Modules and Bounded Linear Operators

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Abstract

An associated R-module of T, which is denoted by V_{T,T^*} is given, Where V is an inner product space and T is bounded linear operator on V. We study in this paper properties of T which effects V_{T,T^*} and conversely.

Keywords**:** The module of an operator, algebraic elements of T, torsion elements of H_{T,T^*}.

1. Introduction

SALMA M. FARIS in [1] described a left R-module V where R the polynomials ring in x and V is vector space as follows:- $\emptyset: R \times V \to V$ by $\emptyset(P, v) = P.v = P(T)v$ this function makes V a left R −module denoted by V_T. In this paper we start by introducing a left R- module on the ring of polynomials in x, y and V is an inner product space as follows:- $\Psi: R \times V \to V$ by $\Psi(P, v) = P(T, T^*)v$ this function makes V a left R −module and denote this module by V_{T,T^*} .In proposition (3.2) we give form of elements of V_{T,T^*}. We prove that V_{S,S^*} is isomorphic to V_{T,T^*} if and only if S is similar to T, we study the relation between the $*-$ algebraic elements and the torsion elements of H_{T,T^*} ,and the module associated with the unilateral shift operator we prove that H_{U,U^*} is acyclic R-module.

2. Preliminaries

In this section the fundamental basic concepts and primitive results are Given.

Definition (2.1) [1]:

Let V be a vector space over a field F. Let T be a linear operator acting on the elements of V on the left .Let $\mathrm{R} = \mathrm{F[x]}$ be the ring of polynomials in x with coefficients in F . Define $\emptyset: R \times V \to V$ by $\emptyset(P, v) = P.v = P(T)v$.

It is clear that $\emptyset$ makes V a left R-module denoted V_T, and call it the associated R −module.

The form of every element in V_T is illustrated in the following proposition.

Proposition (2.2) [1]:

If $S = \{V_j : j \in \Lambda\}$ is a basis for V, then each element of V_T Can be written in the form $\sum_{i=0}^{n} \sum_{j\in\Lambda} c_{ij}\, T^i v_j$, where $c_{ij} \in F$

The symbol $\sum_{j\in\Lambda}$ means that the sum is taken over a finite subset of Λ .

Remark (2.3) [1]:

V_I is a finitely generated R-module if and only if V is a finite dimensional vector space.

In this remark there is a relation between a finite dimensional vector space V and V_T

Remark (2.4) [1]:

Let V be a finite dimensional vector space. Let T be an operator on V, then V_T is a finitely generated R- module.

Recall that if T and S two operators on V.S is similar to T if there exists an invertible operator h on V such that $hSh^{-1} = T$ [2].

Proposition (2.5) [1]:

Let T and S be two operators on V .Then V_S is isomorphic to V_T if and only if S is similar to T.

Definition (2.6) [2]:

Let T be an operator on a vector space. T is said to be of finite rank if the image of T is finite dimensional.

It is shown in (2.4) that if V is a finite dimensional vector space, then V_T is a finitely generated R- module. Also if V is finite dimensional vector space, and T is any operator on V, then TV is finite dimensional. Hence T is of finite rank. Following proposition give the converse.

Proposition (2.7) [1]:

If T is of finite rank, and V_T is finitely generated, then V is finite dimensional.

Definition (2.8) [3]:

Let $T: V \to V$ be an operator. $v \epsilon V$ is said to be an algebraic element (or T-algebraic) if there exists a non zero polynomial $P \epsilon R$ such that $P(T)v = 0$.

T is said to be algebraic if there exists $P \neq 0$ in R such that $P(T)v = 0, \forall v \in V$

Proposition (2.9)[1]:

Let $T: V \to V$ be an operator. Let $A = A(T)$ be the set of all T −algebraic elements. Then A is a subspace of V.

There is a relation between the T −algebraic elements and the torsion elements of V_T this relation is studied in the next proposition.

Recall that an element m of S-module where S is a ring is torsion element if there exists $0 \neq t \in S$ such that $tm = 0$, M is torsion S-module if $\tau(M) = M$

where $\tau(M)$ the set of all torsion elements .[3]

Proposition (2.10)[1]:

Let T be an operator on V ,then $A_T = \tau(V_T)$

Recall that for any ring S and any S −module $M, ann(M) = \{t \in S : tm = 0, \forall m \in M\}$, and $ann(M) = 0$ then M is a faithful S −module.[4].

Proposition (2.11)[1]:

V_T is faithful R −module if and only if T is not an algebraic operator.

The module of the Unilateral shift operator is given finally.

Let $U: l_2(R) \to l_2(R)$ be the operator defined by $U(x_1, x_2, \ldots) = U(0, x_1, x_2, \ldots.)$

This operator called the Unilateral shift operator.[5]

Remark (2.12)[1]:

$\forall i, K \in N$, one can easily see that:

1. $U e_k = e_{k+1}$ 2. $U^i e_k = e_{i+k}$ 3. $U^i e_k = U^{i+k-1} e_1$.

Recall that a left R-module M is called acyclic if M can be generated by a single element. $M(x) = Rx = \{rx / r \in R\}$ for some x in M.

Theorem (2.13)[1]:

Let U be the Unilateral shift operator on H. Then H_U is a cyclic faithful R −module. Hence a free R −module.

3. Main Results

Definition (3.1):

Let $R = F[x, y]$ be the ring of polynomials in x, y with coefficients in F. Let V be an inner product space over afield F and T be a bounded linear operator acting on the elements of V on the left .We will define a left R-module on V as follows: $\Psi: R \times V \to V$

by $\Psi(P, v) = P(T, T^*)v$ i.e $P(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j$, $a_{ij} \in F$. [6] It is clear that Ψ makes V a unitary left R –module. We shall denote this module by V_{T,T^*}.

In the following proposition we introduce the form of each element of V_{T,T^*}.

Proposition (3.2):

If S= $\{v_l : l \in \Lambda\}$ is a basis for V. then each element of V_{T,T^*} can be Written in the form $\sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l \in \Lambda} c_{il} T^i T^{*j} v_l$, $c_{il} \in F$

The symbol $\sum_{l \in \Lambda}$ means that the sum is taken over a finite subset of Λ

***Proof*:-** let $w \in V_{T,T^*}$, then

$w = \sum_{k=1}^{m'} P_k \cdot w_k$, where

$$P_k(x,y) = \sum_{j=0}^{m} (P_k(x))\, y^j,$$

$P_k(x) = \sum_{i=0}^{n_k} a_{ik} x^i$

$P_k(x,y) = \sum_{j=0}^{m} \left(\sum_{i=0}^{n_k} a_{ik} x^i\right) y^j \in R$

$w_K = \sum_{l \in \Lambda} b_{Kl} v_l \in V$,then

$$w = \sum_{k=1}^{m'} \sum_{j=0}^{m} \left(\sum_{i=0}^{n_k} a_{ik} T^i T^{*j}\right) \left(\sum_{l \in \Lambda} b_{kl} v_l\right)$$

Let $n = \max\{n_1, n_2, \cdots, n_m\}$, $a_{ik} = 0, \forall i > n_k$, k =1,2, ⋯ , m'

Then $w = \sum_{j=0}^{m} \sum_{i=0}^{n} a_{ik} T^i T^{*j} \left(\sum_{k=1}^{m'} \sum_{l \in \Lambda} b_{kl} v_l\right)$

$= \sum_{j=0}^{m} \sum_{i=0}^{n} T^i T^{*j} \left(\sum_{l \in \Lambda} \sum_{k=1}^{m'} a_{ik} b_{kl} v_l\right)$

$= \sum_{j=0}^{m} \sum_{i=0}^{n} T^i T^{*j} \left(\sum_{l \in \Lambda} c_{il} v_l\right)$

Where $c_{il} = \sum_{k=1}^{m'} a_{ik} b_{kl}$

Thus $w = \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l \in \Lambda} c_{il} T^i T^{*j} v_l$

Examples (3.3):

1. Let $\{v_l : l \in \Lambda\}$ be abasis for an inner product space V.

(a) Let 0 be the zero operator on V .If $w \in V_{0,0^*}$ then by proposition (3.2)

$w=\sum_{j=0}^{m}\sum_{i=0}^{n}\sum_{l\in\wedge} c_{il}0^{i}0^{*^{j}}$, $c_{il}\in F$. Recall that $0^0=$ I, then $w=\sum_{l\in\wedge} c_{0l}v_l$

(b) Let I be the Identity operator on V.If $w\in V_{I,I^*}$ then by proposition (3.2)
, $w=\sum_{j=0}^{m}\sum_{i=0}^{n}\sum_{l\in\wedge} c_{il}I^iI^jv_l$
put $c_l=\sum_{i+j=0}^{n+m} c_{il}$
Then $w=\sum_{l\in\wedge} c_l\ v_l$

2. Let T be a bounded linear operator on a Hilbert space.
(a) T is a Self –adjoint operator, if $T=T^*$.[2]
Then by proposition (3.2)
$w=\sum_{i+j=0}^{n+m}\sum_{l\in\wedge} c_{il}T^{i+j}v_l$

(b) T is Normal operator ,if $TT^*=T^*T$.[2]
Then by proposition (3.2) ,
$w=\sum_{j=0}^{m}\sum_{i=0}^{n}\sum_{l\in\wedge} c_{il}T^{*^{i}}T^{j}v_l$

Remark (3.4):

V_{I,I^*} is a finitely generated R −module if and only if V is a finite dimensional an inner product space.

***Proof*:**

Let V_{I,I^*} is finitly generated R −module with generators $\{u_1,u_2,\ldots,u_m\}$ we prove by contradiction suppose that V is not finite Dimensional. Let $\{e_l : l\in\wedge\}$ be a basis for V by Ex :1.(b) ,$u_j\in$ V
$u_j=\sum_{k\in\wedge}c_ke_k$,j=1,2,…,m . Thus V_{I,I^*} can be generated by a finite number of elements of the set $\{e_l: l\in\wedge\}$, say, $\{e_1,e_2,\ldots,e_n\}$
Therefore if $K>n$ then $e_k=\sum_{t=1}^{m} P_t\cdot e_t$
Where $P_t(x,y)=\sum_{i=0}^{n}\left(\sum_{j=0}^{k_t}a_{tj}\ x^j\right)y^i$

$$e_k=\sum_{t=1}^{m}\sum_{i=0}^{n}\left(\sum_{j=0}^{k_t}a_{tj}\ x^j\right)y^i\cdot e_t$$

$$=\sum_{j=0}^{k_t}a_{tj}e_t$$

Put $a_t=\sum_{j=0}^{k_t}a_{tj}$
then $p_t\cdot e_t=a_t\cdot e_t$,t=1,2,…,m
Therefore , $e_k=\sum_{t=1}^{n}a_te_t$
Which is a contradiction, thus V is a finite dimensional an inner product space.
Assume V is an n-dimensional normed space with basis $\{v_1,v_2,\ldots,v_n\}$. Let $w\in V_{I,I^*}$ by Ex:1.(b) $w=\sum_{l=1}^{n}c_lv_l$
This shows that V_{I,I^*} is a finitely generated R −module.
Compare the following with proposition (2-5)

Proposition (3.5):

Let T, S be two bounded operators on V.then V_{S,S^*} and V_{T,T^*} are isomorphic R-module iff S and T are similar.

***Proof*:**

If V_{S,S^*} is isomorphic to V_{T,T^*}
Let $h: V_{S,S^*}\to V_{T,T^*}$ be an R-isomorphisim
Thus $h(w_1+w_2)=h(w_1)+h(w_2)$, $\forall w_1,w_2\in V_{S,S^*}$
$h(P(x,y)\cdot w)=P(x,y)\cdot h(w)$, $\forall P\in R, w\in V_{S,S^*}$
i.e h is homomorphisim .then we can define h as:
$h[P(S,S^*)w]=P(T,T^*)h(w)$
If P is a constant polynomial a, $a\in F$, then $h(av)=ah(v)$
Thus h is a linear operator call it again h, if $P(x,y)=x+y$
Then $h(P(x,y)w)=P(x,y)h(w)$
$h((x+y)w)=(x+y)\ h(w)$
$h(S+S^*)=(T+T^*)h$
$hSh^{-1}+hS^*h^{-1}=h^{-1}Th+h^{-1}T^*h$
Then $hSh^{-1}=T$, $hS^*h^{-1}=T^*$
Then S is similar to T

If S and T are similar then there exists an operator h on V s.t
$h(S+S^*)h^{-1}=T+T^*$ it is easy to cheack that

$hP(S,S^*)=P(T,T^*)h\ \forall P\in R$ .... (1)

Define $h': V_{S,S^*}\to V_{T,T^*}$
By $h'[P(S,S^*)v]=P(T,T^*)h(v)$ .... (2)

If $P_1(S,S^*)v_1=P_2(S,S^*)v_2$
Then $h[P_1(S,S^*)v_1]=h[P_2(S,S^*)v_2]$
(since h operator)
Then by
(1) $P_1(T,T^*)h(v_1)=P_2(T,T^*)h(v_2)$
By
(2) $h'[P_1(S,S^*)v_1]=h'[P_2(S,S^*)v_2]$.thus h' is well define.
If $h'[P(S,S^*)v]=0$,
then $P(T,T^*)h(v)=0$
By (1) $hp(S,S^*)v=0$ but h is invertible then $p(S,S^*)v=0$

Therefore h' is 1-1

Let $P(T,T^*)v \in V_{T,T^*}$ since $v \in V$

Then $h^{-1}(v) \in V$ and $P(S,S^*)h^{-1}(v) \in V_{S,S^*}$

$h'[P(S,S^*)h^{-1}(v)] = P(T,T^*)hh^{-1}(v) = P(T,T^*)v$

Thus h' is on to

Note $h'[P(S,S^*)v] = h[P(S,S^*)v]$,but h is an operator, hence

h' is an R-homomorphism, therefore h' is an R-isomorphism.

Remark (3.6):

If V is a finite dimensional an inner product space, then V_{T,T^*} is finitely generated R-module.

We show in (3.6) that if V is a finite dimensional an inner product space, then V_T is finitely generated R-module, also if V is finite dimensional and T is any operator on V, then TV is finite dimensional, hence T is of finite rank.

<u>Proposition (3.7):</u>

If T is of finite rank, and V_{T,T^*} is finitely generated, then V is finite dimensioal.

***Proof*:**

Let $K = K(T\,T^*) = \{w \in V: TT^*w = 0\}$ it is clear that K is an invariant subspaces of V, and $TT^*\,V \cong \frac{V}{K}$

We prove by contradiction way .Assume V is not finite dimensional. TT^*V is finite dimensional since T is finite rank,thus K must be infinite dimensional but K is an invariant subspace of V,then the submodule K_{T,T^*} is generated by the set $\{T^iT^{*^j}w_l: l \in \wedge;\ i = 0,1,\cdots; j = 0,1,\cdots\}$ where $\{w_l: l \in \wedge\}$ is abasis for K.$w_l \in k$ means that $T\,T^*w_l = 0$ .Hence the restriction of $T\,T^*$ on K is the zero operator,thus $K_{T,T^*} = K_{0,0^*}$ by (3.2) K_{T,T^*} cannot be finitely generated, and since R Noetherian [7] ,V_{T,T^*} is finitely generated then K_{T,T^*} is finitely generated .this contradiction shows that V is finite dimensional.

<u>Definition (3.8) [8]:</u>

An operator $T \in B(H)$ is said to be *-algebraic operator if there exists non-zero polynomial of two variables P such that $P(T,T^*)x = 0 \quad, \forall x \in H$. $x \in H$ is called *-algebraic element if there exists non zero polynomial of two variables P such that $P(T,T^*)x = 0$.

<u>Proposition (3.9):</u>

Let $T: H \to H$ and $A = A(T,T^*)$ be the set of all *-algebraic elements then A is a subspace of H.

***Proof*:**

Let $u, v \in A$ then there exist non-zero polynomial p,q inR such that

$P(T,T^*)u = 0$ and $q(T,T^*)\quad v = 0$, then $P(T,T^*)q(T,T^*)(u+v)=0$

Since $R = F[x,y]$ is an integral domain [9], hence $Pq \neq 0$, therefore

$u + v \in A$ if $a \in F$ then $P(T,T^*)au = aP(T,T^*)u = 0$ thus $au \in A$ therefore A subspace of H.

<u>Proposition (3.10):</u>

Let T be an operator on H, then $A_{T,T^*} = \tau(H_{T,T^*})$

***Proof*:**

let $0 \neq w \in A_{T,T^*}$. then $w = \sum_{i=0}^{n} P_i\ v_i$ for some $P_i \in R$, $v_i \in A\ \forall i$

There exists $q_i \neq 0$ in R such that $q_i(T,T^*)v_i = 0$

hence$q(T,T^*)\,w = q.w = 0$ where $q = q_1q_2\cdots q_n$Thus $w \in \tau(H_{T,T^*})$

And let $u \in \tau(H)$,then there exists $P \neq 0$ in R Such that $P.u = 0$ therefore $P(T,T^*)u = 0$, thus $u \in A_{T,T^*}$

Therefore $A_{T,T^*} = \tau(H_{T,T^*})$

In the following proposition we give the relation between faithful R-module and *-algebraic operator.

<u>Proposition (3.11):</u>

H_{T,T^*} is a faithful R −module if and only if T is not *-algebraic operator.

***Proof*:**

Let $P \in R$ such that $P(T,T^*)v = 0\ \forall v \in H$

Then $P.v = 0\ \forall x \in H$. Thus $P.v = 0\ \forall v \in H_{T,T^*}$ hence $P \in ann(H_{T,T^*})$

Therefor $P = 0$ and T is not *-algebraic operator.

Conversely, let $P \in ann\,(H_{T,T^*})$

Then $P.v = 0\ \forall v \in H_{T,T^*}$, thus $P(T, T^*)v = 0\ \forall v \in H$

If T is not *-algebraic operator, then $P = 0$. Therefor H_{T,T^*} is faithful.

Finally, we study the module of Unilateral shift operator in the following.

Theorem (3.12) :

Let U be the Unilateral shift operator on H. then H_{U,U^*} is a cyclic R- module .hence a free R-module.

***Proof*:**

Let $w \in H_{U,U^*}$, then

$w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^i {U^*}^j e_l$

Since $U^* = B$, $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^i B^j e_l$ $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^i e_{l-j}$.[1] $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^{i+l-1} U^{-j} e_1$. By(2.12) remark 3,

Thus $w = P.e_1$,

where $P(x,y) = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} x^{i+l-1} y^j$

Therefore H_{U,U^*} is cyclic R-module generated by e_1.thus H_{U,U^*} is afree R-module. [10]

Corollary (3.13):

Let U be the unilateral shift operator on H. then H_{U,U^*} is a faithful R-module.

***Proof*:**

Let $P(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j \in \text{ann}(H_{U,U^*})$ then $P(x,y).e_1=0$

Hence

$\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} U^i B^j e_1 = 0, \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} U^i (e_{1-j}) = 0$.[1]

By (2.12) remark 2 we have $\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} e_{i-j+1} = 0$.

But $e_1, e_2, \dots, e_{m-n+1}$ are linearly Independent hence $a_{ij} = 0$ $\forall i = 0,1,\dots,m$, $j = 0,1,\dots,n$ thus P=0

Therefore H_{U,U^*} is a faithful R-module.

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الخلاصة

الموديول التابع للمؤثر T. الذي يرمز له بالرمز V_{T,T^*} أعطي, عندما V فضاء الجداء الداخلي وT مؤثر خطي مقيد على V. سندرس في هذا البحث صفات للمؤثر T التي تؤثر على V_{T,T^*} وبالعكس.