



r-Small Compressible Modules and r-Small Retractable Modules

Mohammed Baqer Hashim Al Hakeem¹, Hiba R. Baanoon^{2,*}

¹ Ministry of Higher Education and Scientific Research, Baghdad, Iraq.

² Department of Mathematics, College of Education, University of Misan, Misan, Iraq.

Article's Information

Received:30.07.2025

Accepted:03.12.2025

Published:15.09.2026

Keywords:

Small submodules,
r-Small submodules,
Small compressible modules,
Small retractable modules.

Abstract

Assuming that R is a commutative ring with unity, and X be a left unitary R -module. This work presents a generalization of the concepts of compressible and retractable modules. We specifically define and examine r-small compressible and r-small retractable modules. We examine their essential aspects, offer characterizations, and present instructive examples. Additionally, we investigate the relationships between these novel classes and existing established types of modules.

<http://doi.org/10.22401/ANJS.29.3.18>

*Corresponding author: hibabaanoon@uomisan.edu.iq



This work is licensed under a [Creative Commons Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/)

1. Introduction

All rings in this work are commutative with an identity element, and all modules are unitary left R -modules. Suppose X is a module, O and U be two submodules in X . Then O is termed a small submodule ($O \ll X$) if $O + U = X$ up to $U = X$; see [1,2]. Remember that module X 's maximal submodule L is its proper submodule such that whenever $L < S \leq X$ implies that $S = X$, where S is a submodule of X . See [2]. The Jacobson radical of a module T , symbolized by ($Rad(T)$), is expressed as the intersection of each maximal submodules of T , provided that such submodules exist, which is also a submodule of T [2]. If $A \ll Rad(X)$, then A is termed a radical small (r-small) submodule and denoted by ($A \ll_r X$). See [3]. A module S is termed to be compressible if, for each non-zero submodule C of S , there exists an injective (one-to-one) homomorphism $\sigma: S \rightarrow C$. In other words, the module S can be embedded into every of its non-zero submodules through a monomorphism [4]. A module X is termed retractable if, for each non-zero submodule $A \subseteq X$, there exists a non-zero homomorphism from X to A ; that is, $Hom_R(X, A) \neq 0$ [5]. Many authors generalize the compressible module and retractable module as [6-15]. This work will utilize the r-small submodule to generalize the

compressible module and retractable module as r-small compressible modules and r-retractable modules. We will review several examples and characteristics of r-small submodules, compressible modules, and retractable modules. For additional information, one may refer to references [3], [4], and [5].

Examples (1.1):

1. It is clear that $0 \ll_r X$ for all R -module X .
2. $Rad(Z) = 0$, $Rad(Q) = Q$.
3. In the Z -module Z_6 , the radical $Rad(Z_6)$ is the zero submodule $\langle 0 \rangle$; therefore, the submodule $\langle 3 \rangle$ is r-small in Z_6 .
4. In the Z -module Z_8 , the radical $Rad(Z_8)$ is the submodule $\langle 2 \rangle$.
5. $\langle 6 \rangle \ll_r Z_{12}$, Z_{12} as Z -module
6. In the Z -module Q , $Z \ll_r Q$.
7. The Z -module Z is compressible module, because $Hom_R(Z, nZ) \neq 0$, and it is a retractable module.
8. In the Z -module Q , the set of all homomorphism from Q into Z is zero, so it is not compressible module and it is not a retractable module.

9. The Z -module Z_n is not a compressible module, for all $n \in \mathbb{Z}^+, n > 1$ because $\text{Hom}_R(Z_n, A) = 0$, where A is a submodule of Z_n , while it is a retractable module.
10. The Prüfer p -group Z_{p^∞} , when viewed as a module over Z , fails to satisfy the conditions of being compressible or retractable.
11. The homomorphic image of a compressible module does not inherently preserve the attribute of compressibility. For instance, the module, $\frac{Z}{6Z} \simeq Z_6$ serves as a counterexample, as it is not compressible.
12. Each compressible module is retractable module. Conversely, therefore it is not true (see 9).

Remarks (1.2): Suppose that X is a module and O, U are two submodules in X , then :

1. If $O \ll_r X$, then $O \ll X$.
2. If $O \ll X$ and $U \ll_r O$, then $U \ll_r X$.
3. $Rx \ll X$ if and only if $x \in \text{Rad}(X)$; $x \in X$.
4. Any submodule $U \leq S$. $\text{Rad}(U) \leq \text{Rad}(S)$.
5. If $O_1 \ll_r S_1$ and $O_2 \ll_r S_2$, then $O_1 + O_2 \ll_r S_1 + S_2$; S_1 and S_2 are two modules.
6. If $O \ll_r U \leq X$, then $O \ll_r X$.
7. If $O \leq U \ll_r X$, then $O \ll_r X$.
8. If $X = \bigoplus_{i \in I} X_i$; X_i is submodules in X for all i , then $\text{Rad}(X) = \bigoplus_{i \in I} \text{Rad}(X_i)$.

2. r-Small Compressible Modules

This paper introduces the concept of r-small compressible modules as a logical extension of compressible modules. We present multiple examples to elucidate the concept and delineate certain essential qualities that are pivotal to our inquiry.

Definition (2.1): The module X is considered r-small compressible when it admits an embedding into each of its nontrivial r-small submodules, meaning there exists a monomorphism $\sigma: X \rightarrow A$; $A \ll_r X$.

Examples and Remarks (2.2):

1. Every compressible module qualifies as an r-small compressible module. Typically, the converse is not applicable. For example, $\mathbb{Z}Z_6$, regarded as a Z -module, is r-small compressible but not compressible (see examples(1.1,(9))).
2. The Z -module Z is an r-small compressible module, as it is compressible.

3. The Z -module Z_8 is not r-small compressible module, as Z_8 cannot be embedded in $\langle 4 \rangle$ where $\langle 4 \rangle \ll_r Z_8$ with $\text{Rad}(Z_8) = \langle 2 \rangle$.
4. The Z -module, Q cannot be r-small compressible module, as $\text{Hom}_Z(Q, Z) = 0$ where $Z \ll_r Q$.
5. All simple module is an r-small compressible module; however, the converse is not true. For instance, Z , as a Z -module, is an r-small compressible module, although it is not simple.

A module X is said to be small compressible if it admits an embedding into each non-zero small submodule of itself. [9].

Proposition (2.3): Each module that is small compressible can be regarded as an r-small compressible module.

Proof. Take S to be an arbitrary module and $0 \neq O \ll_r S$. By hypotheses S is small compressible module and by remark(1.2)(1) $O \ll S$, hence there is a monomorphism $\theta: S \rightarrow O$. Thus we can be embedded S in O . Therefore, S is r-small compressible module. ■

The converse of proposition(2.3) need not to be true in general. For example, Z_4 as Z -module has no r-small submodule since $\text{Rad}(Z_4) = \langle 2 \rangle$ so it is r-small compressible module but not small compressible, since $\langle 2 \rangle$ is small submodule and Z_4 cannot be embedded in $\langle 2 \rangle$.

Proposition (2.4)

If a module is r-small compressible, then all of its submodules are likewise r-small compressible.

Proof. Suppose that O is a submodule of a module S and $0 \neq U \ll_r O$. We like to show that the submodule O can be embedded in U . So $0 \neq U \ll_r O \leq S$. By remark (1.2)(6), then $0 \neq U \ll_r S$, since S is r-small compressible module. Then there exists a monomorphism $\rho: S \rightarrow U$. We have that $i: O \rightarrow S$ is the inclusion monomorphism, hence $\rho \circ i: O \rightarrow U$ is a monomorphism. Thus, O is r-small compressible module. ■

Remark (2.5): In general, a homomorphic image of an r-small compressible module does not necessarily inherit the r-small compressibility property. For instance, the module $\frac{Z}{8Z} \simeq Z_8$ provides a counterexample, as it is not r-small compressible.

Proposition (2.6): If S_1 and S_2 are two modules such that $U_1 \leq S_1$ and $U_2 \leq S_2$ with $U_1 \oplus U_2 \ll_r S_1 \oplus S_2$, then $U_1 \ll_r S_1$ and $U_2 \ll_r S_2$.

Proof. Suppose that $U_1 \leq U_1 \oplus U_2 \ll_r S_1 \oplus S_2$. Thus by definition of r -small submodule $U_1 \oplus U_2 \ll \text{Rad}(S_1 \oplus S_2) \leq \text{Rad}(S_1) \oplus \text{Rad}(S_2)$. Thus $U_1 \oplus U_2 \ll \text{Rad}(S_1) \oplus \text{Rad}(S_2)$. So $U_1 \ll \text{Rad}(S_1)$ and $U_2 \ll \text{Rad}(S_2)$. Therefore, $U_1 \ll_r S_1$ and $U_2 \ll_r S_2$. ■

Proposition (2.7): Let $X = A_1 \oplus A_2$ be a module such that $\text{ann}A_1 + \text{ann}A_2 = R$. Then module X is said to be r -small compressible if and only if its components A_1 and A_2 are themselves r -small compressible.

Proof. $\Rightarrow$) Let $0 \neq P_1 \ll_r A_1 \leq A_1 \oplus A_2 = X$. Then $0 \neq P_1 \ll_r X$, there exists a monomorphism $\delta: A_1 \oplus A_2 \rightarrow P_1$ with the inclusion monomorphism $i: A_1 \rightarrow A_1 \oplus A_2$. Hence there is a monomorphism $\delta \circ i: A_1 \rightarrow P_1$. Thus, A_1 is an r -small compressible module; similar for A_2 .

$\Leftarrow$) Let $0 \neq P \ll_r A_1 \oplus A_2 = X$. Then $P = P_1 \oplus P_2$ for some $0 \neq P_1 \ll_r A_1 \leq X$ and $0 \neq P_2 \ll_r A_2 \leq X$, so P_1 and P_2 are r -small submodules in X . But A_1 and A_2 are r -small compressible modules, then there exists a monomorphism $\sigma_1: A_1 \rightarrow P_1$ and $\sigma_2: A_2 \rightarrow P_2$. Now, we can define $\gamma: X \rightarrow P$ by $\gamma(s, d) = (\sigma_1(s), \sigma_2(d))$; $s \in A_1$ and $d \in A_2$. It easily to show that γ is a monomorphism and so X is r -small compressible module ■

3. r -Small Retractable Modules

This paper introduces the notion of r -small retractable modules as a generalization of retractable modules, accompanied by several examples, and presents fundamental aspects of this module type utilized in our research.

Definition (3.1): A module X qualifies as r -small retractable when each of its nonzero r -small submodules A satisfies $\text{Hom}_R(X, A) \neq 0$.

Examples and Remarks (3.2):

1. Any retractable module is, in particular, an r -small retractable module.
2. The Z -module, Z is an r -small retractable module, as it is retractable.
3. The Z -module, Z_n is an r -small retractable module, for each $n \in Z^+, n > 1$, (because $\text{Hom}_Z(Z_n, A) = 0$), where $A \ll_r Z_n$.
4. The Z -module Q is not an r -small retractable module, (because $\text{Hom}_R(Q, Z) = 0$).
5. Each semi-simple module is r -small retractable; however, the reverse is not necessarily true. For instance, the Z -module Z_8 is a retractable module of r -small type, although it is not semi-simple.
6. The Z -module, $Q \oplus Z$ is r -small retractable module.

7. In general, the homomorphic image of an r -small retractable module does not necessarily remain r -small retractable. For instance, the module, $\frac{Q \oplus Z}{Z} \simeq Q$ is not r -small retractable module.
8. The direct summand of an r -small retractable module does not necessarily inherit the r -small retractability property. (since $Q \oplus Z$ is r -small retractable module, but Q is not r -small retractable module.

Proposition (3.3): If S is an r -small retractable module, then any r -small submodule of S is itself r -small retractable

Proof. Assuming O is an r -small submodule of S . We have to show O is also r -small retractable module. Let $0 \neq U \ll_r O \leq S$. Thus, $U \ll_r S$, because S is an r -small retractable module, hence there is a nonzero homomorphism $\sigma: S \rightarrow U$ and $i: O \rightarrow S$ is the inclusion homomorphism. Thus, $\sigma \circ i: O \rightarrow U$ is a nonzero homomorphism ($\text{Hom}_R(O, U) \neq 0$). Therefore O is an r -small retractable module. ■

Corollary (3.4): Any r -small submodule contained in a retractable module is itself r -small retractable.

Proof: Suppose O is an r -small submodule of S , where S is a retractable module, by examples and remarks (3.2.1) S is r -small retractable module. Then by proposition(3.3) O is an r -small retractable module. ■

Remark (3.5): In general, a direct summand, homomorphic image, or quotient of an r -small retractable module may fail to be r -small retractable. For instance, consider the quotient module $\frac{Z \oplus Z_{p^\infty}}{Z} \simeq Z_{p^\infty}$, which is not an r -small retractable module, since Z_{p^∞} , when regarded as a Z -module, does not satisfy the r -small retractability condition.

Proposition (3.6): Let $S = S_1 \oplus S_2$ be a module with $\text{ann}S_1 + \text{ann}S_2 = R$. Then S is r -small retractable if and only if S_1 and S_2 are r -small retractable modules.

Proof. $\Rightarrow$) Suppose $0 \neq P_1 \ll_r S_1 \leq S_1 \oplus S_2 = S$. So $0 \neq P_1 \ll_r S$, there is a nonzero homomorphism $\delta: S_1 \oplus S_2 \rightarrow P_1$ with the inclusion homomorphism $i: S_1 \rightarrow S_1 \oplus S_2$. Hence there is a nonzero homomorphism $\delta \circ i: S_1 \rightarrow P_1$. Thus, S_1 is an r -small retractable module; similar for S_2 .

$\Leftarrow$) Let $0 \neq P \ll_r S_1 \oplus S_2$. Since $\text{ann}S_1 + \text{ann}S_2 = R$, we can give $P = P_1 \oplus P_2$ with $P_1 \leq S_1$ and $P_2 \leq S_2$, but $P_1 \oplus P_2 \ll_r S_1 \oplus S_2$ implies that $P_1 \ll_r S_1$ and $P_2 \ll_r S_2$. But S_1 and S_2 are r -small retractable modules, then

there exists nonzero homomorphism $\sigma_1: S_1 \rightarrow P_1$ and $\sigma_2: S_2 \rightarrow P_2$. Now, we can define $\gamma: S \rightarrow P$ by $\gamma(a, b) = (\sigma_1(a), \sigma_2(b))$; $a \in S_1$ and $b \in S_2$. It easily to show that γ is a nonzero homomorphism and hence S is r-small retractable module ■

Hence, we can define a new concept rs-hollow modules

Definition (3.7): A module X is said to be rs-hollow if each proper submodule in X is r-small.

Proposition (3.8): Let S be an rs-hollow module. Then the following are equal :

1. S is an retractable module.
2. S is a small retractable module.
3. S is an r-small retractable module.

Proof. (1) $\Rightarrow$ (2) Let O be a nonzero small submodule of S and assume that S is a retractable module. Then $O \leq S$, and there exists a nonzero homomorphism $\sigma: S \rightarrow O$. Therefore S is small retractable module.

(2) $\Rightarrow$ (3) Suppose $0 \neq B \ll_r S$ and assume that S is small retractable module. Then $B \ll S$, and there exists a nonzero homomorphism $\sigma: S \rightarrow B$. Therefore S is r-small retractable module.

(3) $\Rightarrow$ (1) Let $0 \neq B \leq X$. Since X is rs-hollow module, then $0 \neq B \ll_r X$ and there exists a nonzero homomorphism $\sigma: X \rightarrow B$. Therefore, X is retractable module.

4. Discussion

In this paper, we have introduced and studied the notions of r-Small Compressible Modules and r-Small Retractable Modules as natural generalizations of compressible and retractable modules. Several properties, examples, and characterizations have been provided, together with some relations to other well-known classes of modules.

1. Limitations of this study: The present work is theoretical in nature and limited to commutative rings with unity. The results are not yet extended to non-commutative settings or to modules with additional structures, such as graded modules. Moreover, the study focuses mainly on structural properties rather than applications to other branches of algebra.
2. Disadvantages and possible improvements: One disadvantage is that some proofs rely heavily on previously known results without providing deeper structural characterizations. This may limit accessibility for readers less familiar with the area. Future work could aim at giving more

constructive proofs or explicit examples, possibly supported by computational algebra systems, to make the theory more transparent.

3. Features and comparison with existing results: The proposed concepts extend the classical ideas of compressibility and retractability by replacing the role of small submodules with r-small submodules. This generalization allows for a wider class of modules to be studied under similar frameworks. Compared with existing literature, our results unify and broaden the scope of analysis while preserving many useful properties of the classical notions.

5. Conclusions

In this paper, we alluded that each compressible module is r-small compressible module. However in general conversely is not true, a submodule of an r-small compressible module is r-small compressible module. But, a homomorphic image of an r-small compressible need not be r-small compressible in general, every retractable modules is r-small retractable module, An r-small submodule of an r-small retractable module is also r-small retractable module, and a direct summand (a homomorphic image or quotient module) of an r-small retractable module it not may be r-small retractable module in general.

Conflicts of Interest: Conflicts of Interest: The authors declare no conflict of interest.

Funding: No financial support was received for this research.

References

- [1] Leonard, W. W.; "Small modules". Proc. Amer. Math. Soc.,17(2): 527-531, 1966.
- [2] Kasch, F.;" Modules and Rings". 3rd ed; Academic Press Inc.: London, UK, 1982.
- [3] Nebiyev, C.; Ökten, H.H.; "r-Small Submodules". In: Conference Proceedings of Science and Technology, 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020), Online, Turkey, June 24–27, 2020; Tosun, M.; Ed.;CPOST: Turkey, 2020.
- [4] Zelmanowitz, J.; "An extension of the Jacobson density theorem". Bull. Amer. Math. Soc., 82(4): 551-553, 1976.
- [5] Khuri, S. M.; "Endomorphism rings and lattice isomorphism.", J. Algebra, 59: 401-408, 1979.
- [6] Al Hakeem, M.B.H.; Al-Mothafar, N.S.; "S-Essentially Compressible Modules and S-

-
- Essentially Retractable Modules". Iraqi J. Sci., 64(8): 4102–4108, 2023.
- [7] Al-Aeashi, S. N.; Al-Bakaa, F.H.; "Essentially retractable modules relative to a submodule and some generalizations". Iraqi J. Comput. Sci. Math., 2(2): 1–5, 2021.
- [8] Alhossaini, A. M.; "s-Compressible and s-Prime Modules". Iraqi J. Sci., 63(3): 1200–1207, 2022.
- [9] Muslem, I. H.; "Some types Of Retractable and Compressible Modules". M.Sc. thesis, College of Education for pure Science, Univ. of Baghdad, 2016.
- [10] Rajaei, S.; Farzalipour, F.; Poyan, M.; "A new class of small submodules". J. Algebraic Syst., 13(3): 157–174, 2025.
- [11] Kabban, A. A.; Khalid, W.; "e* Singular–Hollow Modules and e* Singular–Coclosed Submodules". Iraqi J. Sci., 66(6): 2477–2486, 2025.
- [12] Shihab, B. N.; Hatam, Y. K.; "On Quasi-Maximal Modules". Al-Nahrain J. Sci., 13(4): 205-210, 2010.
- [13] Abdul-Alkalik, A. J.; "Small Quasi Primary Submodules", Al-Nahrain J. Sci., 28(3): 125 – 129, 2025.
- [14] Bader, R.F.; Al-Mothafar, N.S.; "Semi-hollow modules". J. Discrete Math. Sci. Cryptogr., 28(4-B): 1385–1390, 2025.
- [15] Diop, P. Ch.; Dia M. L.; "On E-Small Compressible Modules". J. Algebra Relat. Top., 9(2): 69-81, 2021.