

Commuting with Higher Centralizers on Prime Rings

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Abstract:

The main purpose of this paper is studied the commuting on prime ring by using higher left (right) centralizers we have obtained several results that ensure a ring becomes commutative as assume $J \neq \{0\}$ an ideal of E . If E admits a higher left (right) centralizers $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ s.t. $\kappa_n(pv) - \kappa_{n-1}(pv) \in Z(E)$ or $\kappa_n(pv) + \kappa_{n-1}(pv) \in Z(E)$, $\forall p, v \in J$. Then E is commutative.

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1. Introduction

In this article, let E be an associative prime ring equipped with higher centralizers κ_n , unless otherwise stated. E is named prime if $hep=0$, $\forall e \in E$ we have h or p equal to zero, E is named semiprime if $heh=0$, $\forall e \in E$, obtained h equal to zero, we mean by $\text{char.}(E)$ not equal to two i.e any element of E multiply by two that mean its element equal to zero. For more see references [2, 8-10]. In this article, E is prime ring and $J \neq \{0\}$ an ideal of E unless otherwise stated. Many researchers studied the commuting and centralizing on ring under some mapping [1,2,3,5,8,11], as derivation, reverse derivations, centralizer [4,10] in this paper, we investigate the commuting properties of prime rings with higher left (or right) centralizers, such that assume $J \neq \{0\}$ an ideal of E . If E admits a higher left (right) centralizers $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ s.t. $\kappa_n(pv) - \kappa_{n-1}(pv) \in Z(E)$ or $\kappa_n(pv) + \kappa_{n-1}(pv) \in Z(E)$, $\forall p, v \in J$. Then E is commutative.

2. Preliminaries

In this section, some basic results are collected.

Lemma 2.1 [6]: If there exists a nonzero commutative right ideal J in E , then E is commutative.

Lemma 2.2 [7]: Every $0 \neq p \in Z(E)$, where $pv \in Z(E)$, then $v \in Z(E)$.

Lemma 2.3 [11]: If $\text{char.}(E) \neq 2$, $p, v \in E$, and J a lie ideal of E which non-central s.t. $pJv = (0)$ then $p=0$ or $v=0$.

3. Commuting Higher Centralizers on Prime Rings

In the following, we give the results for commuting ring by using higher left (or right) centralizers.

Lemma 3.1: A nonzero ideal J of a ring E and $\kappa = (\kappa_i)_{i \in \mathbb{N}}$ be higher left (right)-centralizers of E such that $[\kappa_n(p), \kappa_{n-1}(p)]$ is central, $\forall p \in J$, then E is commutative.

Proof: Since $[\kappa_n(p), \kappa_{n-1}(p)]$ is central $\forall p \in J$ Then

$$\sum_{i=1}^n [[\kappa_i(p), \kappa_{i-1}(p)] \kappa_{i-1}(v), \kappa_{i-1}(p)] = 0, \forall p, v \in J$$
$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(p)] [\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0, \forall p, v \in J$$

Replace pv for v in above relation we obtain

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(p)] [\kappa_{i-1}(pv), \kappa_{i-1}(p)] = 0$$

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(p)] \kappa_{i-1}(p) [\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0$$

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(p)] JE[\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0$$

Since E is prime ring, $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(p)]$ is non-zero central and J is nonzero ideal then

$$[\kappa_n(v), \kappa_{n-1}(p)] = 0 \quad \forall p, v \in J$$

Therefore, by Lemma 2.1, E must be commutative.

Theorem 3.2: Assume $J \neq \{0\}$. If E admits a left (right) centralizer such that $\kappa_n(pv) \cdot \kappa_{n-1}(pv) \in Z(E)$, $\forall p, v \in J$, then E is commutative.

Proof: by assumption $\sum_{i=1}^n \kappa_i(pv) \cdot \kappa_{i-1}(pv) \in Z(E)$, $\forall p, v \in J$. If $\kappa_n(pv) = 0$, then $\kappa_n(pv) \in Z(E)$, $\forall p, v \in J$. In particular $[\kappa_n(pv), \kappa_{n-1}(p)] = 0$, for all $p, v \in J$

$$\sum_{i=1}^n \kappa_i(p) [\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0$$

Replace v by vc in last relation

$$\sum_{i=1}^n \kappa_i(p) [\kappa_{i-1}(vc), \kappa_{i-1}(p)] = 0$$

$$\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(v) [\kappa_{i-1}(c), \kappa_{i-1}(p)]$$

$$+ \kappa_{i-1}(p) [\kappa_{i-1}(v), \kappa_{i-1}(p)] \kappa_{i-1}(c) = 0$$

$$\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(v) [\kappa_{i-1}(c), \kappa_{i-1}(p)] = 0, \text{ for all } p, v, c \in J$$

Thus it follows that

$$\sum_{i=1}^n \kappa_i(p) E J [\kappa_{i-1}(c), \kappa_{i-1}(p)] = \{0\}, \quad \text{for all } p, c \in J$$

Since E is prime we obtain

$$\kappa_n(p) = 0 \text{ or } J[\kappa_{n-1}(c), \kappa_{n-1}(p)] = \{0\}$$

$$\text{If } \kappa_n(p) = 0 \Rightarrow J[\kappa_{n-1}(c), \kappa_{n-1}(p)] = \{0\}$$

Since $J \neq \{0\}$ and E is prime the above relation yield that

$$[\kappa_{n-1}(c), \kappa_{n-1}(p)] = 0 \quad \forall p, c \in J$$

By lemma 2.1, E is commutative.

Presume $\kappa_n \neq 0$

$$\kappa_n(pv) - \kappa_{n-1}(pv) \in Z(E)$$

$$\sum_{i=1}^n \kappa_i(pv) - \kappa_{i-1}(pv) \in Z(E)$$

$$\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(v) - \kappa_{i-1}(p) \kappa_{i-1}(v) \in Z(E)$$

Replace v by vc we obtain

$$\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(vc) - \kappa_{i-1}(p) \kappa_{i-1}(vc) \in Z(E), \quad \forall p, v, c \in Z(E)$$

$$\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(v) \kappa_{i-1}(c) - \kappa_{i-1}(p) \kappa_{i-1}(v) \kappa_{i-1}(c) \in Z(E)$$

In particular

$$[\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(v) \kappa_{i-1}(c) - \kappa_{i-1}(p) \kappa_{i-1}(v) \kappa_{i-1}(c)], \kappa_{i-1}(c) = 0$$

$$[\sum_{i=1}^n (\kappa_i(p) - \kappa_{i-1}(p)) \kappa_{i-1}(v) \kappa_{i-1}(c), \kappa_{i-1}(c)] = 0$$

$$\sum_{i=1}^n [\kappa_i(p) - \kappa_{i-1}(p), \kappa_{i-1}(c)] \kappa_{i-1}(v) \kappa_{i-1}(c) + (\kappa_i(p) - \kappa_{i-1}(p)) [\kappa_{i-1}(v) \kappa_{i-1}(c), \kappa_{i-1}(c)] = 0$$

$$\sum_{i=1}^n (\kappa_i(p) - \kappa_{i-1}(p)) [\kappa_{i-1}(v) \kappa_{i-1}(c), \kappa_{i-1}(c)] = 0$$

$$\sum_{i=1}^n (\kappa_i(p) - \kappa_{i-1}(p)) [\kappa_{i-1}(v), \kappa_{i-1}(c)] \kappa_{i-1}(c) = 0$$

Replace v by ev in above relation we have

$$\sum_{i=1}^n (\kappa_i(p) - \kappa_{i-1}(p)) \kappa_{i-1}(e) [\kappa_{i-1}(v), \kappa_{i-1}(c)] \kappa_{i-1}(c) = 0$$

$$\sum_{i=1}^n (\kappa_i(p) - \kappa_{i-1}(p)) E [\kappa_{i-1}(v), \kappa_{i-1}(c)] \kappa_{i-1}(c) = 0$$

Since E is prime, and κ_n is not identity we patience

$$\sum_{i=1}^n [\kappa_{i-1}(v), \kappa_{i-1}(c)] = 0$$

Thus E is commutative.

Corollary 3.3: Presume $J \neq \{0\}$, if E admits a higher left (right) centralizer κ_n s.t. $\kappa_n(pv) + \kappa_{n-1}(pv) \in Z(E)$, for all $p, v \in J$. Then E is commutative.

Proof: By using the same technique of theorem 3.2.

Theorem 3.4: If E admits higher left (right) centralizers $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ such that $\kappa_n(pv) \cdot \kappa_{n-1}(vp) \in Z(E)$, $\forall p, v \in J$, then E is commutative.

Proof: $\kappa_n(pv) \cdot \kappa_{n-1}(vp) \in Z(E)$

If $\kappa_n = 0 \Rightarrow \kappa_{n-1}(vp) \in Z(E)$

Employing the same technique as applied in proof of theorem 2, we obtain the required result.

Presume $\kappa_n \neq 0$

$\forall p, v \in J$, we obtain

$$\kappa_n(pv) \cdot \kappa_{n-1}(vp) \in Z(E)$$

$$\sum_{i=1}^n \kappa_i(p) \kappa_{i-1}(v) - \kappa_{i-1}(v) \kappa_{i-1}(p) \in Z(E). \text{ For all } p, v \in J.$$

$$\sum_{i=1}^n [\kappa_i(p) \kappa_{i-1}(v) - \kappa_{i-1}(v) \kappa_{i-1}(p), \kappa_{i-1}(e)] = 0$$

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(e)] \kappa_{i-1}(v) + \kappa_i(p) [\kappa_{i-1}(v), \kappa_{i-1}(e)] = \sum_{i=1}^n [\kappa_{i-1}(v), \kappa_{i-1}(e)] \kappa_{i-1}(p) +$$

$$\kappa_{i-1}(v) [\kappa_{i-1}(p), \kappa_{i-1}(e)]$$

Replace v by ve in last relation

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(e)] \kappa_{i-1}(v) \kappa_{i-1}(e) + \kappa_{i-1}(p) [\kappa_{i-1}(v), \kappa_{i-1}(e)] \kappa_{i-1}(e) =$$

$$\sum_{i=1}^n [\kappa_{i-1}(v), \kappa_{i-1}(e)] \kappa_{i-1}(e) \kappa_{i-1}(p) + \kappa_{i-1}(v) \kappa_{i-1}(e) [\kappa_{i-1}(p), \kappa_{i-1}(e)]$$

$$\Rightarrow \sum_{i=1}^n [\kappa_{i-1}(v), \kappa_{i-1}(e)] \kappa_{i-1}(p) \kappa_{i-1}(e) + \kappa_{i-1}(v) [\kappa_{i-1}(p), \kappa_{i-1}(e)] \kappa_{i-1}(e) =$$

$$\sum_{i=1}^n [\kappa_{i-1}(v), \kappa_{i-1}(e)] \kappa_{i-1}(e) \kappa_{i-1}(p) + \kappa_{i-1}(v) \kappa_{i-1}(e) [\kappa_{i-1}(p), \kappa_{i-1}(e)]$$

$\Rightarrow$

$$\sum_{i=1}^n \kappa_{i-1}(v) [\kappa_{i-1}(p), \kappa_{i-1}(e)] \kappa_{i-1}(e) - \kappa_{i-1}(v) \kappa_{i-1}(e) [\kappa_{i-1}(p), \kappa_{i-1}(e)] = 0$$

$$\sum_{i=1}^n \kappa_{i-1}(v) (\kappa_{i-1}(e) [\kappa_{i-1}(p), \kappa_{i-1}(e)] - \kappa_{i-1}(e) [\kappa_{i-1}(p), \kappa_{i-1}(e)]) = 0$$

$$\sum_{i=1}^n \kappa_{i-1}(v) [\kappa_{i-1}(p), \kappa_{i-1}(e)] \kappa_{i-1}(e) = 0$$

Replace v by vm and $[\kappa_{i-1}(p), \kappa_{i-1}(e)]$ by $\kappa_{i-1}(m)$, $\forall m \in J, e \in E$.

$$\sum_{i=1}^n \kappa_{i-1}(v) E[\kappa_{i-1}(m), \kappa_{i-1}(e)] = 0$$

By prime of E

Since $\kappa_{i-1}(v) \neq 0$, then $[\kappa_{i-1}(m), \kappa_{i-1}(e)] = 0$

Thus E is commutative.

By using same technique of theorem 3.4 we can proof the subsequent result

Corollary 3.5: If E admits higher left (right) centralizers $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ such that $\kappa_n(pv) + \kappa_{n-1}(vp) \in Z(E)$, for all $p, v \in J$. Then E is commutative.

Theorem 3.6: If E admits a higher left (right) centralizers $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ s.t. $\kappa_n(pv) - \kappa_{n-1}(pv) \in Z(E)$ or $\kappa_n(pv) + \kappa_{n-1}(pv) \in Z(E)$, $\forall p, v \in J$. Then E is commutative.

Proof: For each $p \in J$.

We put

$$J_1 = \{p \in J: \kappa_n(pv) - \kappa_{n-1}(pv) \in Z(E)\}$$

$$J_2 = \{p \in J: \kappa_n(pv) + \kappa_{n-1}(pv) \in Z(E)\}$$

It's clear J_1, J_2 are additive subgroups of J where

$$J_1 \cup J_2 = J. \text{ Hence either } J_1 = J \text{ or } J_2 = J$$

Further using same arguments of theorem 2 and corollary 3 we find

$$J = \{p \in J: J = J_1\} \text{ or } J = \{p \in J: J = J_2\}$$

By theorem 3.2 and corollary 3.3, E is commutative.

Theorem 3.7: Assume $J \neq \{0\}$ an ideal of E if admits a higher left (right) centralizers

$\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ s.t. $\kappa_n(p) \kappa_n(v) - \kappa_{n-1}(pv) \in Z(E)$ or $\kappa_n(p) \kappa_n(v) + \kappa_{n-1}(pv) \in Z(E)$, for all $p, v \in J$, then E is commutative.

Proof: we earn $\kappa_n(p) \kappa_{n-1}(v) - \kappa_{n-1}(pv) \in Z(E)$ Or $\kappa_n(p) \kappa_{n-1}(v) + \kappa_{n-1}(pv) \in Z(E), \forall p, v \in J$.

If $\kappa_n = 0$, then either case $\kappa_{n-1}(pv) \in Z(E), \forall p, v \in J$.

Employ the same technique just as the proof of theorem 2, we obtain the wanted outturn.

If $\kappa_n \neq 0$

For any $c \in E$, replacing v by vc we get

$$\kappa_n(p) \kappa_{n-1}(vc) - \kappa_{n-1}(pvc) \in Z(E)$$

$$\sum_{i=1}^n (\kappa_i(p) \kappa_{i-1}(v) - \kappa_{i-1}(p) \kappa_{i-1}(v)) \kappa_{i-1}(c) + \kappa_i(p) \kappa_{i-1}(v) \kappa_{i-1}(c) \in Z(E).$$

$$\text{Or } \sum_{i=1}^n (\kappa_i(p) \kappa_{i-1}(v) + \kappa_{i-1}(p) \kappa_{i-1}(v)) \kappa_{i-1}(c) + \kappa_i(p) \kappa_{i-1}(v) \kappa_{i-1}(c) \in Z(E).$$

Thus in every case we obtain

$$\sum_{i=1}^n [\kappa_i(p) \kappa_{i-1}(v) \kappa_{i-1}(c), \kappa_{i-1}(c)] = 0 \quad \dots (1)$$

$$\Rightarrow \sum_{i=1}^n \kappa_i(p) [\kappa_{i-1}(v) \kappa_{i-1}(c), \kappa_{i-1}(c)] + [\kappa_i(p), \kappa_{i-1}(c)] \kappa_{i-1}(v) \kappa_{i-1}(c) = 0$$

Replace $\kappa_{i-1}(v)$ by $\kappa_i(p) \kappa_{i-1}(v)$ in the above expression and use (1) to obtain

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] \kappa_i(p) \kappa_{i-1}(v) \kappa_{i-1}(c) = 0$$

That's mean

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] \kappa_i(p) E J \kappa_{i-1}(c) = 0$$

By Primness of E

Either $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] \kappa_i(p) = 0, \forall p \in J, c \in E$.

Or $\sum_{i=1}^n J \kappa_{i-1}(c) = \{0\}, \forall c \in E$.

If $\sum_{i=1}^n J \kappa_{i-1}(c) = \{0\}, \forall c \in E$.

$$\Rightarrow J E \sum_{i=1}^n \kappa_{i-1}(c) = \{0\}, \forall c \in E.$$

Since $J \neq \{0\}$ and E is prime, the above relation yields that $\kappa_n = 0$, which is a contradiction.

Therefore, assume the remaining possibility that

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] \kappa_i(p) = 0, \text{ for all } p \in J, c \in E.$$

For any $s \in E$, replace cs for c to obtain

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] E \kappa_i(p) = \{0\}, \text{ for all } p \in J, c \in E.$$

The primness of E implies that for each $p \in J$,

$$\text{Either } \sum_{i=1}^n \kappa_i(p) = 0 \text{ or } \sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] = 0$$

Hence in every case $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(c)] = 0$, for all $c \in E$.

Thus in particular

$$[\kappa_n(p), \kappa_{n-1}(p)] = 0, \forall p \in J.$$

Thus by lemma 3.1, E is commutative.

4. Main Results

Theorem 4.1: Assume $\kappa_n = (\kappa_i)_{i \in N}$ be higher left (right) centralizer of E s.t.

$[\kappa_n(p), \kappa_{i-1}(p)] \in Z(E)$, $\forall p$ belong to left ideal J of E , then for all $p \in J \cap Z(E)$, $\kappa_n(p) \in Z(E)$.

Proof: Since

$$[\kappa_n(p), \kappa_{i-1}(p)] \in Z(E), \forall p \in J. \quad \dots (2)$$

By linearizing (2) we obtain

$$[\kappa_n(p+v), \kappa_{n-1}(p+v)] \in Z(E)$$

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] + [\kappa_i(v), \kappa_{i-1}(p)] \in Z(E) \quad \dots (3)$$

Now, if $p \in Z(E)$, then from (3) we obtain

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] \in Z(E)$$

Replace $\kappa_{i-1}(v)$ by $\kappa_i(p)\kappa_{i-1}(v)$ we obtain

$$\sum_{i=1}^n \kappa_i(p) [\kappa_i(p), \kappa_{i-1}(v)] \in Z(E)$$

If $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] = 0$ then $\sum_{i=1}^n \kappa_i(p) \in C_E(J)$ the central of J in E .

Thus $\kappa_n(p) \in Z(E)$.

But on the other hand $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] \neq 0$, it follow from lemma 2.2, that $\kappa_n(p) \in Z(E)$

Lemma 4.2: If $\kappa_n = (\kappa_i)_{i \in N}$ of E is higher reverse left (right) centralizers then it's central higher centralizer.

Proof: Assume that $\kappa_n = (\kappa_i)_{i \in N}$ is higher reverse left (right) centralizer.

$$\begin{aligned} \kappa_n(pv^2) &= \sum_{i=1}^n \kappa_i(v^2) \kappa_{i-1}(p) \\ &= \sum_{i=1}^n \kappa_i(v) \kappa_{i-1}(v) \kappa_{i-1}(p) \quad \dots (4) \end{aligned}$$

That is

$$\kappa_n((pv)v) = \sum_{i=1}^n \kappa_i(v) \kappa_{i-1}(pv)$$

Replace $\kappa_{i-1}(pv)$ by $\kappa_{i-1}(vp)$ in last relation

$$\kappa_n((pv)v) = \sum_{i=1}^n \kappa_i(v) \kappa_{i-1}(p) \kappa_{i-1}(v) \quad \dots (5)$$

From (4) and (5) we achieve

$$\sum_{i=1}^n \kappa_i(v) [\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0, \forall p, v \in E \quad \dots (6)$$

Replacing p by ep in (6) and using (6) we get

$$\sum_{i=1}^n \kappa_i(v) \kappa_{i-1}(e) [\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0 \quad \dots (7)$$

On the other hand a linearization of (6) lead to

$$\begin{aligned} \sum_{i=1}^n \kappa_i(p) [\kappa_{i-1}(u), \kappa_{i-1}(p)] &= - \\ \sum_{i=1}^n \kappa_i(p) [\kappa_{i-1}(v), \kappa_{i-1}(p)] &, \forall p, v, u \in E. \end{aligned}$$

Replacing $\kappa_{i-1}(e)$ by $[\kappa_i(u), \kappa_{i-1}(p)] \kappa_{i-1}(e) \kappa_i(u)$ in (7) with primness E gives

$$\sum_{i=1}^n \kappa_i(u) [\kappa_{i-1}(v), \kappa_{i-1}(p)] = 0 \quad \forall p, v, u \in E.$$

Then we get $\kappa_n(u) \in Z(E)$, $\forall u \in E$.

Lemma 4.3: Assume $\text{char.}(E) \neq 2$, $p \in E$. If $J \neq (0)$ a central Lie ideal of E such that $u^2 \in J$,

$\forall u \in J$ and $\kappa_n(pJ) \subseteq Z(E)$ ($\kappa_n(Jp) \subseteq Z(E)$) then $\kappa_n(p) \in Z(E)$.

Proof: By the hypothesis

$$[\kappa_n(), \kappa_{n-1}(p)] = 0$$

And so

$$\sum_{j=1}^n \kappa_j(p) [\kappa_{j-1}(u), \kappa_{j-1}(p)] = 0, \forall u \in J. \quad \dots (8)$$

Replace u by $2ue$ in (8)

$$\sum_{i=1}^n \kappa_i(p) J [\kappa_{i-1}(e), \kappa_{i-1}(p)] = 0, \forall e \in J.$$

We acquire $\kappa_i(p) = 0$ or $[\kappa_{i-1}(e), \kappa_{i-1}(p)] = 0$, $\forall e \in J$.
By lemma 2.3 and $\text{sok}_n(p) \in Z(E)$.

Lemma 4.4: Assume $\text{char.}(E) \neq 2$, a left ideal $J \neq \{0\}$ on E , $\kappa_n \neq 0$, $\forall n \in \mathbb{N}$ is a left (right) centralizer of E . If J is not commutative such that $[\kappa_{n-1}(p), \kappa_n(p)] = 0$, $\forall p \in J$, then $\kappa_n = 0$.

Proof: by linearizing the equation $[\kappa_n(p), \kappa_{n-1}(p)] = 0$ which gives

$$\sum_{i=1}^n [\kappa_i(p+v), \kappa_{i-1}(p+v)] = 0$$

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(p)] + [\kappa_i(v), \kappa_{i-1}(v)] = 0$$

Replacing $\kappa_i(p)$ by $\kappa_{i-1}(p) \kappa_i(p)$ and $\kappa_i(v)$ by $\kappa_{i-1}(p) \kappa_i(v)$

$$\sum_{i=1}^n [\kappa_{i-1}(p) \kappa_i(p), \kappa_{i-1}(p)] + [\kappa_{i-1}(p) \kappa_i(v), \kappa_{i-1}(p)] = 0$$

$$\Rightarrow \sum_{i=1}^n \kappa_{i-1}(p) [\kappa_i(p), \kappa_{i-1}(p)] + \kappa_{i-1}(p) [\kappa_i(v), \kappa_{i-1}(v)] + [\kappa_{i-1}(p), \kappa_{i-1}(v)] \kappa_i(v) = 0$$

$$\sum_{i=1}^n [\kappa_{i-1}(p), \kappa_{i-1}(v)] \kappa_i(v) = 0 \quad \dots (9)$$

Replacing v by ve in (9), $\forall e \in J$, and using (9) we obtain

$$\sum_{i=1}^n [\kappa_{i-1}(p), \kappa_{i-1}(v) \kappa_{i-1}(e)] \kappa_i(v) \kappa_{i-1}(e) = 0$$

$$\sum_{i=1}^n \kappa_{i-1}(v) [\kappa_{i-1}(p), \kappa_{i-1}(e)] \kappa_i(v) \kappa_{i-1}(e) + [\kappa_{i-1}(p), \kappa_{i-1}(v)] \kappa_{i-1}(e) \kappa_i(v) \kappa_{i-1}(e) = 0$$

$$\sum_{i=1}^n [\kappa_{i-1}(p), \kappa_{i-1}(v)] \kappa_{i-1}(e) \kappa_i(v) \kappa_{i-1}(e) = 0 \quad \dots (10)$$

Replacing $\kappa_{i-1}(e)$ by $\kappa_{i-1}(r) \kappa_{i-1}(e)$, $\forall r \in E$ in (10)

$$\sum_{i=1}^n [\kappa_{i-1}(p), \kappa_{i-1}(v)] E \kappa_{i-1}(e) \kappa_i(v) \kappa_{i-1}(r) \kappa_{i-1}(e) = 0$$

By primness of E we acquire

$$\sum_{i=1}^n [\kappa_{i-1}(p), \kappa_{i-1}(v)] = 0 \text{ or}$$

$$\sum_{i=1}^n \kappa_{i-1}(e) \kappa_i(v) \kappa_{i-1}(r) \kappa_{i-1}(e) = 0$$

Since J is not commutative then $\kappa_n = 0$.

Theorem 4.5: Assume $\text{char.}(E) \neq 2$, J a left ideal of E , $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ is nonzero higher left (right) centralizers of E . If $[\kappa_n(v), \kappa_n(p)] = [\kappa_{i-1}(v), \kappa_{i-1}(p)] \forall p, v \in J$, then $[\kappa_{n-1}(p), \kappa_n(p)] = 0$ and hence E is commutative.

Proof: By $[\kappa_n(v), \kappa_n(p)] = [\kappa_{n-1}(v), \kappa_{n-1}(p)]$, $\forall p, v \in J$.

By taking vp instead of v in the hypothesis, then we acquire

$$\begin{aligned} [\kappa_{n-1}(vp), \kappa_{n-1}(p)] &= \sum_{i=1}^n [\kappa_i(v) \kappa_{i-1}(p), \kappa_i(p)] \\ &= \sum_{i=1}^n [\kappa_i(v), \kappa_i(p)] \kappa_{i-1}(p) + \kappa_i(v) [\kappa_{i-1}(p), \kappa_i(p)] \end{aligned}$$

Replacing $\kappa_{i-1}(v)$ by $\kappa_{i-1}(v) \kappa_{i-1}(r)$, $\forall r \in E$ in last relation we obtain

$$\sum_{i=1}^n \kappa_i(v) \kappa_{i-1}(r) [\kappa_{i-1}(p), \kappa_i(p)] = 0$$

By primness of E and $\kappa_n \neq 0$, we obtain

$$\sum_{i=1}^n [\kappa_{i-1}(p), \kappa_i(p)] = 0$$

Theorem 4.6: Let E be a ring of characteristic different from 2, and let J be a nonzero left ideal of E , $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ is nonzero higher left (right) centralizer of E . If $[\kappa_n(v), \kappa_n(p)] = 0$, $\forall p, v \in J$ then E is commutative.

Proof: Given $[\kappa_n(v), \kappa_n(p)] = 0$, $\forall p, v \in J$.

By taking vp instead of v for hypothesis we getting

$$\sum_{i=1}^n \kappa_i(v) [\kappa_{i-1}(p), \kappa_i(p)] + [\kappa_i(v), \kappa_i(p)] \kappa_{i-1}(p) = 0$$

$$\text{Then } \sum_{i=1}^n \kappa_i(v) [\kappa_{i-1}(p), \kappa_i(p)] = 0$$

Replacing $\kappa_i(v)$ by $\kappa_i(v) \kappa_i(r)$ in last relation, $\forall r \in E$.

$$\sum_{i=1}^n \kappa_i(v) \kappa_i(r) [\kappa_{i-1}(p), \kappa_i(p)] = 0$$

By primness of E and κ_n is nonzero we acquire

$$[\kappa_{n-1}(p), \kappa_n(p)] = 0$$

By theorem 4.5 we get the require result.

Example 4.7: Assume $E =$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ v & c & 0 \end{pmatrix} : p, v, c \text{ are integer numbers} \right\} \text{ be ring}$$

define an additive mappings $\kappa_n(\kappa_i)_{i \in \mathbb{N}}$ by

$$\kappa_n \left(\begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ v & c & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ np & nc & 0 \end{pmatrix}, \forall \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ v & c & 0 \end{pmatrix} \in E$$

Hence it's commuting on E .

Lemma 4.8: Assume $\kappa_n = (\kappa_i)_{i \in \mathbb{N}}$ is a higher left (right) centralizers on E such that

$$[\kappa_n(p), \kappa_{n-1}(p)] \in Z(E), \forall p \in J, \text{ where } J \text{ is a left ideal of } E, \text{ then for each } \kappa_{n-1}(p) \in J \cap Z(E), \kappa_n(p) \in Z(E).$$

Proof: Since $[\kappa_n(p), \kappa_{n-1}(p)] \in Z(E)$, $\forall p \in E$.

$$[\kappa_n(p+v), \kappa_{n-1}(p+v)] \in Z(E), \forall p, v \in J.$$

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] + [\kappa_i(v), \kappa_{i-1}(p)] \in Z(E)$$

But, $\kappa_{n-1}(p) \in Z(E)$, then from above relation we obtain

$$\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] \in Z(E).$$

Replace $\kappa_{i-1}(v)$ by $\kappa_i(p)\kappa_{i-1}(v)$ in last relation we obtain

$$\sum_{i=1}^n [\kappa_i(p), \kappa_i(p)\kappa_{i-1}(v)] \in Z(E)$$

$$\sum_{i=1}^n \kappa_i(p) [\kappa_i(p), \kappa_{i-1}(v)] \in Z(E)$$

If $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] = 0$, then $\kappa_n(p) \in C_E(J)$, the central of J in E .

Hence it's belong to $Z(E)$.

But on the other hand if $\sum_{i=1}^n [\kappa_i(p), \kappa_{i-1}(v)] \neq 0$.

It again follow from lemma 2.2 that is $\kappa_n(p) \in Z(E)$.

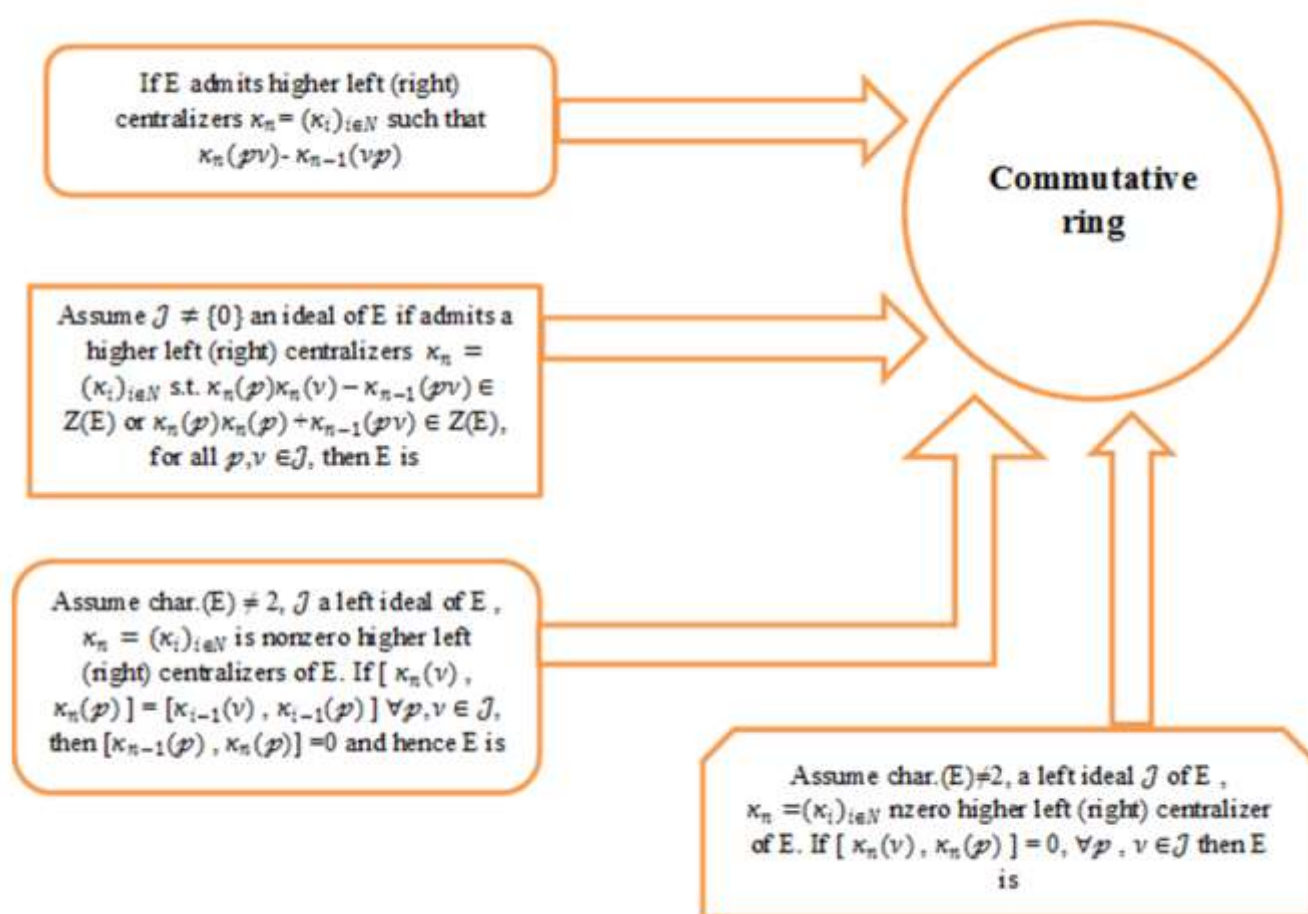


Figure 1. Sample of results obtained in the present work.

5. Conclusion

Higher centralizers are a fundamental topic in non-commutative algebra, as they play an important role in studying the conditions that make rings satisfy the commutativity and centrality conditions. We have reached several results, including: assume $\text{char.}(E) \neq 2$, a left ideal J of E , $\kappa_n = (\kappa_i)_{i \in N}$ is higher left (right) centralizers of E . If $[\kappa_n(v), \kappa_n(p)] = 0$, $\forall p, v \in J$ then E is commutative.

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