Al-Nahrain Journal of Science
ANJS, Vol.27(3), September, 2024, pp. 126-141

On Modified Weighted Exponential Rayleigh Distribution

Lamyaa Khalid Hussein[1,*], Iden Hasan Hussein[2], Huda Abdullah Rasheed[3]

[1] Department of Mathematics and Computer Applications, College of Sciences, Al-Nahrain University, Jadriya, Baghdad, Iraq.
[2] Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq.
[3] Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.

Article's Information

Received: 24.03.2024
Accepted: 21.05.2024
Published: 15.09.2024

Keywords:
Modified Weighted Exponential Rayleigh Distribution
Maximum Likelihood Estimation
Rank Set Sampling Estimation
Maximum Entropy Estimation

Abstract

This study seeks for classical estimations to estimating the anonymous parameters and reliability function of Modified Weighted Exponential Rayleigh $MWER$ distribution. These classical methods are chosen precisely because all of these methods maximize the probability density function. Newton-Raphson technique was used to derive estimation methods.

http://doi.org/10.22401/ANJS.27.3.14
*Corresponding author: lamyaa.khalid@nahrainuniv.edu.iq

1. Introduction

Many science disciplines, especially in observational studies of humans, wildlife, insects, fisheries, and plants, have the problem that observations recorded on individuals in these populations are biased and would not have the original distribution if each observation was not given an equal opportunity to be recorded. Weighted distribution theory provides a unified approach to correcting biases in unequally weighted sample data. Due to the importance of weighted distributions in many fields such as reliability (survival), biomedicine, environment, and fields of practical interest such as mathematics, probability, and statistics, there are different ways to add a shape parameter to the probability distribution model. Azzalini's (1985) introduced form to add a shape parameter to several distributions, some of the researchers used this form such as, Hussian (2013) [1] generalized version of the inverted exponential distribution created on Azzalini's approach to obtain a new class which is named as weighted inverted exponential distribution. Nasiru (2015) [6] obtained a different weighted version of Weibull distribution named weighted Weibull distribution and derived some of its several statistical properties. Oguntunde et al. (2016) [7] presented a new weighted exponential distribution and discussed essential statistical properties of this distribution. Mudasir et al. (2018) [5] suggested weighted power distribution. Hussein, L. K. and et al. (2023) [2] presented a different distribution titled Modified Weighted Exponential Rayleigh $\boldsymbol{MWER}$ distribution and derived some of its essential statistical properties.

The pdf of $MWER$ distribution is knowledge of the following formula [2]:

$$f(x;\mu,\rho,\varphi) = \quad (\mu(1+\varphi)+\rho(1+\varphi)x)\, e^{-\left(\mu(1+\varphi)x+\frac{\rho(1+\varphi)}{2}x^2\right)} \quad (1)$$

where $x > 0$; $\mu, \rho > 0$ are scale parameters and $\varphi > 0$ is shape parameter.

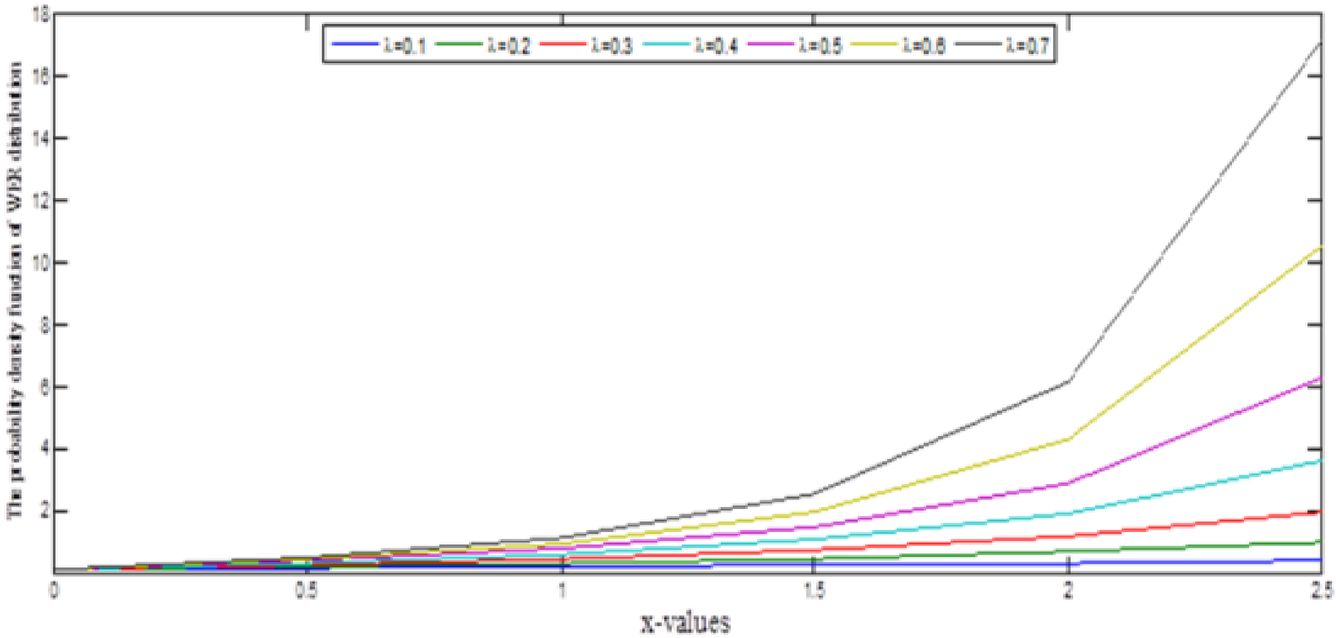

Figure 1. Graph of pdf of $MWER$ distribution for $\mu = \varphi = 0.1$ and changed values of $(\rho = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7)$

Equation of the cdf of $MWER$ is [2]:

$$\mathcal{F}(x;\mu,\rho,\varphi) = 1 - e^{-\left(\mu(1+\varphi)x+\frac{\rho(1+\varphi)}{2}x^2\right)} \ldots (2)$$

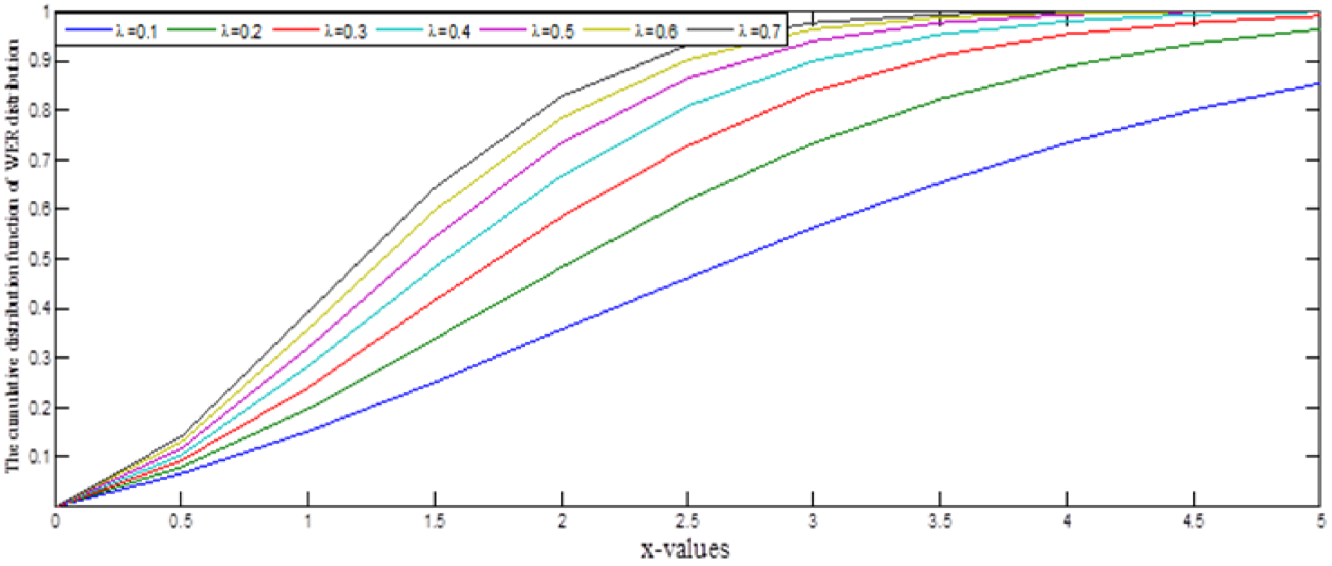

Figure 2. Graph of cdf of $MWER$ distribution for $\mu = \varphi = 0.1$ and changed values of $(\rho = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7)$

The reliability function of $\boldsymbol{MWER}$ is known by [2]

$$\mathcal{R}(t;\mu,\rho,\varphi) = e^{-\left(\mu(1+\varphi)t + \frac{\rho(1+\varphi)}{2}t^2\right)} \quad \ldots (3)$$

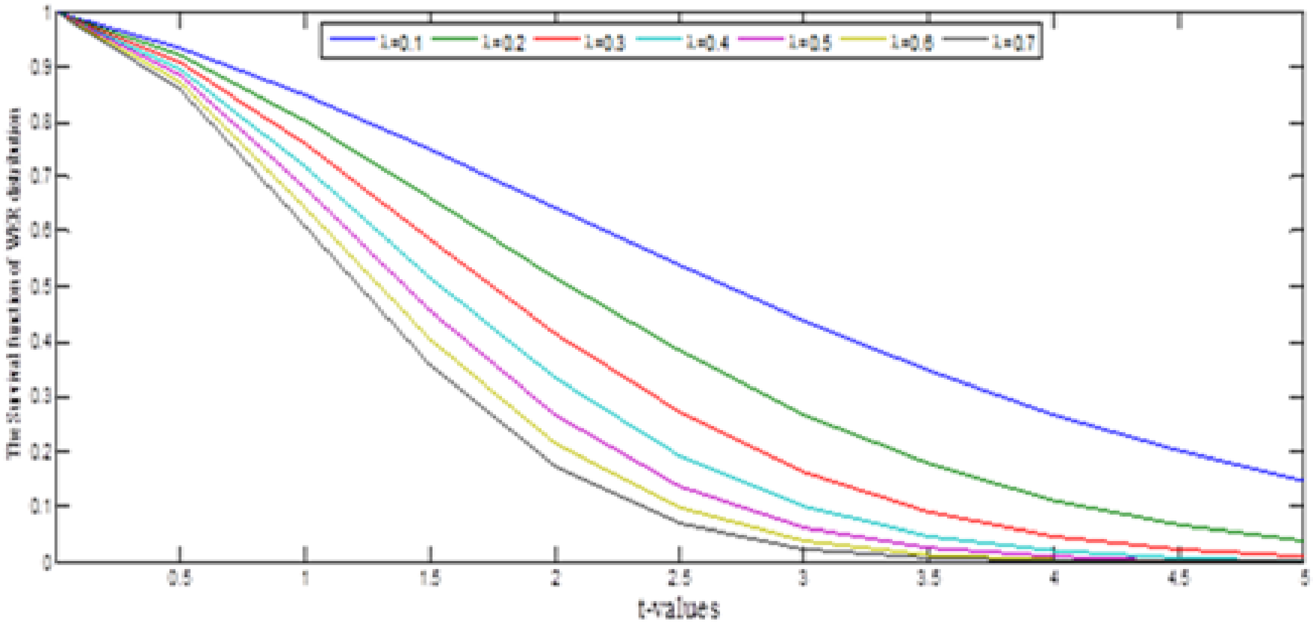

Figure 3. Graph of reliability function of $\boldsymbol{MWER}$ distribution for $\boldsymbol{\mu = \varphi = 0.1}$ and changed values of ($\boldsymbol{\rho = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7}$)

The hazard rate function of $\boldsymbol{MWER}$ is given in the following formula [2]:

$$\mathcal{H}(t;\mu,\rho,\varphi) = \mu(1+\varphi) + \rho(1+\varphi)t \quad \ldots (4)$$

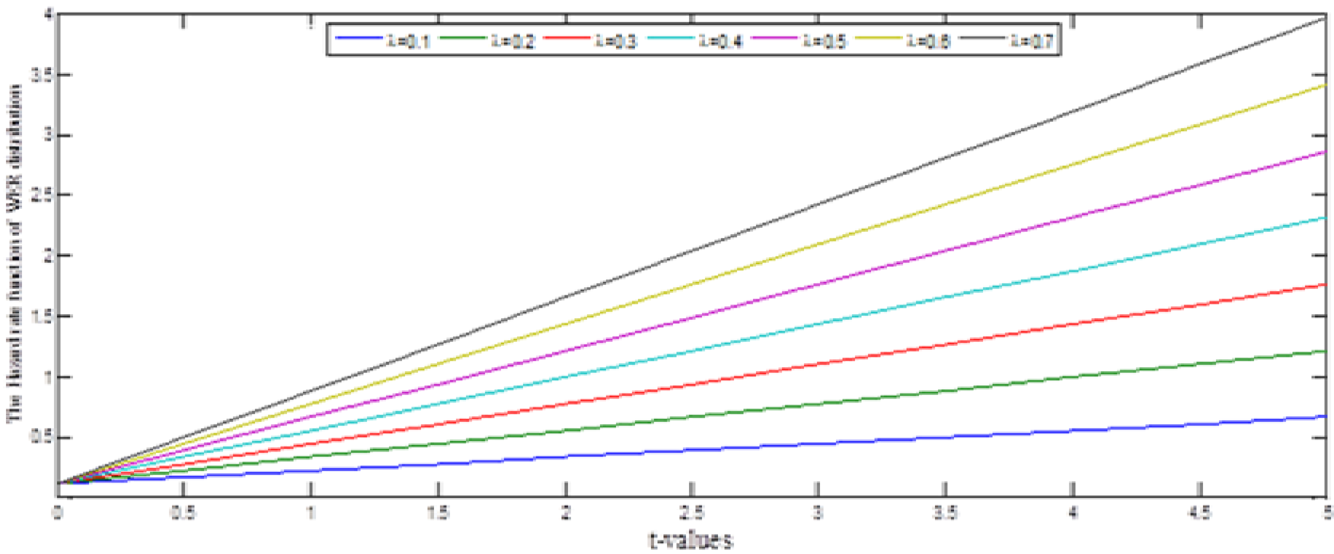

Figure 4. Graph of hazard rate function of $\boldsymbol{MWER}$ distribution for $\boldsymbol{\mu = \varphi = 0.1}$ and changed values of ($\boldsymbol{\rho = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7}$)

The reverse hazard rate function of $\boldsymbol{MWER}$ is Knowledge of the formula [2]:

$$\mathcal{N}(t;\mu,\rho,\varphi)=\frac{(\mu(1+\varphi)+\rho(1+\varphi)t)\,e^{-\left(\mu(1+\varphi)t+\frac{\rho(1+\varphi)}{2}t^2\right)}}{1-\;e^{-\left(\mu(1+\varphi)t+\frac{\rho(1+\varphi)}{2}t^2\right)}}\qquad\ldots(5)$$

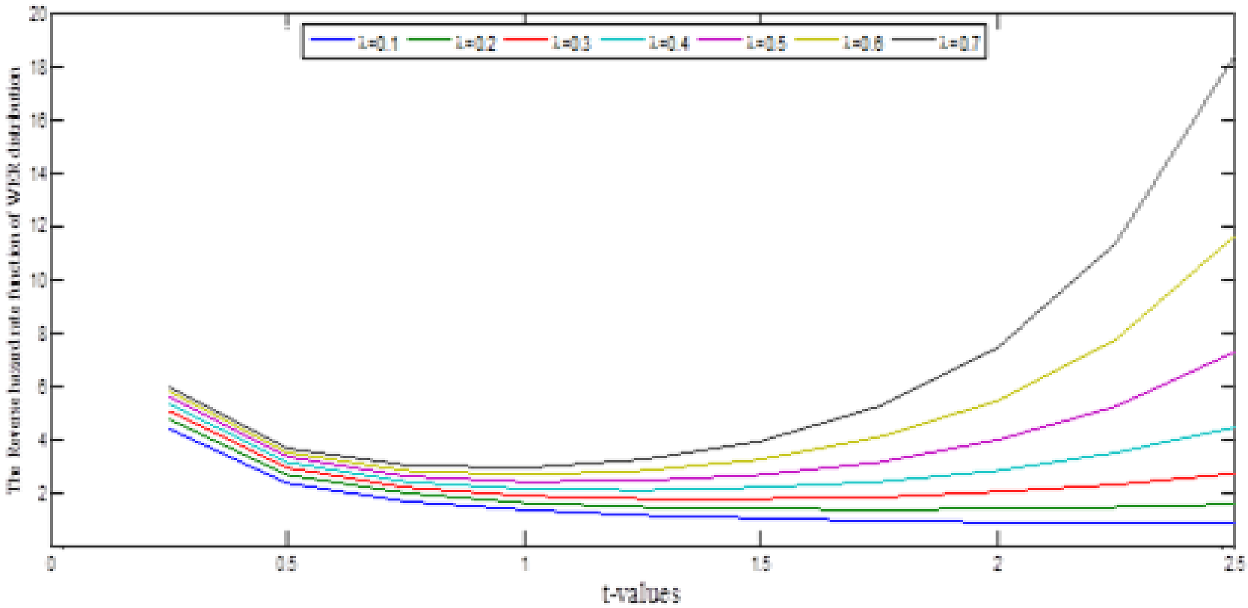

Figure 5. Graph of reverse hazard rate function of $\boldsymbol{MWER}$ distribution for $\boldsymbol{\mu=\varphi=0.1}$ and changed values of $(\boldsymbol{\rho=0.1,0.2,0.3,0.4,0.5,0.6,0.7})$

The general form of the $\boldsymbol{k^{th}}$ moment about the origin where $\boldsymbol{k=1,2,3,\ldots}$ is given by the formula [2]:

$$E(X^k)=\sum_{w=0}^{\infty}\frac{\left(-\mu(1+\;\varphi)\right)^w}{w!}\frac{2^{\frac{k+w}{2}}}{\rho^{\frac{k+w}{2}}\;(1+\;\varphi)^{\frac{k+w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\;\Gamma\left(\frac{k+w+1}{2}\right)+\;\Gamma\left(\frac{k+w+2}{2}\right)\right]\qquad\ldots(6)$$

The Expected value: When we put $\boldsymbol{k=1}$ in equation (6) get the Expected value thus:

$$E(X)=\sum_{w=0}^{\infty}\frac{\left(-\mu(1+\;\varphi)\right)^w}{w!}\frac{2^{\frac{1+w}{2}}}{\rho^{\frac{1+w}{2}}\;(1+\;\varphi)^{\frac{1+w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\;\Gamma\left(\frac{w+2}{2}\right)+\;\Gamma\left(\frac{w+3}{2}\right)\right]\qquad\ldots(7)$$

The Variance: The variance equation of $\boldsymbol{MWER}$ distribution is given by the formula:

$$V(X)=E(X^2)-[E(X)]^2$$

$$\boldsymbol{V(X)} = \left[\sum_{w=0}^{\infty} \frac{(-\mu(1+\varphi))^{w}}{w!} \frac{2^{\frac{2+w}{2}}}{\rho^{\frac{2+w}{2}}(1+\varphi)^{\frac{2+w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\,\Gamma\left(\frac{w+3}{2}\right) + \Gamma\left(\frac{w+4}{2}\right)\right]\right] - \left[\sum_{w=0}^{\infty} \frac{(-\mu(1+\varphi))^{w}}{w!} \frac{2^{\frac{1+w}{2}}}{\rho^{\frac{1+w}{2}}(1+\varphi)^{\frac{1+w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\,\Gamma\left(\frac{w+2}{2}\right) + \Gamma\left(\frac{w+3}{2}\right)\right]\right]^{2} \quad \dots(8)$$

Equation (9) gives the formula of the moment generating function of ***MWER*** distribution [2]:

$$\mathcal{M}_{X}(t) = \sum_{w=0}^{\infty} \frac{\left(-(\mu(1+\varphi)-t)\right)^{w}}{w!} \frac{2^{\frac{w}{2}}}{\rho^{\frac{w}{2}}(1+\varphi)^{\frac{w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\,\Gamma\left(\frac{w+1}{2}\right) + \Gamma\left(\frac{w+2}{2}\right)\right] \quad \dots(9)$$

The factorial moment generating function equation of ***MWER*** distribution can be found by [2]:

$$\mathcal{M}(t) = \sum_{w=0}^{\infty} \frac{\left(-(\mu(1+\varphi)-\ln(t))\right)^{w}}{w!} \frac{2^{\frac{w}{2}}}{\rho^{\frac{w}{2}}(1+\varphi)^{\frac{w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\,\Gamma\left(\frac{w+1}{2}\right) + \Gamma\left(\frac{w+2}{2}\right)\right] \quad \dots(10)$$

Equation of the characteristic function of ***MWER*** distribution is Knowledge of the formula [2]:

$$\mathcal{C}_{x}(it) = \sum_{w=0}^{\infty} \frac{(-(\mu(1+\varphi)-\mathrm{i}t))^{w}}{w!} \frac{2^{\frac{w}{2}}}{\rho^{\frac{w}{2}}(1+\varphi)^{\frac{w}{2}}}\left[\frac{\mu\sqrt{\varphi}}{\sqrt{2\rho}}\,\Gamma\left(\frac{w+1}{2}\right) + \Gamma\left(\frac{w+2}{2}\right)\right] \quad \dots(11)$$

2. Classical Estimation

2.1. Maximum likelihood estimation

Let $\underline{x} = (x_1, x_2, \dots, x_n)$ be a random sample drawn independently from $MWER$ distribution and it has pdf known in equation (1). So the formula of the complete data likelihood function $\mathcal{L}(\mu, \rho, \varphi \mid \underline{x})^{MLE}$ is,

$$\mathcal{L}(\mu, \rho, \varphi \mid \underline{x})^{MLE} = \prod_{q=1}^{w} f(x_q; \mu, \rho, \varphi)$$

$$\mathcal{L}(\mu, \rho, \varphi \mid \underline{x})^{MLE} = \prod_{q=1}^{w}\left[\left(\mu(1+\varphi) + \rho(1+\varphi)x_j\right) e^{-\left(\mu(1+\varphi)x_q + \frac{\rho(1+\varphi)}{2}x_q^2\right)}\right] \quad \dots(12)$$

From which we calculate the natural log – likelihood function:

$$\ell^{MLE} = \ln \mathcal{L}(\mu, \rho, \varphi \mid \underline{x})^{MLE}$$

$$\ell^{MLE} = \sum_{q=1}^{w} \ln\left(\mu(1+\varphi) + \rho(1+\varphi)x_q\right) - \sum_{q=1}^{w}\left(\mu(1+\varphi)x_q + \frac{\rho(1+\varphi)}{2}x_q^2\right) \quad \dots(13)$$

We derive an equation (13) partially with respect to μ, ρ and φ and then we set it equal to zero to produce,

$$\frac{\partial \ell^{MLE}}{\partial \mu} = (1+\varphi)\sum_{q=1}^{w} \frac{1}{\mu(1+\varphi) + \rho(1+\varphi)x_q} - (1+\varphi)\sum_{q=1}^{w} x_q = 0 \quad \dots(14)$$

$$\frac{\partial \ell^{MLE}}{\partial \rho} = (1+\varphi)\sum_{q=1}^{w} \frac{x_j}{\mu(1+\varphi) + \rho(1+\varphi)x_q} - \frac{(1+\varphi)}{2}\sum_{q=1}^{w} x_q^2 = 0 \quad \dots(15)$$

$$\frac{\partial \ell^{MLE}}{\partial \varphi} = \sum_{q=1}^{w} \frac{\mu + \rho x_j}{\mu(1+\varphi) + \rho\,(1+\varphi)x_q} - \mu \sum_{q=1}^{w} x_q - \frac{\rho}{2} \sum_{q=1}^{w} x_q^2 = 0 \qquad \dots (16)$$

By resolving equations (14), (15) and (16) gotten the MLE symbolized by $\hat{\mu}^{MLE}, \hat{\rho}^{MLE}$ and $\hat{\varphi}^{MLE}$. Since these equations are non-linear, so, Newton – Raphson process can be used to get the resolve as follows such that $h = 0,1,2,\dots$

$$\begin{bmatrix} \hat{\mu}_{MLE}^{(h+1)} \\ \hat{\rho}_{MLE}^{(h+1)} \\ \hat{\varphi}_{MLE}^{(h+1)} \end{bmatrix} = \begin{bmatrix} \hat{\mu}_{MLE}^{(h)} \\ \hat{\rho}_{MLE}^{(h)} \\ \hat{\varphi}_{MLE}^{(h)} \end{bmatrix} - \mathcal{J}_{(h)}^{-1} \begin{bmatrix} \frac{\partial \ell^{MLE}}{\partial \mu} \\ \frac{\partial \ell^{MLE}}{\partial \rho} \\ \frac{\partial \ell^{MLE}}{\partial \varphi} \end{bmatrix}_{\substack{\mu=\hat{\mu}_{MLE}^{(h)} \\ \rho=\hat{\rho}_{MLE}^{(h)} \\ \varphi=\hat{\varphi}_{MLE}^{(h)}}}$$

where

$$\mathcal{J}_{(h)} = \begin{bmatrix} \frac{\partial^2 \ell^{MLE}}{\partial \mu^2} & \frac{\partial^2 \ell^{MLE}}{\partial \mu \partial \rho} & \frac{\partial^2 \ell^{MLE}}{\partial \mu \partial \varphi} \\ \frac{\partial^2 \ell^{MLE}}{\partial \rho \partial \mu} & \frac{\partial^2 \ell^{MLE}}{\partial \rho^2} & \frac{\partial^2 \ell^{MLE}}{\partial \rho \partial \varphi} \\ \frac{\partial^2 \ell^{MLE}}{\partial \varphi \partial \mu} & \frac{\partial^2 \ell^{MLE}}{\partial \varphi \partial \rho} & \frac{\partial^2 \ell^{MLE}}{\partial \varphi^2} \end{bmatrix}_{\substack{\mu=\hat{\mu}_{MLE}^{(h)} \\ \rho=\hat{\rho}_{MLE}^{(h)} \\ \varphi=\hat{\varphi}_{MLE}^{(h)}}}$$

Equations (14), (15) and (16) represent the first partial derivatives and the second partial derivatives are given as follows,

$$\frac{\partial^2 \ell^{MLE}}{\partial \mu^2} = -(1+\varphi)^2 \sum_{q=1}^{w} \frac{1}{\left(\mu(1+\varphi) + \rho(1+\varphi)x_q\right)^2} \qquad \dots (17)$$

$$\frac{\partial^2 \ell^{MLE}}{\partial \rho^2} = -(1+\varphi)^2 \sum_{q=1}^{w} \frac{{x_j}^2}{\left(\mu(1+\varphi) + \rho(1+\varphi)x_q\right)^2} \qquad \dots (18)$$

$$\frac{\partial^2 \ell^{MLE}}{\partial \varphi^2} = -\sum_{q=1}^{w} \frac{\left(\alpha + \rho x_j\right)^2}{\left(\mu(1+\varphi) + \rho(1+\varphi)x_q\right)^2} \qquad \dots (19)$$

$$\frac{\partial^2 \ell^{MLE}}{\partial \mu \partial \rho} = \frac{\partial^2 \ell^{MLE}}{\partial \rho \partial \mu} = -(1+\varphi)^2 \sum_{q=1}^{w} \frac{x_j}{\left(\mu(1+\varphi) + \rho(1+\varphi)x_q\right)^2} \qquad \dots (20)$$

$$\frac{\partial^2 \ell^{MLE}}{\partial \mu \partial \varphi} = \frac{\partial^2 \ell^{MLE}}{\partial \varphi \partial \mu} = -\sum_{q=1}^{w} x_q \qquad \dots (21)$$

$$\frac{\partial^2 \ell^{MLE}}{\partial \rho \partial \varphi} = \frac{\partial^2 \ell^{MLE}}{\partial \varphi \partial \rho} = -\frac{1}{2} \sum_{q=1}^{w} x_q^2 \qquad \dots (22)$$

Now, Now, based on an invariant property of the MLE estimator, the reliability function at mission time (t) of the $MWER$ distribution can be obtained by replacing μ, ρ and φ in equation (3), by their MLE estimators as follows:

$$\hat{\mathcal{R}}^{MLE}(t;\mu,\rho,\varphi) = \exp\left(-(\hat{\mu}^{MLE}(1+\hat{\varphi}^{MLE})t + \frac{\hat{\rho}^{MLE}(1+\hat{\varphi}^{MLE})}{2} t^2)\right) \quad \dots(23)$$

2.2. Rank set sampling estimation [3]

In this estimation the p.d.f. of $MWER$ distribution is Knowledge as follows:

$$f(x_{(q)}) = \frac{w!}{(q-1)!\,(w-q)!}[\mathcal{F}(x_{(q)})]^{q-1}[1-\mathcal{F}(x_{(q)})]^{w-q} f(x_{(q)}) \quad \dots(24)$$

Let

$$B = \frac{w!}{(q-1)!\,(w-q)!}$$

$$f(x_{(q)}) = B\left[\mu(1+\varphi) + \rho(1+\varphi)x_{(q)}\right]\left[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}\right]^{q-1}\left[e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}\right]^{w-j+1} \quad \dots(25)$$

The complete data likelihood function $\mathcal{L}(\mu,\rho,\varphi \mid \underline{x})^{RSS}$ for order sample $(x_{(1)}, x_{(2)}, \dots, x_{(w)})$ is,

$$\mathcal{L}(\mu,\rho,\varphi \mid \underline{x})^{RSS} = B^w \prod_{q=1}^{w}[\mu(1+\varphi) + \rho(1+\varphi)x_{(q)}] \prod_{q=1}^{w}\left[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}\right]^{q-1} \prod_{q=1}^{w}\left[e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}\right]^{w-q+1}$$

Taking the natural log – likelihood function we get:

$$\ell^{RSS} = ln\,\mathcal{L}(\mu,\rho,\varphi \mid \underline{x})^{RSS}$$

$$\ell^{RSS} = w\ln B + \sum_{q=1}^{w} ln\left[\mu(1+\varphi) + \rho(1+\varphi)x_{(q)}\right] + \sum_{q=1}^{w}(q-1)ln\left[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}\right] - \sum_{q=1}^{w}(w-q+1)\left[\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right] \quad \dots(26)$$

We derive equation (26) partially with respect to μ, ρ and φ and then we set it equal to zero to produce,

$$\frac{\partial \ell^{RSS}}{\partial \mu} = (1+\varphi)\sum_{q=1}^{w}\frac{1}{\mu(1+\varphi) + \rho(1+\varphi)x_{(q)}} - (1+\varphi)\sum_{q=1}^{w}(w-q+1)\,x_{(q)} + (1+\varphi)\sum_{q=1}^{w}\frac{(q-1)\,x_{(q)}\,e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}}{1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2}x_{(q)}^2\right)}} = 0 \quad \dots(27)$$

$$\frac{\partial \ell^{RSS}}{\partial \rho} = (1+\varphi)\sum_{q=1}^{w}\frac{x_{(j)}}{\mu(1+\varphi)+\ \rho(1+\varphi)x_{(q)}} - (1+\varphi)\sum_{q=1}^{w}(w-q+1)\left(\frac{x_{(q)}^{2}}{2}\right)$$
$$+ (1+\varphi)\sum_{q=1}^{w}\frac{(q-1)\left(\frac{x_{(q)}^{2}}{2}\right)\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}}{1\ -\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}} = 0 \qquad \dots(28)$$

$$\frac{\partial \ell^{RSS}}{\partial \varphi} = \sum_{q=1}^{w}\frac{\mu+\rho\, x_{(q)}}{\mu(1+\varphi)+\ \rho(1+\varphi)x_{(q)}} - \sum_{q=1}^{w}(w-q+1)\left(\mu x_{(q)}\ +\ \frac{\rho}{2}x_{(q)}^{2}\right)$$
$$+ \sum_{q=1}^{w}\frac{(q-1)\left(\mu x_{(q)}\ +\ \frac{\rho}{2}x_{(q)}^{2}\right)\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}}{1\ -\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}} = 0 \qquad \dots(29)$$

And by resolving equations (27), (28) and (29) we get the RSS estimators symbolized by $\hat{\mu}^{RSS}$, $\hat{\rho}^{RSS}$ $and\ \hat{\varphi}^{RSS}$. Since these equations are non-linear, so, Newton – Raphson process can be used to get the resolve as follows such that $h = 0,1,2,\dots$

$$\begin{bmatrix}\hat{\mu}_{RSS}^{(h+1)}\\ \hat{\rho}_{RSS}^{(h+1)}\\ \hat{\varphi}_{RSS}^{(h+1)}\end{bmatrix} = \begin{bmatrix}\hat{\mu}_{RSS}^{(h)}\\ \hat{\rho}_{RSS}^{(h)}\\ \hat{\varphi}_{RSS}^{(h)}\end{bmatrix} - \mathcal{J}_{(h)}^{-1}\begin{bmatrix}\frac{\partial \ell^{RSS}}{\partial \mu}\\ \frac{\partial \ell^{RSS}}{\partial \rho}\\ \frac{\partial \ell^{RSS}}{\partial \varphi}\end{bmatrix}_{\substack{\mu=\hat{\mu}_{RSS}^{(h)}\\ \rho=\hat{\rho}_{RSS}^{(h)}\\ \varphi=\hat{\varphi}_{RSS}^{(h)}}}$$

where

$$\mathcal{J}_{(h)} = \begin{bmatrix}\frac{\partial^2 \ell^{RSS}}{\partial \mu^2} & \frac{\partial^2 \ell^{RSS}}{\partial \mu \partial \rho} & \frac{\partial^2 \ell^{RSS}}{\partial \mu \partial \varphi}\\ \frac{\partial^2 \ell^{RSS}}{\partial \rho \partial \mu} & \frac{\partial^2 \ell^{RSS}}{\partial \rho^2} & \frac{\partial^2 \ell^{RSS}}{\partial \rho \partial \varphi}\\ \frac{\partial^2 \ell^{RSS}}{\partial \varphi \partial \mu} & \frac{\partial^2 \ell^{RSS}}{\partial \varphi \partial \rho} & \frac{\partial^2 \ell^{RSS}}{\partial \varphi^2}\end{bmatrix}_{\substack{\mu=\hat{\mu}_{RSS}^{(h)}\\ \rho=\hat{\rho}_{RSS}^{(h)}\\ \varphi=\hat{\varphi}_{RSS}^{(h)}}}$$

Equations (27), (28) and (29) represent the first partial derivatives and the second partial derivatives are known,

$$\frac{\partial^2 \ell^{RSS}}{\partial \mu^2} = -(1+\varphi)^2\sum_{q=1}^{w}\frac{1}{\left(\mu(1+\varphi)+\rho(1+\varphi)x_{(q)}\right)^2} - (1+\varphi)^2\sum_{q=1}^{w}\frac{(j-1)\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}\left(x_{(q)}^{2}\right)}{1\ -\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}}$$
$$- (1+\varphi)^2\sum_{q=1}^{w}\frac{(q-1)\ e^{-2\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}\left(x_{(q)}^{2}\right)}{[1\ -\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}]^2} \qquad \dots(30)$$

$$\frac{\partial^2 \ell^{RSS}}{\partial \rho^2} = -(1+\varphi)^2\sum_{q=1}^{w}\frac{x_{(q)}^{2}}{\left[\mu(1+\varphi)+\rho\,(1+\varphi)x_{(q)}\right]^2} - (1+\varphi)^2\sum_{q=1}^{w}\frac{(q-1)\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}\left(\frac{x_{(q)}^{4}}{4}\right)}{1-e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}}$$
$$-\ (1+\ \varphi)^2\sum_{q=1}^{w}\frac{(q-1)\ e^{-2\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}\left(\frac{x_{(q)}^{4}}{4}\right)}{[1-\ e^{-\left(\mu(1+\varphi)x_{(q)}+\frac{\rho(1+\varphi)}{2}x_{(q)}^{2}\right)}]^2} \qquad \dots(31)$$

$$\frac{\partial^2 \ell^{RSS}}{\partial \varphi^2} = -\sum_{q=1}^{w} \frac{\left(\mu + \rho x_q\right)^2}{\left[\mu(1+\varphi) + \rho(1+\varphi)x_q\right]^2} - \sum_{q=1}^{w} \frac{(q-1)\left(\mu x_{(q)} + \frac{\rho}{2} x_{(q)}^2\right)^2 e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}}{1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}} - \sum_{q=1}^{w} \frac{(q-1)(\mu x_{(q)} + \frac{\rho}{2} x_{(q)}^2)^2 e^{-2\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}}{[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}]^2} \quad \dots (32)$$

$$\frac{\partial^2 \ell^{RSS}}{\partial \mu \partial \rho} = \frac{\partial^2 \ell^{RSS}}{\partial \rho \partial \mu} = -(1+\varphi)^2 \sum_{q=1}^{w} \frac{x_{(q)}}{\left[\mu(1+\varphi) + \rho(1+\varphi)x_{(q)}\right]^2} - (1+\varphi)^2 \sum_{q=1}^{w} \frac{(q-1)\, e^{-\left(\mu(1+\theta)x_{(q)} + \frac{\rho(1+\theta)}{2} x_{(q)}^2\right)} \left(\frac{x_{(q)}^3}{2}\right)}{1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}} - (1+\varphi)^2 \sum_{q=1}^{w} \frac{(q-1)\, e^{-2\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)} \left(\frac{x_{(q)}^3}{2}\right)}{[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}]^2} \quad \dots (33)$$

$$\frac{\partial^2 \ell^{RSS}}{\partial \mu \partial \varphi} = \frac{\partial^2 \ell^{RSS}}{\partial \varphi \partial \mu} = \sum_{q=1}^{w} \frac{(q-1)\, x_{(q)}\, e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}}{1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}} \left[1 - \left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)\right] - \sum_{q=1}^{w} (w-q+1)x_{(q)} - (1+\varphi) \sum_{q=1}^{w} \frac{(q-1)\left(\mu x_{(q)} + \frac{\rho}{2} x_{(q)}^2\right) x_{(q)}\, e^{-2\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}}{[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}]^2} \quad \dots (34)$$

$$\frac{\partial^2 \ell^{RSS}}{\partial \varphi \partial \rho} = \frac{\partial^2 \ell^{RSS}}{\partial \rho \partial \varphi} = \sum_{q=1}^{w} \frac{(q-1)\left(\frac{x_{(q)}^2}{2}\right) e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}}{1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}} \left[1 - \left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)\right] - (1+\varphi) \sum_{q=1}^{w} \frac{(q-1)\left(\mu\, x_{(q)} + \frac{\rho}{2} x_{(q)}^2\right)\left(\frac{x_{(q)}^2}{2}\right) e^{-2\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}}{[1 - e^{-\left(\mu(1+\varphi)x_{(q)} + \frac{\rho(1+\varphi)}{2} x_{(q)}^2\right)}]^2} - \sum_{q=1}^{w} (w-q+1)\left(\frac{x_{(q)}^2}{2}\right) \quad \dots (35)$$

Now, Now, based on an invariant property of the RSS estimator, the reliability function at mission time (t) of the $MWER$ distribution can be obtained by replacing μ, ρ and φ in equation (3), by their RSS estimators as follows:

$$\hat{\mathcal{R}}^{RSS}(t; \mu, \rho, \varphi) = \exp\left(-(\hat{\mu}^{RSS}(1+\hat{\varphi}^{RSS})\, t + \frac{\hat{\rho}^{RSS}(1+\hat{\varphi}^{RSS})}{2}\, t^2)\right) \quad \dots (36)$$

2.3. Maximum entropy estimation [4][8]

The term of entropy was discussed by Ludwig Boltzmann in statistical mechanics. One of the important procedures is the Boltzmann-Gibbs - Shannon entropy

$$Sh = -\int_0^\infty \ln f(x)\, f(x)\, dx \quad \dots (37)$$

where $f(x)$ is the probability density function of X and X is continuous non-negative random variable. The information used in the principle of maximum entropy ME is defined as a set of constraints C_j established expectations of functions $T_i(x)$ as follows:

$$E[T_i(x)] = \int_0^\infty T_i(x)\, f(x)\, dx = C_i \quad ; i = 1,2,3, \dots, n \quad \dots (38)$$

The ME distributions appear by maximizing the particular form of entropy, subject to Equation (38) and the additional constraint:

$$\int_0^\infty f(x)dx = 1 \qquad \dots(39)$$

The maximization is reached by the Lagrange multipliers method, such that the general solution form of the ME distribution from maximizing equation (37) is given by:

$$f(x) = \exp\left[-\rho_0 - \sum_{i=1}^{w} \rho_i \; T_i(x)\right] \qquad \dots(40)$$

Such that the restraint in equation (38) linked to the Lagrange multipliers ρ_i and the added restraint in equation (39) linked to the Lagrange multiplier ρ_0. To estimate the unknown parameters and the reliability function of the $MWER$ distribution via maximum entropy ME method there exist four stages:

Stage 1: Explanation of appropriate restraints. Take the normal logarithm from equation (1), we become:

$$ln\, f(x;\mu,\rho,\varphi) = ln(\mu(1+\varphi)+\rho(1+\varphi)x) - \left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}x^2\right) \qquad \dots(41)$$

Now equation (41) multiplying it by $[-f(x;\mu,\rho,\varphi)]$ and integrating on $0 \le x < \infty$ we become:

$$Sh = -\int_0^\infty \left[ln(\mu(1+\varphi)+\rho(1+\varphi)x) - \left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}x^2\right)\right] f(x;\mu,\rho,\varphi)\, dx \qquad \dots(42)$$

The $MWER$ density function in equation (42) must be achieve

$$\int_0^\infty f(x;\mu,\rho,\varphi)\, dx = 1 = C_1 \qquad \dots(43)$$

$$\int_0^\infty ln(\mu(1+\varphi)+\rho(1+\varphi)x)\, f(x;\mu,\rho,\varphi)dx = E[\,ln(\mu(1+\varphi)+\rho(1+\varphi)x)] = C_2 \qquad \dots(44)$$

$$-\int_0^\infty \left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}x^2\right) f(x;\mu,\rho,\varphi)\, dx = -E[\left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}x^2\right)] = C_3 \qquad \dots(45)$$

where C_1, C_2 and C_3 are restraints.

Stage 2: Building of the Lagrange multipliers
By

$$f(x) = \exp[-\rho_0 - \textstyle\sum_{i=1}^{w} \rho_i \; T_i(x)] \text{ and when } w = 2$$

$$f(x;\mu,\rho,\varphi) = \exp\left[-\rho_0 - \rho_1 ln(\mu(1+\varphi)+\rho(1+\varphi)x) + \rho_2\left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}x^2\right)\right] \qquad \dots(46)$$

Substituting equation (46) in equation (43) results:

$$\int_0^\infty exp[-\rho_0 - \rho_1 ln(\mu(1+\varphi)+\rho(1+\varphi)x) + \rho_2\left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}x^2\right)]\, dx = 1 \qquad \dots(47)$$

$$exp(\rho_0) = \int_0^\infty (\mu(1+\varphi)+\rho(1+\varphi)x)^{-\rho_1}\, exp[\;\rho_2\left(\mu(1+\varphi)\,x + \frac{\rho(1+\varphi)}{2}x^2\right)]\, dx$$

$$exp(\rho_0) = \int_0^\infty \frac{exp\;[\;\rho_2\left(\mu(1+\varphi)x + \frac{\rho(1+\varphi)}{2}\,x^2\right)]}{(\mu(1+\varphi)+\rho(1+\varphi)x)^{\rho_1}}\, dx \qquad \dots(48)$$

Let

$$u = \mu(1+\varphi) + \rho(1+\varphi)x \quad , \ x = \frac{u-\mu(1+\varphi)}{\rho(1+\varphi)} \quad , \ dx = \frac{1}{\rho(1+\varphi)}\, du$$

$$exp(\rho_0) = \frac{1}{\rho(1+\varphi)} \int_{\mu(1+\varphi)}^{\infty} \frac{exp[\ \rho_2 \left(\mu(1+\varphi)\left(\frac{u-\mu(1+\varphi)}{\rho(1+\varphi)}\right) + \frac{\rho(1+\varphi)}{2}\left(\frac{(u-\mu(1+\varphi))^2}{\rho^2(1+\varphi)^2}\right)\right)]}{u^{\rho_1}}\, du$$

$$exp(\rho_0) = \frac{1}{\rho(1+\varphi)} \int_{\mu(1+\varphi)}^{\infty} \frac{e^{-[-\rho_2\left(\frac{u^2-\mu^2(1+\varphi)^2}{2\rho(1+\varphi)}\right)]}}{u^{\rho_1}}\, du$$

$$exp(\rho_0) = \frac{e^{-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}}}{\rho(1+\varphi)} \int_{\mu(1+\varphi)}^{\infty} \frac{e^{-(-\frac{\rho_2\,u^2}{2\,\rho(1+\varphi)})}}{u^{\rho_1}}\, du \qquad \dots(49)$$

Now, we will solve

$$\int_{\mu(1+\varphi)}^{\infty} \frac{e^{-(-\frac{\rho_2\,u^2}{2\rho(1+\varphi)})}}{u^{\rho_1}}\, du$$

since

$$\int_{v}^{\infty} \frac{e^{-B\,x^a}}{x^b}\, dx = \frac{\Gamma(z,\ Bv^a)}{a\,B^z} \qquad ; v > 0$$

where

$$z = \frac{1-\ b}{a}$$

$$\text{If } \ b = \rho_1 \ \ , a = 2 \ \ , B = -\ \frac{\rho_2}{2\,\rho(1+\varphi)} \ \ , v = \mu(1+\varphi)$$

$$z = \frac{1-\ \rho_1}{2}$$

The result it:

$$exp(\rho_0) = \frac{e^{-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}}}{\rho(1+\varphi)} \int_{\mu(1+\varphi)}^{\infty} \frac{e^{-\left(-\frac{\rho_2\,u^2}{2\rho(1+\varphi)}\right)}}{u^{\rho_1}}\, du$$

$$= \frac{e^{-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}}}{\rho(1+\varphi)} \frac{\Gamma[\frac{1-\rho_1}{2}\ \ ,\ \ -\ \frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}]}{2\ (-\ \frac{\rho_2}{2\,\rho(1+\varphi)})^{\frac{1-\rho_1}{2}}} \qquad \dots(50)$$

Since

$$\Gamma(a\,,x) = (a-1)!\,e^{-x} \sum_{m=0}^{a-1} \frac{x^m}{m!}$$

$$\Gamma\left[\frac{1-\rho_1}{2}\,, \quad -\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right] = \left(\frac{1-\rho_1}{2}-1\right)!\,e^{\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}} \sum_{m=0}^{\frac{1-\rho_1}{2}-1} \frac{\left[-\ \frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^m}{m!} \qquad \dots(51)$$

Substituting equation (51) in equation (50) yields:

$$exp(\rho_0) = \frac{\left(\frac{1-\rho_1}{2}-1\right)!\ \sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}]^m}{m!}}{2\,\rho(1+\varphi)\ (-\ \frac{\rho_2}{2\,\rho(1+\varphi)})^{\frac{1-\rho_1}{2}}} \qquad \dots(52)$$

Stage 3: Derivation of the entropy function of the distribution.
Substituting equation (52) in equation (46) yields:

$$f(x)=\frac{2\,\rho(1+\varphi)\,(-\frac{\rho_2}{2\,\rho(1+\varphi)})^{\frac{1-\rho_1}{2}}\,(\mu(1+\varphi)+\rho(1+\varphi)x)^{-\rho_1}\,e^{[\rho_2\left(\mu(1+\varphi)x+\frac{\rho(1+\varphi)}{2}x^2\right)]}}{\left(\frac{1-\rho_1}{2}-1\right)!\,\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}]^m}{m!}}\quad \dots(53)$$

Taking the natural logarithm, thus we get:

$$\ln f(x)=\ln(2\,\rho)+ln(1+\varphi)+\left(\frac{1-\rho_1}{2}\right)ln(-\rho_2)-\left(\frac{1-\rho_1}{2}\right)ln(2\,\rho)-\left(\frac{1-\rho_1}{2}\right)ln(1+\varphi)-\rho_1\,ln(\mu(1+\varphi)+\rho(1+\varphi)x)+\left[\rho_2\left(\mu(1+\varphi)x+\frac{\rho(1+\varphi)}{2}x^2\right)\right]-ln\left(\frac{1-\rho_1}{2}-1\right)!-ln\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}]^m}{m!}\quad \dots(54)$$

Depending on equation

$$Sh=-\int_0^{\infty} ln\,f(x)\,f(x)\,dx$$

$$Sh=-ln(2\,\rho)-ln(1+\varphi)-\left(\frac{1-\rho_1}{2}\right)ln(-\rho_2)+\left(\frac{1-\rho_1}{2}\right)ln(2\,\rho)+\left(\frac{1-\rho_1}{2}\right)ln(1+\varphi)+\rho_1 E\,[ln(\mu(1+\varphi)+\rho(1+\varphi)x)]-\rho_2\mu(1+\varphi)\,E\,(x)-\frac{\rho_2\,\rho\,(1+\varphi)}{2}E(x^2)+ln\left(\frac{1-\rho_1}{2}-1\right)!+ln\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}\right]^m}{m!}\quad \dots(55)$$

Stage 4: Derivation of the relative among the Lagrange multiplier and the restraints.
To maximize equation (55) We derive it partially with respect to μ,ρ and φ and then we set it equal to zero to produce,

$$\frac{\partial Sh^{ME}}{\partial\mu}=\rho_1 E\left(\frac{1+\varphi}{\mu(1+\varphi)+\rho(1+\varphi)x}\right)-\rho_2(1+\varphi)\,E(x)+\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[-\frac{\rho_2\,\mu\,(1+\varphi)}{\rho}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^m}{m!}}=0\quad \dots(56)$$

$$\frac{\partial Sh^{ME}}{\partial\rho}=\frac{1-\rho_1}{2\,\rho}-\frac{1}{\rho}+\rho_1 E\left(\frac{(1+\varphi)x}{\mu(1+\varphi)+\rho(1+\varphi)x}\right)-\frac{\rho_2\,(1+\varphi)}{2}E(x^2)+\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho^2}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^m}{m!}}=0\quad \dots(57)$$

$$\frac{\partial Sh^{ME}}{\partial\varphi}=\frac{1-\rho_1}{2(1+\varphi)}-\frac{1}{1+\varphi}+\rho_1 E\left(\frac{\mu+\rho x}{\mu(1+\varphi)+\rho(1+\varphi)x}\right)-\rho_2\mu\,E(x)-\frac{\rho_2\,\rho}{2}E(x^2)+\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[-\frac{\rho_2\,\mu^2}{2\,\rho}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}\right]^m}{m!}}=0\quad \dots(58)$$

Since

$$E(X)=\sum_{all\ x}x.p(X=x)$$

Since

$$p(X=x)=1$$

$$E\left(\frac{1+\varphi}{\mu(1+\varphi)+\rho(1+\varphi)x}\right) \cong \sum_{q=1}^{w} \frac{1+\varphi}{\mu(1+\varphi)+\rho(1+\varphi)x_q} \quad \dots(59)$$

$$E\left(\frac{(1+\varphi)x}{\mu(1+\varphi)+\rho(1+\varphi)x}\right) \cong \sum_{q=1}^{w} \frac{(1+\varphi)x_q}{\mu(1+\varphi)+\rho(1+\varphi)x_q} \quad \dots(60)$$

$$E\left(\frac{\mu+\rho x}{\mu(1+\varphi)+\rho(1+\varphi)x}\right) \cong \sum_{q=1}^{w} \frac{\mu+\rho x_q}{\mu(1+\varphi)+\rho(1+\varphi)x_q} \quad \dots(61)$$

$$E(x) = \sum_{q=1}^{w} x_q \quad \dots(62)$$

$$E(x^2) \cong \sum_{q=1}^{w} x_q^2 \quad \dots(63)$$

Substituting equations (59), (60), (61), (62) and (63) in equations (56), (57) and (58) we get:

$$\frac{\partial Sh^{ME}}{\partial \mu} = \rho_1(1+\varphi)\sum_{q=1}^{w}\frac{1}{\mu(1+\varphi)+\rho(1+\varphi)x_q} - \rho_2(1+\varphi)\sum_{q=1}^{w} x_q + \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[-\frac{\rho_2\,\mu\,(1+\varphi)}{\rho}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m}}{m!}} = 0 \quad \dots(64)$$

$$\frac{\partial Sh^{ME}}{\partial \rho} = \frac{1-\rho_1}{2\,\rho} - \frac{1}{\rho} + \rho_1(1+\varphi)\sum_{q=1}^{w}\frac{x_q}{\mu(1+\varphi)+\rho(1+\varphi)x_q} - \frac{\rho_2\,(1+\varphi)}{2}\sum_{q=1}^{w} x_q^2 + \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho^2}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m}}{m!}} = 0 \quad \dots(65)$$

$$\frac{\partial Sh^{ME}}{\partial \varphi} = \frac{1-\rho_1}{2(1+\varphi)} - \frac{1}{1+\varphi} + \rho_1\sum_{q=1}^{w}\frac{\mu+\rho x_q}{\mu(1+\varphi)+\rho(1+\varphi)x_q} - \rho_2\mu\sum_{q=1}^{w} x_q - \frac{\rho_2\,\rho}{2}\sum_{q=1}^{w} x_q^2 + \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[-\frac{\rho_2\,\mu^2}{2\,\rho}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m}}{m!}} = 0 \quad \dots(66)$$

And by solving equations (64), (65) and (66) we get the ME estimators denoted by $\hat{\mu}^{ME}$, $\hat{\rho}^{ME}$ $and\, \hat{\varphi}^{ME}$. Since these equations are non-linear, so, Newton – Raphson process can be used to get the resolve as follows such that $h = 0,1,2,\dots$

$$\begin{bmatrix} \hat{\mu}_{ME}^{(h+1)} \\ \hat{\rho}_{ME}^{(h+1)} \\ \hat{\varphi}_{ME}^{(h+1)} \end{bmatrix} = \begin{bmatrix} \hat{\mu}_{ME}^{(h)} \\ \hat{\rho}_{ME}^{(h)} \\ \hat{\varphi}_{ME}^{(h)} \end{bmatrix} - \mathcal{J}_{(h)}^{-1} \begin{bmatrix} \frac{\partial Sh^{ME}}{\partial \mu} \\ \frac{\partial Sh^{ME}}{\partial \rho} \\ \frac{\partial Sh^{ME}}{\partial \varphi} \end{bmatrix}_{\substack{\mu=\hat{\mu}_{ME}^{(h)} \\ \rho=\hat{\rho}_{ME}^{(h)} \\ \varphi=\hat{\varphi}_{ME}^{(h)}}}$$

where

$$\mathcal{J}_{(h)} = \begin{bmatrix} \frac{\partial^2 Sh^{ME}}{\partial\mu^2} & \frac{\partial^2 Sh^{ME}}{\partial\mu\partial\rho} & \frac{\partial^2 Sh^{ME}}{\partial\mu\partial\varphi} \\ \frac{\partial^2 Sh^{ME}}{\partial\rho\partial\mu} & \frac{\partial^2 Sh^{ME}}{\partial\rho^2} & \frac{\partial^2 Sh^{ME}}{\partial\rho\partial\varphi} \\ \frac{\partial^2 Sh^{ME}}{\partial\varphi\partial\mu} & \frac{\partial^2 Sh^{ME}}{\partial\varphi\partial\rho} & \frac{\partial^2 Sh^{ME}}{\partial\varphi^2} \end{bmatrix}_{\substack{\mu=\hat{\mu}_{ME}^{(h)} \\ \rho=\hat{\rho}_{ME}^{(h)} \\ \varphi=\hat{\varphi}_{ME}^{(h)}}}$$

Such that

$$\frac{\partial^2 Sh^{ME}}{\partial\mu^2} = -\rho_1(1+\varphi)^2 \sum_{q=1}^{w} \frac{1}{\left(\mu(1+\varphi)+\rho(1+\varphi)x_q\right)^2} - \frac{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[-\frac{\rho_2\,\mu\,(1+\varphi)}{\rho}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}\right]^2}{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m}}{m!}\right]^2} + \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m(m-1)\left[\frac{\rho_2^2\mu^2(1+\varphi)^2}{\rho^2}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-2} - m\left[\frac{\rho_2\,(1+\varphi)}{\rho}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m}}{m!}} \quad \ldots (67)$$

$$\frac{\partial^2 Sh^{ME}}{\partial\rho^2} = \frac{1}{\rho^2} - \frac{1-\rho_1}{2\,\rho^2} - \rho_1(1+\varphi)^2 \sum_{q=1}^{w} \frac{x_i^2}{\left(\mu(1+\varphi)+\rho(1+\varphi)x_q\right)^2} + \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m(m-1)\left[\frac{\rho_2^2\mu^4(1+\varphi)^2}{4\,\rho^4}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-2} - m\left[\frac{\rho_2\,\mu^2(1+\varphi)}{\rho^3}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m}}{m!}} - \frac{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho^2}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}\right]^2}{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}\right]^2} \quad \ldots (68)$$

$$\frac{\partial^2 Sh^{ME}}{\partial \varphi^2}=\frac{1}{(1+\varphi)^2}-\frac{1-\rho_1}{2(1+\varphi)^2}-\rho_1\sum_{q=1}^{w}\frac{\left(\mu+\rho x_q\right)^2}{\left[\mu(1+\varphi)+\rho(1+\varphi)x_q\right]^2}+\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m(m-1)\left[\frac{\rho_2^2\mu^4}{4\rho^2}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-2}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}}-\frac{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[-\frac{\rho_2\mu^2}{2\rho}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}\right]^2}{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}\right]^2}\quad \ldots(69)$$

$$\frac{\partial^2 Sh^{ME}}{\partial \mu\partial\rho}=\frac{\partial^2 Sh^{ME}}{\partial \rho\partial\mu}=-\rho_1(1+\varphi)^2\sum_{q=1}^{w}\frac{x_q}{\left[\mu(1+\varphi)+\rho(1+\varphi)x_q\right]^2}+\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m(m-1)\left[-\frac{\rho_2^2\mu^3(1+\varphi)^2}{2\rho^3}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-2}+m\left[\frac{\rho_2\mu(1+\varphi)}{\rho^2}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}}-\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[-\frac{\rho_2\mu(1+\varphi)}{\rho}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[\frac{\rho_2\mu^2(1+\varphi)}{2\rho^2}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}}{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}\right]^2}\quad \ldots(70)$$

$$\frac{\partial^2 Sh^{ME}}{\partial \mu\partial\varphi}=\frac{\partial^2 Sh^{ME}}{\partial \varphi\partial\mu}=\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m(m-1)\left[\frac{\rho_2^2\mu^3(1+\varphi)}{2\rho^2}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-2}-m\left[\frac{\rho_2\mu}{\rho}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}}-\frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[-\frac{\rho_2\mu^2}{2\rho}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{m\left[-\frac{\rho_2\mu(1+\varphi)}{\rho}\right]\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m-1}}{m!}}{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}}\frac{\left[-\frac{\rho_2\mu^2(1+\varphi)}{2\rho}\right]^{m}}{m!}\right]^2}-\rho_2\sum_{q=1}^{w}x_q\quad \ldots(71)$$

$$\frac{\partial^2 Sh^{ME}}{\partial\varphi\partial\rho} = \frac{\partial^2 Sh^{ME}}{\partial\rho\partial\varphi}$$

$$= \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m(m-1)\left[-\frac{\rho_2^2\,\mu^4(1+\varphi)}{4\,\rho^3}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-2} + m\left[\frac{\rho_2\,\mu^2}{2\,\rho^2}\right]\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^m}{m!}}$$

$$- \frac{\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[\frac{\rho_2\,\mu^2\,(1+\varphi)}{2\,\rho^2}\right]\left[-\frac{\rho_2\,\mu^2\,(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!} \sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{m\left[-\frac{\rho_2\,\mu^2}{2\,\rho}\right]\left[-\frac{\rho_2\,\mu^2\,(1+\varphi)}{2\,\rho}\right]^{m-1}}{m!}}{\left[\sum_{m=0}^{\frac{-\rho_1-1}{2}} \frac{\left[-\frac{\rho_2\,\mu^2(1+\varphi)}{2\,\rho}\right]^m}{m!}\right]^2} - \frac{\rho_2}{2}\sum_{q=1}^{w} x_q^2 \qquad \ldots(72)$$

Now, Now, based on an invariant property of the ME estimator, the reliability function at mission time (t) of the $MWER$ distribution can be obtained by replacing μ, ρ and φ in equation (3), by their ME estimators as follows:

$$\hat{\mathcal{R}}^{ME}(t;\mu,\rho,\varphi) = \exp\left(-(\hat{\mu}^{ME}(1+\hat{\varphi}^{ME})t + \frac{\hat{\rho}^{ME}(1+\hat{\varphi}^{ME})}{2}\,t^2)\right) \qquad \ldots(73)$$

3. Recommendations

For future studies, one can put the following recommendations:

i. Using same estimation methods and applying simulation studies or applications with different real-life data sets to evaluate the performance of the anonymous parameters and reliability function of $MWER$ distribution.
ii. Using complete (censored) data to estimate the anonymous parameters and reliability function of $MWER$ distribution taking into account changing: estimation methods, approximation technique.
iii. From a different continuous probability distribution used same estimation methods.

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