac{n}{n-1}\right)^2 \left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 \left[2 + \frac{E(\bar{X}^4)-\{E(\bar{X}^2)\}^2}{\{E(\bar{X}^2)\}^2} + \frac{var(S^2)}{\{E(S^2)\}^2} - \frac{2E(\bar{X}^2S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

$$= \left(\frac{n}{n-1}\right)^2 \left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 \left[1 + \frac{E(\bar{X}^4)}{\{E(\bar{X}^2)\}^2} + \frac{var(S^2)}{\{E(S^2)\}^2} - \frac{2E(\bar{X}^2S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

Where

$$\left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 = \left[\frac{n\alpha(n\alpha+1)(\frac{\beta}{n})^2}{\alpha\beta^2}\right]^2 = \left(\frac{n\alpha+1}{n}\right)^2$$

$$\frac{E(\bar{X}^4)}{\{E(\bar{X}^2)\}^2} = \frac{n\alpha(n\alpha+1)(n\alpha+2)(n\alpha+3)\left(\frac{\beta}{n}\right)^4}{\left[n\alpha(n\alpha+1)\left(\frac{\beta}{n}\right)^2\right]^2} = \frac{(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)}$$

$$\frac{var(S^2)}{\{E(S^2)\}^2} = \frac{6n-6+2n\alpha}{n\alpha(n-1)}$$

$$\frac{2E(\bar{X}^2S^2)}{E(\bar{X}^2)E(S^2)} = \frac{\frac{2\alpha\beta^4}{n^2}(n\alpha+2)(n\alpha+3)}{\alpha^2(n\alpha+1)\frac{\beta^4}{n}} = \frac{2(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)}$$

Therefore

$$var(\hat{\alpha}) = \left(\frac{n}{n-1}\right)^2 \left(\frac{n\alpha+1}{n}\right)^2 \left[1 + \frac{(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)} + \frac{6n-6+2n\alpha}{n\alpha(n-1)} - \frac{2(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)}\right]$$

$$= \frac{2n(n\alpha+1)(\alpha+1)}{(n-1)^3} \quad \text{.............................} (16)$$

(5.3) Approximation to the mean of $\hat{\beta}$

Consider the expectation of $\hat{\beta}$ given by Eq.(9)

$$E(\hat{\beta}) = E\left[\frac{(n-1)S^2}{n\bar{X}}\right] = \left(\frac{n-1}{n}\right) E\left(\frac{S^2}{\bar{X}}\right)$$

Use of Eq.(12), with $X = S^2$ and $Y = \bar{X}$, we have:

$$E(\hat{\beta}) \approx \left(\frac{n-1}{n}\right) \left[\frac{E(S^2)}{E(\bar{X})}\right] \left[2 + \frac{var(\bar{X})}{\{E(\bar{X})\}^2} - \frac{E(\bar{X}S^2)}{E(\bar{X})E(S^2)}\right]$$

Where

$E(\bar{X}) = \alpha\beta$, $var(\bar{X}) = \frac{\alpha\beta^2}{n}$, $E(S^2) = \alpha\beta^2$ and

$$var(S^2) = \frac{\alpha\beta^4(6n-6+2n\alpha)}{n(n-1)}$$

Consider

$$E(\bar{X}S^2) = \mathrm{E}\left[\bar{X}\frac{1}{n-1}\left\{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right\}\right] = \frac{1}{n-1}\mathrm{E}\left(\bar{X}\sum_{i=1}^{n} X_i^2\right) - \frac{n}{n-1}E(\bar{X}^3)$$

$$= \frac{1}{n(n-1)}E\left[\left\{\sum_{i=1}^{n} X_i\right\}\left\{\sum_{i=1}^{n} X_i^2\right\}\right] - \frac{n}{n-1}E(\bar{X}^3)$$

$$= A - B$$

Where

$A = \frac{1}{n(n-1)}E[\sum_{i=1}^{n} X_i^3 + \sum_{i<j}\sum X_i^2 X_j$, and

$$B = \frac{n}{n-1}E(\bar{X}^3)$$

Consider

$$A = \frac{1}{n(n-1)}\left[\sum_{i=1}^{n} E(X_i^3) + \sum_{i<j}\sum E(X_i^2)E(X_J)\right]$$

Use of Eq.(2), with r=2,3, we have:

$$A = \frac{1}{n(n-1)}\left[\sum_{i=1}^{n} \alpha(\alpha+1)(\alpha+2)\beta^3 + \sum_{i<j}\sum \alpha(\alpha+1)\beta^2\alpha\beta\right]$$

$$= \frac{1}{n(n-1)}\left[n\alpha(\alpha+1)(\alpha+2)\beta^3 + n(n-1)\alpha^2(\alpha+1)\beta^3\right] = \frac{n\alpha(\alpha+1)(n\alpha+2)\beta^3}{n(n-1)}$$

and

$$B = \frac{n}{n-1}E(\bar{X}^3)$$

Use of Eq.(4), with r=3, we have:

$$B = \frac{n}{n-1}n\alpha(n\alpha+1)(n\alpha+2)\left(\frac{\beta}{n}\right)^3 = \frac{\alpha(n\alpha+1)(n\alpha+2)\beta^3}{n(n-1)}$$

Therefore

$$E(\bar{X}S^2) = \frac{n\alpha(\alpha+1)(n\alpha+2)\beta^3}{n(n-1)} - \frac{\alpha(n\alpha+1)(n\alpha+2)\beta^3}{n(n-1)} = \frac{\alpha(n\alpha+2)\beta^3}{n}$$

So

$$E(\hat{\beta}) \approx \left(\frac{n-1}{n}\right)\left(\frac{\alpha\beta^2}{\alpha\beta}\right)\left[2+\frac{\left(\frac{\alpha\beta^2}{n}\right)}{(\alpha\beta)^2}-\frac{\frac{\alpha(n\alpha+2)\beta^3}{n}}{(\alpha\beta)(\alpha\beta^2)}\right] = \left(\frac{n-1}{n}\right)\beta\left[1-\frac{1}{n\alpha}\right] = \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\alpha}\right)\beta \quad \text{.............................. (17)}$$

By taking the limit as $n \to \infty$ to both sides of Eq.(17), we have:

$$\lim_{n\to\infty} E(\hat{\beta}) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\alpha}\right)\beta = \beta$$

Therefore

$\hat{\beta}$ is asymptotically unbiased estimator for β. Where the bias of

$\hat{\beta}$ is $b(\hat{\beta}) = E(\hat{\beta}) - \beta$ .... (18)

(5.4) Approximation to the variance of $\hat{\beta}$

Use of Eq.(13) with $X = S^2$ and $Y = \bar{X}$, we have:

$$var(\hat{\beta}) = var\left[\frac{(n-1)S^2}{n\bar{X}}\right]$$

$$= \left(\frac{n-1}{n}\right)^2 \text{var}\left(\frac{S^2}{\bar{X}}\right)$$

$$\approx \left(\frac{n-1}{n}\right)^2 \left[\frac{E(S^2)}{E(\bar{X})}\right]^2 \left[2+\frac{var(S^2)}{\{E(S^2)\}^2}+\frac{var(\bar{X})}{\{E(\bar{X})\}^2}-\frac{2E(\bar{X}S^2)}{E(\bar{X})E(S^2)}\right]$$

$$= \left(\frac{n-1}{n}\right)^2\left(\frac{\alpha\beta^2}{\alpha\beta}\right)^2\left[2+\frac{\frac{\alpha\beta^4(6n-6+2n\alpha)}{n(n-1)}}{(\alpha\beta^2)^2}+\frac{\frac{\alpha\beta^2}{n}}{(\alpha\beta)^2}-\frac{\frac{2\alpha(n\alpha+2)\beta^3}{n}}{(\alpha\beta)(\alpha\beta^2)}\right] = \left(\frac{n-1}{n}\right)^2\left(\frac{3}{n\alpha}+\frac{2}{n-1}\right)\beta^2 \quad \text{....... (19)}$$

6. Monte-Carlo Simulation:[2]

A large scale monte-carlo investigation is considered by generating a random sample of size n=5(1) 10(2) 20(5) 30 by utilizing the relation given by Eq.(5) and run size 500 is used, and the initial values of $\alpha = 3$ and $\beta = 1$.

Table (1)
Show the values of the simulated biases of $\hat{\alpha}$, $\hat{\beta}$ as given by equations (8), (9) together with the values of the approximation given by equations (15), (18).

N	$b(\hat{\alpha})$		$b(\hat{\beta})$	
	Simulation	*Approximation*	*Simulation*	*Approximation*
5	4.09254	3.5	-0.22205	-0.253
6	2.47371	2.72	-0.81094	-0.213
7	1.29739	2.222	-0.12144	-0.184
8	1.58142	1.878	-0.11824	-0.161
9	1.50682	1.625	-0.14461	-0.144
10	1.1535	1.432	-0.08094	-0.13
12	1.11699	1.157	-0.12609	-0.109
14	0.86187	0.97	-0.09131	-0.094
16	0.63403	0.836	-0.05489	-0.082
18	0.66235	0.734	-0.05964	-0.073
20	0.53683	0.654	-0.04449	-0.066
25	0.53612	0.514	-0.06614	-0.053
30	0.34809	0.423	-0.0032	-0.044

Table (2)
Show the values of the simulated variances of $\hat{\alpha}$, $\hat{\beta}$ together with the values of the approximated variances.

N	$Var(\hat{\alpha})$		$Var(\hat{\beta})$	
	Simulation	*Approximation*	*Simulation*	*Approximation*
5	33.9146	10	0.50406	0.448
6	19.84355	7.296	0.27187	0.39352
7	11.71693	5.7037	0.30003	0.34985
8	6.95881	4.66472	0.26405	0.31445
9	6.34074	3.9375	0.21868	0.28532
10	4.90011	3.40192	0.26033	0.261
12	4.17264	2.66867	0.1629	0.2228
14	2.99714	2.19208	0.1556	0.19424
16	2.12534	1.85837	0.15705	0.17212
18	2.02785	1.61205	0.13634	0.15449
20	1.66254	1.42295	0.13213	0.14012
25	1.30061	1.09954	0.10039	0.11366
30	0.96436	0.89549	0.08979	0.09559

Table (3)
Show the values of the simulated MSE of $\hat{\alpha}, \hat{\beta}$ together with values of the approximated MSE.

N	$MSE(\hat{\alpha})$		$MSE(\hat{\beta})$	
	Simulation	*Approximation*	*Simulation*	*Approximation*
5	46.67385	22.25	0.55337	0.51218
6	25.96278	14.694	0.30762	0.43887
7	15.43175	10.642	0.31478	0.38359
8	9.45969	8.19	0.27803	0.34052
9	8.61123	6.578	0.23959	0.30607
10	6.23068	5.453	0.26688	0.2779
12	5.42031	4.007	0.17879	0.23464
14	3.73996	3.134	0.16393	0.20299
16	2.52734	2.557	0.16006	0.17885
18	2.46657	2.15	0.1399	0.15983
20	1.95073	1.85	0.13411	0.14446
25	1.58804	1.364	0.10476	0.11645
30	1.08553	1.075	0.09082	0.09754

Conclusions

1. Table (1) show that the biases of $\hat{\alpha}, \hat{\beta}$ as obtained by simulation are very close to biases values of the approximation and for large samples size both biases approach zero.
2. The approximated values of the biases and variances become more accurate if we consider the approximation to the expectation and variance up to third order derivation of Taylor series expansion.
3. Table (2) and (3) show that there a considerable difference between the simulated and approximated variances and mean square error of $\hat{\alpha}$ for small sample sizes but it becomes adequate for large sample size, while the simulated and approximated variances and mean square error of $\hat{\beta}$ are excellent for all sample size.

References

[1] Alexander M. Mood, "Introduction to the Theory of Statistics", Franklin A. Gray bill, Pittenger Duane C. Bose, McGraw-Hill, 3rd edition, (2005).

[2] Reuven Y. Rubinstein, "Simulation and the Monte Carlo Method", John Wiley and Sons, 2nd edition, Inc. U.S.A, (2007).

[3] Keijo Ruohonen, "Statistic 1", Tampere University of Technology, p.97, (2011).

[4] Ingmar R. Prucha, "Instrumental Variables/Method of Moments Estimation", Handbook of Regional Science, Springer Berlin Heidelberg, pp 1597-1617, (2014).

[5] Oscar A. Rosas-Jaimes, Araceli C. Campero-carmona, Oscar L. Sanchez-Flores, "Prediction under Bayesian approach of car accident in urban intersections", 3rd International Conference on Road Safety and Simulation, September 14-16, USA, (2011).

[6] Smith,O.E; Adelfang, S.I.; Tubbs, J.D, "A bivariate gamma probability distribution with application to gust modeling", [for the ascent flight of the space shuttle., Technical Report, NASA-TM-82483, NAS 1.15:82483, (1982).

[7] Singh, Bhupendra---Sharma, K.K.---Kumar, Anuj, "Analyzing the dynamic system model with discrete failure time distribution", Statistical Methods and Applications, Volume 18, 4Pages: 521-542, Springer DOI: 10.1007/s10260-008-0111-Y, (2009).

[8] Sudeshna Mitra [a], Simon Washington [b], "On the nature of over-dispersion in motor vehicle crash prediction models", [a] civil and Environmental Engineering Department, Cal Poly. State University, San Luis Obsipo,CA 93407-0353, United states, [b] Department of civil and Environmental Engineering. Arizona State University, PO Box 875306, Tempe, AZ 85287-53006, United states, (2003).

[9] Hafzullah AKSOY, "Use of Gamma Distribution in Hydrological Analysis", Istanbul Technical University, civil Engineering Faculty, Hydraulics Division, 80626 Ayazaga, Istanbul-TURKEY, (1999).

الخلاصة

في هذا البحث تطرقنا الى تقريب المعدل والتباين لمخمنات طريقه العزوم لمعلمات توزيع كاما بأستخدام مفكوك تايلر.

هذا التقريب اظهر ان المخمنات غير متحيزه بالتحاذي وبمعدل مربع خطأ يقترب الى الصفر عندما تقترب سعه العينه الى المالانهايه. التقريب النظري دعم عمليا بأستخدام طرائق محاكاه مونت كارلو.