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Simulation Study for Estimating the Parameters and Reliability Function of Weighted Exponential Distribution with Fuzzy Data

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Abstract

This paper investigates the estimation of the two unknown parameters and the reliability function of the weighted exponential distribution. It explores Bayesian and non-Bayesian (maximum likelihood) estimation methods when the information available is in the form of fuzzy data. The Newton-Raphson algorithm is used to obtain the maximum likelihood estimates. In Bayes estimation, the symmetric squared error loss function is used. This loss function links equal importance to the losses due to overestimating and underestimating equal magnitude. Lindley approximation procedure in Bayesian estimation theory is used to evaluate the ratio of integrals. A comparative analysis using simulation is carried out to evaluate the performance of the obtained parameters estimators using mean squared error criteria and the performance of the obtained reliability estimators using integrated mean squared error criteria. The simulation results demonstrate that, for different sample sizes, the performance of Bayes estimates surpasses the maximum likelihood, and that all estimators perform consistently.

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1. Introduction

The weighted exponential (WE) distribution was introduced by Kupta and Kunda (2009) after adding a shape parameter to the exponential distribution based on the idea of Azzalini [1][2]. Many authors have been interested in this distribution. For example, Zamani and Ismail (2010) mixed the WE distribution with the Poisson distribution to form a new distribution called the Poisson-WE distribution [3]. Makhdoom and Nasiri (2011) experimentally compared the performance of the maximum likelihood and moment parameters estimators with the existence of outliers [4]. Roy and Adnan (2012) presented a class of circular distribution which is titled wrapped WE distribution [5]. Abed Al-Kadim and Hantoosh (2013) derived the double-weighted distribution in addition to the double WE distribution and studied several statistical properties [6]. Khorram and Farahani (2014) discussed the maximum likelihood (ML) and Bayes parameters estimators with the censored data [7]. Badmus et al. (2015) introduced a new distribution called Beta WE distribution [8]. Oguntunde et al. (2016) derived the main mathematical properties of a new version of WE [9]. Al-Noor and Hussein (2018) discussed approximate Bayes estimations of WE parameters with fuzzy data [10]. Al-Noor and Hussein (2018) proposed a new family called the WE-G family of probability distributions also, two sub-models, WE-uniform and WE-Kumaraswamy, were presented [11]. Abd El-Bar and Ragab (2019) used the WE distribution to introduce a new distribution called the WE-Gompertz [12]. Al-Noor and Hussein (2020) introduced Kumaraswamy WE distribution as a new version of WE with four parameters [13]. Mallick et al. (2021) discussed the

ANJS, Vol. 27(2), June, 2024, pp. 145-154

bounded WE Distribution and its applications [14]. Tiana and Yang (2022) studied a change-point problem of WE distribution, the techniques established on the likelihood ratio test, modified information criterion, and Schwarz information criterion [15]. Niu et al. (2023) considered the differentiating between the WE distribution and two positively skewed lifetime distributions, generalized exponential and Weibull [16]. In this paper, a simulation study is conducted to examine the behavior of maximum likelihood and Bayes parameters and reliability estimators of the WE distribution under fuzzy data.

2. Conditional Density of a Random Variable Given Fuzzy Event

Lotfi Zadeh (1965) introduced the concept of fuzzy set and fuzzy logic [32][35], but it has not spread and it was used in a wide range until (1990). The theory of the fuzzy set is a mathematical way to represent the uncertain nature of the information. This theory is now used to problems widely in functional analysis, in engineering, business, medical and related health sciences, and the natural sciences [33,36]. The fuzzy set theory is a generalization of the classical set theory and that means the classical set is a special case of fuzzy set. A fuzzy set $\tilde{B}$ in χ if χ is a collection of objects with universal element x is a set of ordered pairs $\tilde{B} = \{(x, \mathrm{M}_{\tilde{B}}(x)) | x \in \chi\}$, where $\mathrm{M}_{\tilde{B}}(x)$ is called the membership function of x in $\tilde{B}$ which maps χ to the membership space M , $\mathrm{M}_{\tilde{B}}(x) : \chi \to [0,1]$. (When M contains only two points 0 and 1, $\tilde{B}$ is non-fuzzy (crisp) and $\mathrm{M}_{\tilde{B}}(x)$ is identical to the characteristic function of a non-fuzzy set) [34].

Let $(\mathfrak{R}^n, B, P)$ be a probability space. The probability of a fuzzy event $\tilde{B}$ in $\mathfrak{R}^n$ according to Zadeh's definition is [17] [18]:

$$P(\tilde{B}) = \int \mathrm{M}_{\tilde{B}}(x)\, dP \quad ; \ \forall\, x \in \mathfrak{R}^n \quad \ldots (1)$$

Equation (1) means a fuzzy event $\tilde{B}$ in $\mathfrak{R}^n$ is defined as the expectation of $\mathrm{M}_{\tilde{B}}$ with respect to P.

If $P(\tilde{B}\tilde{F}) = P(\tilde{B})\, P(\tilde{F})$ where $\tilde{B}\tilde{F}$ is the fuzzy subset of $\mathfrak{R}^n$ with membership function, $\mu_{\tilde{B}\tilde{F}}(x) = \mu_{\tilde{B}}(x)\, \mu_{\tilde{F}}(x)$; $\forall x \in \mathfrak{R}^n$ in this case two fuzzy events $\tilde{B}$ and $\tilde{F}$ are said to be independent. The conditional probability of $\tilde{B}$ given $\tilde{F}$ is defined by,

$$P(\tilde{B}|\tilde{F}) = \frac{P(\tilde{B}\,\tilde{F})}{P(\tilde{F})};\ P(\tilde{F}) > 0$$

Now, let P is the probability distribution of a continuous random variable X with probability density function (pdf) $g(x)$. The conditional probability of a crisp subset D given a fuzzy subset $\tilde{B}$ can be defined as,

$$P(D|\tilde{B}) = \frac{\int \mathrm{M}_D(x)\, \mathrm{M}_{\tilde{B}}(x)\, g(x)\, dx}{\int \mathrm{M}_{\tilde{B}}(u)\, g(u) du}$$

$$= \int_D \frac{\mathrm{M}_{\tilde{B}}(x)\, g(x)}{\int \mathrm{M}_{\tilde{B}}(u)\, g(u) du} dx$$

Therefore, the conditional density of X given $\tilde{B}$ can be defined as:

$$g(x|\tilde{B}) = \frac{\mu_{\tilde{B}}(x) g(x)}{\int \mu_{\tilde{B}}(u)\, g(u) du}.$$

3. Weighted Exponential Distribution with some properties and some special cases

The pdf of WE distribution for $x > 0$; $\delta, \xi > 0$ is given by [1]:

$$g(x; \delta, \xi) = \begin{cases} \frac{\delta+1}{\delta}\, \xi\, e^{-\xi x}\left(1 - e^{-\delta \xi x}\right) \\ 0 \qquad ; otherwise \end{cases} \quad \ldots (2)$$

where δ denote the shape parameter and ξ denote the scale parameter.

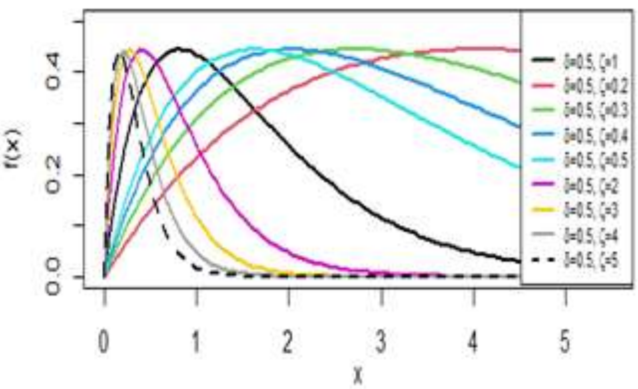

Figure 1. graph of the pdf of WE distribution for $\delta = 0.5$ and different values of ξ.

From equation (2), the random variable X has a WE distribution if and only if $X = \mathcal{Z} + \mathcal{Y}$, where $\mathcal{Z}$ and $\mathcal{Y}$ are two independent random variables distributed as exponential distribution with parameters (ξ) and (ξ(δ+1)) respectively. The cumulative distribution function of WE distribution is knowledge of the following formula [11]:

$$G(x; \delta, \xi) = \frac{1}{\delta}\left[(\delta + 1)\left(1 - e^{-\xi x}\right) + e^{-\xi(\delta+1)x} - 1\right]$$

$$G(x;\delta,\xi) = 1 - \frac{1}{\delta}\, e^{-\xi x}\left(\delta + 1 - e^{-\delta \xi x}\right) \qquad \dots(3)$$

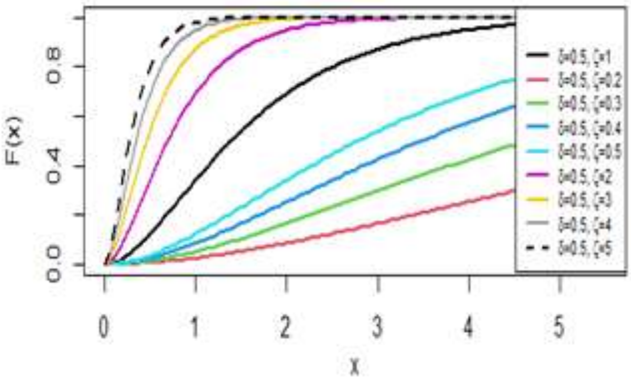

Figure 2. graph of the cdf of the WE distribution for $\delta = 0.5$ and different values of ξ .

The reliability function of the WE distribution is given by the formula:

$$\mathcal{R}(t;\delta,\xi) = 1 - G(t;\delta,\xi)$$
$$= \frac{1}{\delta}\, e^{-\xi t}\left(\delta + 1 - e^{-\delta \xi t}\right) \qquad \dots(4)$$

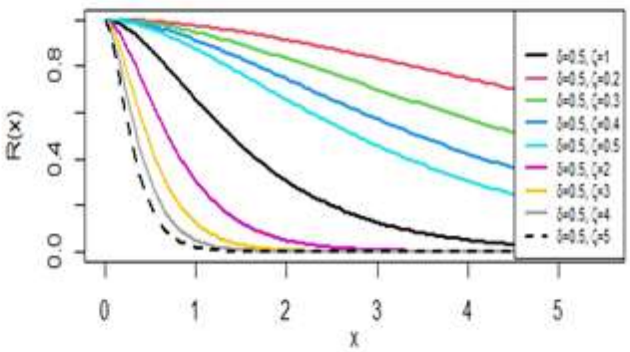

Figure 3. graph of the reliability function of the WE distribution for $\delta = 0.5$ and different values of ξ.

The r^{th} moment about the origin $(r = 1,2,3,\dots)$ is given by the formula [19]:

$$E(X^r) = \frac{(\delta+1)\,\Gamma(r+1)}{\delta\,\xi^r}\left(1 - \frac{1}{(\delta+1)^{r+1}}\right) \qquad \dots(5)$$

where,

$$E(X) = \frac{\delta+2}{\xi(\delta+1)} \qquad \dots(6)$$

$$E(X^2) = \frac{2(\delta^2+3\delta+3)}{\xi^2(\delta+1)^2} \qquad \dots(7)$$

Then,

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \frac{1}{\xi^2}\left[1 + \frac{1}{(\delta+1)^2}\right] \qquad \dots(8)$$

With certain values of the parameters, the pdf of WE distribution can be found as a special case of the following distributions:

- Jones' model [1] for $x > 0; a,b,c > 0$ when $a = \frac{1}{\delta}, b = 2, c = \delta\xi$;
$$f(x;a,b,c) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, c\, e^{-acx}(1 - e^{-cx})^{b-1}$$
- Weighted gamma-exponential distribution [20] for $x > 0;\ \delta,\beta,\xi > 0$ when $\beta = 1$
$$f(x;\delta,\beta,\xi) = \frac{\xi^\beta\, x^{\beta-1}\, e^{-\xi x}\left(1 - e^{-\delta\xi x}\right)}{\Gamma(\beta)[1 - (1+\delta)^{-\beta}]}$$
- Weighted generalized exponential-exponential distribution [20] for $x > 0;\ \theta,\delta,\xi > 0$ when $\theta = 1$;
$$f(x;\theta,\delta,\xi) = \frac{\theta\xi\, e^{-\xi x}\left(1 - e^{-\xi x}\right)^{\theta-1}\left(1 - e^{-\delta\xi x}\right)}{1 - \theta\, B(\delta+1,\ \theta)}$$
where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ is the beta function.
- Weighted Weibull distribution [21] for $x > 0;\ \delta,\beta,\xi > 0$ when $\beta = 1$;
$$f(x;\delta,\beta,\xi) = \frac{\delta+1}{\delta}\,\xi\beta\, x^{\beta-1}\, e^{-\xi x^\beta}\left(1 - e^{-\delta\xi x^\beta}\right)$$
- Exponentiated WE distribution [22] for $x > 0;\ \theta,\delta,\xi > 0$ when $\theta = 1$;
$$f(x;\theta,\delta,\xi) = \theta\left[\frac{\delta+1}{\delta}\xi e^{-\xi x}\left(1 - e^{-\xi\delta x}\right)\right]\left[\frac{\delta+1}{\delta}\left(1 - e^{-\xi x} - \frac{1}{\delta+1}\left(1 - e^{-\xi x(\delta+1)}\right)\right)\right]^{\theta-1}$$
- Weighted gamma distribution [23] for $x > 0;\ \delta,k,\xi > 0$ when $k = 1, \theta = \frac{1}{\xi}$;
$$f(x;\delta,k,\xi) = \frac{\frac{x^{k-1}e^{-\frac{x}{\xi}}}{\xi^k\Gamma(k)}\left(1 - e^{-\frac{\delta x}{\xi}}\sum_{i=0}^{k-1}\frac{\left(\frac{\delta x}{\xi}\right)^i}{i!}\right)}{1 - \sum_{i=0}^{k-1}\frac{\delta^i\Gamma(k+i)}{(\delta+1)^{k+i}\Gamma(k)i!}}$$
- Modified double WE distribution [24] for $x > 0;\ \delta > 0, \xi > \beta$ when $\beta = 0$;
$$f(x;\delta,\xi,\beta) = \frac{(\beta-\xi)(\beta-\xi-\xi\delta)}{\xi\delta}\, e^{(\beta-\xi)x}\left(1 - e^{-\delta\xi x}\right)$$
- Weighted gamma distribution [25] for $x > 0;\ \delta,\beta,\xi > 0$; $k^{-1} = 1 - \left(\frac{1}{1+\delta}\right)^\beta$ when $\beta = 1$
$$f(x;\delta,\beta,\xi) = k\,\frac{\left(1 - e^{-\delta\xi x}\right)\xi^\beta x^{\beta-1}e^{-\xi x}}{\Gamma(\beta)}$$

4. Non – Bayesian and Bayesian Estimation under Fuzzy Data

4.1 Maximum Likelihood Estimation

Let a random sample $\underline{x} = (x_1, x_2, x_3, \dots, x_n)$ drawn from a population having a pdf given by equation (2).

The formula of the complete-data likelihood function $\mathcal{L}(\delta, \xi | \underline{x})$ is,

$$\mathcal{L}(\delta, \xi | \underline{x}) = \prod_{i=1}^{n} g(x_i; \delta, \xi)$$

$$= \left(\frac{\delta + 1}{\delta}\right)^n \xi^n e^{-\xi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left(1 - e^{-\delta \xi x_i}\right) \quad \dots (9)$$

Now adopt that $\underline{x}$ is not exactly observed and only partial data exists in the form of a fuzzy subset $\underline{\tilde{x}}$ linked with the membership function $\mathrm{M}_{\tilde{x}_i}(x)$. So, depending on equation (1) the observed data likelihood function and its natural log-likelihood function can be found, respectively, as [31]:

$$\mathcal{L}(\delta, \xi | \underline{\tilde{x}}) = \left(\frac{\delta + 1}{\delta}\right)^n \xi^n \prod_{i=1}^{n} \int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx \quad \dots (10)$$

$$\tilde{\ell} = \ln \mathcal{L}(\delta, \xi | \underline{\tilde{x}}) = n \ln(\delta + 1) - n \ln \delta + n \ln \xi + \sum_{i=1}^{n} \ln \int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx \quad \dots (11)$$

The maximum likelihood estimators of $\hat{\delta}_{ML}$ and $\hat{\xi}_{ML}$ can be gotten as the results of the following first partial derivative for the equation (11);

$$\frac{\partial \tilde{\ell}}{\partial \delta} = \frac{-n}{\delta(\delta + 1)} + \sum_{i=1}^{n} \frac{\int \xi\, x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx} = 0 \quad \dots (12)$$

$$\frac{\partial \tilde{\ell}}{\partial \xi} = \frac{n}{\xi} + \sum_{i=1}^{n} \frac{\int x e^{-\xi x} \left((\delta + 1) e^{-\delta \xi x} - 1\right) \mathrm{M}_{\tilde{x}_i}(x) dx}{\int e^{-\xi x} (1 - e^{-\delta \xi x})\, \mathrm{M}_{\tilde{x}_i}(x) dx} = 0 \quad \dots (13)$$

Since the two equations (12) and (13) are non-linear equations and cannot be solved directly. So, Newton–Raphson's algorithm used to find the solution. The Newton-Raphson algorithm is done by applying the following iterative process,

$$\begin{bmatrix} \hat{\delta} \\ \hat{\xi} \end{bmatrix}^{(h+1)} = \begin{bmatrix} \hat{\delta} \\ \hat{\xi} \end{bmatrix}^{(h)} - \jmath_{(h)}^{-1} \begin{bmatrix} \frac{\partial \tilde{\ell}}{\partial \delta} \\ \frac{\partial \tilde{\ell}}{\partial \xi} \end{bmatrix}^{(h)} \quad ; h = 0,1,2, \dots$$

where,

$$\jmath_{(h)} = \begin{bmatrix} \frac{\partial^2 \tilde{\ell}}{\partial \delta^2} & \frac{\partial^2 \tilde{\ell}}{\partial \delta\, \partial \xi} \\ \frac{\partial^2 \tilde{\ell}}{\partial \xi\, \partial \delta} & \frac{\partial^2 \tilde{\ell}}{\partial \xi^2} \end{bmatrix}^{(h)}$$

So,

$$\frac{\partial^2 \tilde{\ell}}{\partial \delta^2} = \frac{n(2\delta + 1)}{\delta^2 (\delta + 1)^2} - \sum_{i=1}^{n} \left[\frac{\int \xi^2\, x^2\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx} + \left(\frac{\int \xi\, x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, d\,x} \right)^2 \right] \quad \dots (14)$$

$$\frac{\partial^2 \tilde{\ell}}{\partial \xi^2} = \frac{-n}{\xi^2} + \sum_{i=1}^{n} \left[\frac{\int x^2\, e^{-\xi x} \left(1 - (\delta + 1)^2\, e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx} - \left(\frac{\int x\, e^{-\xi x} \left((\delta + 1) e^{-\delta \xi x} - 1\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x} \left(1 - e^{-\delta \xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, d\,x} \right)^2 \right] \quad \dots (15)$$

$$\frac{\partial^2 \tilde{\ell}}{\partial \delta\, \partial \xi} = \frac{\partial^2 \tilde{\ell}}{\partial \xi\, \partial \delta} = \sum_{i=1}^{n} \left[\frac{\int x e^{-\xi x(\delta+1)} \left(1 - \xi x(\delta + 1)\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x} (1 - e^{-\delta \xi x}) \mathrm{M}_{\tilde{x}_i}(x)\, dx} - \frac{\int \xi x\, e^{-\xi x(\delta+1)}\, \mathcal{M}_{\tilde{x}_i}(x)\, dx \int x\, e^{-\xi x} \left((\delta + 1) e^{-\delta \xi x} - 1\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x} (1 - e^{-\delta \xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2} \right] \quad \dots (16)$$

With $h = 0$, the initial values say δ^0 and ξ^0, have been selected to be the moment estimators as [30],

$$\hat{\delta}_{\mathcal{MO}} = \frac{2\mathcal{M} - 3 + \sqrt{2\mathcal{M} - 3}}{2 - \mathcal{M}}$$

$$\hat{\xi}_{\mathcal{MO}} = \frac{\hat{\delta}_{\mathcal{MO}} + 2}{(\hat{\delta}_{\mathcal{MO}} + 1)\bar{x}}$$

where $\mathcal{M} = \frac{E(X^2)}{(E(X))^2}$; $\frac{3}{2} < \mathcal{M} < 2$

Now, the reliability function of the WE distribution can be found by changing δ and ξ in equation (4) by their MLE estimators as in the following equation:

$$\hat{\mathcal{R}}(t; \delta, \xi) = \frac{1}{\hat{\delta}}\, e^{-\hat{\xi} t} \left(\hat{\delta} + 1 - e^{-\hat{\delta} \hat{\xi} t}\right) \quad \dots (17)$$

4.2 Bayes Estimation

The unknown parameters in Bayesian estimation are supposed to be random variables. This means that we need prior distributions for those parameters, which depend on information about the parameters and preceding experience with similar studies. Next, we obtain the joint posterior density function by combining these prior distributions with the likelihood function. Let ρ be the parameter to be estimated by $\hat{\rho}$, then Bayes estimation of ρ is built in minimization of a Bayes loss function. The squared error loss function (SELF) for ρ is given by the formula [26],

$$L(\hat{\rho},\rho) = (\hat{\rho}-\rho)^2$$

such that $\hat{\rho}$ is an estimation of ρ, so Bayes estimator of ρ created on this loss function is found as:

$$\hat{\rho} = E(\rho|\underline{\tilde{x}}) \quad \dots(18)$$

That is, the expectation is taken with respect to the posterior distribution of ρ. Now, let the independent Gamma (a,b) and Gamma (c,d) correspondingly with pdfs as in equation (19) and (20) represent the prior distributions of δ and ξ;

$$\psi(\delta) = \frac{b^a}{\Gamma(a)}\delta^{a-1}e^{-b\delta}; \delta > 0, a, b > 0 \quad \dots(19)$$

$$\psi(\xi) = \frac{d^c}{\Gamma(c)}\xi^{c-1}e^{-d\xi}; \xi > 0, c, d > 0 \quad \dots(20)$$

The joint prior distribution say $\psi(\delta,\xi)$, of unknown parameters can be written as:

$$\psi(\delta,\xi) = \psi(\delta)\,\psi(\xi)$$
$$= \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\delta^{a-1}\,\xi^{c-1}\,e^{-(b\delta+d\xi)} \quad \dots(21)$$

Now, combining equations (10) and (21) to get the joint posterior density function of δ and ξ given fuzzy data,

$$\Pi(\delta,\xi|\underline{\tilde{x}}) = \frac{\mathcal{L}(\delta,\xi|\underline{\tilde{x}})\,\psi(\delta,\xi)}{\int_\xi \int_\delta \mathcal{L}(\delta,\xi|\underline{\tilde{x}})\,\psi(\delta,\xi)\,d\delta\,d\xi} \quad \dots(22)$$

where,

$$\mathcal{L}(\delta,\xi|\underline{\tilde{x}})\,\psi(\delta,\xi) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\delta^{a-n-1}\,\xi^{n+c-1}$$
$$(\delta+1)^n\,e^{-(b\delta+d\xi)}\prod_{i=1}^{n}\int e^{-\xi x}\left(1-e^{-\delta\xi x}\right)\mathrm{M}_{\tilde{x}_i}(x)\,dx$$

So, Bayes estimator of every function say $w(\delta,\xi)$, built on SELF can be written as:

$$\hat{w}(\delta,\xi) = E\left[w(\delta,\xi)\,|\underline{\tilde{x}}\right]$$
$$= \frac{\int_0^\infty \int_0^\infty w(\delta,\xi)\,\mathcal{L}(\delta,\xi|\underline{\tilde{x}})\,\psi(\delta,\xi)d\delta\,d\xi}{\int_0^\infty \int_0^\infty \mathcal{L}(\delta,\xi|\underline{\tilde{x}})\,\psi(\delta,\xi)d\delta\,d\xi} \quad \dots(23)$$

Note that, equation (23) cannot be solved, so we use Lindley's approximation. Lindley (1980) [27] established an approximate technique for evaluating the ratio of two integrals that cannot be solved. Let $I(\underline{\tilde{x}})$ definite as:

$$I(\underline{\tilde{x}}) = \frac{\int_0^\infty \int_0^\infty w(\delta,\xi)\,e^{\tilde{\ell}+\psi(\delta,\xi)}d\delta\,d\xi}{\int_0^\infty \int_0^\infty e^{\tilde{\ell}+\psi(\delta,\xi)}d\delta\,d\xi} \quad \dots(24)$$

where,
$w(\delta,\xi)$: the function of δ and ξ.
$\tilde{\ell}$ represents the natural log-likelihood function defined by equation (11), $\psi(\delta,\xi)$: the natural log-joint prior density function. Then, for a sufficiently large sample size, the ratio of two integrals $I(\underline{\tilde{x}})$ can be approximated as, (see [28]).

$$I(\underline{\tilde{x}}) = w(\hat{\delta},\hat{\xi}) + \frac{1}{2}\left[(\hat{w}_{\delta\delta}+2\hat{w}_\delta\hat{\psi}_\delta)\hat{\varphi}_{\delta\delta} + (\hat{w}_{\delta\xi}+2\hat{w}_\delta\hat{\psi}_\xi)\hat{\varphi}_{\delta\xi} + (\hat{w}_{\xi\delta}+2\hat{w}_\xi\hat{\psi}_\delta)\hat{\varphi}_{\delta\xi} + (\hat{w}_{\xi\xi}+2\hat{w}_\xi\hat{\psi}_\xi)\hat{\varphi}_{\xi\xi}\right] + \frac{1}{2}\left[(\hat{w}_\xi\hat{\varphi}_{\delta\xi}+\hat{w}_\delta\hat{\varphi}_{\delta\delta})(\hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\xi\xi}+\hat{\ell}_{\xi\delta\delta}\hat{\varphi}_{\xi\delta}+\hat{\ell}_{\delta\xi\delta}\hat{\varphi}_{\delta\xi}+\hat{\ell}_{\delta\delta\delta}\hat{\varphi}_{\delta\delta}) + (\hat{w}_\xi\hat{\varphi}_{\xi\xi}+\hat{w}_\delta\hat{\varphi}_{\xi\delta})(\hat{\ell}_{\xi\xi\xi}\hat{\varphi}_{\xi\xi}+\hat{\ell}_{\xi\delta\xi}\hat{\varphi}_{\xi\delta}+\hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\delta\xi}+\hat{\ell}_{\delta\delta\xi}\hat{\varphi}_{\delta\delta})\right] \quad \dots(25)$$

where,
$\hat{\delta}$ and $\hat{\xi}$ are the *MLE's* of δ and ξ correspondingly.
φ_{ij} is the $(i,j)^{th}$ elements of matrix $\left[\frac{-\partial^2\tilde{\ell}}{\partial\delta\partial\xi}\right]^{-1}$; $i,j = 1,2$ such that sub-scripts (i,j) denote to δ,ξ correspondingly.

$\hat{w}_\delta$ represent the first partial derivative of the function $w(\delta,\xi)$ with respect to δ estimated at $\hat{\delta}$.
$\hat{w}_{\delta\delta}$ represent the second partial derivative of the function $w(\delta,\xi)$ with respect to δ estimated at $\hat{\delta}$.Further expressions we can deduce in closely the same way

$$\hat{\psi}_\delta = \frac{\partial\ln\psi(\delta,\xi)}{\partial\delta}\bigg|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \frac{a-1}{\hat{\delta}} - b$$

$$\hat{\psi}_\xi = \frac{\partial\ln\psi(\delta,\xi)}{\partial\xi}\bigg|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \frac{c-1}{\hat{\xi}} - d$$

and from (14) (15) and (16), we can get:

$$\hat{\ell}_{\delta\delta\xi} = \left.\frac{\partial^3\tilde{\ell}}{\partial\delta\partial\delta\partial\xi}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \hat{\ell}_{\delta\xi\delta} = \left.\frac{\partial^3\tilde{\ell}}{\partial\delta\partial\xi\partial\delta}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \hat{\ell}_{\xi\delta\delta} = \left.\frac{\partial^3\tilde{\ell}}{\partial\xi\partial\delta\partial\delta}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}}$$

$$= \sum_{i=1}^{n} \frac{\int \xi\, x^2\, e^{-\xi x(\delta+1)}(\xi(\delta+1)x-2)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}$$

$$+ \sum_{i=1}^{n} \frac{\int \xi^2 x^2\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx \int x\, e^{-\xi x}\left((\delta+1)e^{-\delta\xi x}-1\right)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2}$$

$$+ 2\sum_{i=1}^{n} \frac{\int \xi x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx \int x\, e^{-\xi x(\delta+1)}(\xi(\delta+1)x-1)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2}$$

$$+ 2\sum_{i=1}^{n} \frac{\left(\int \xi\, x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2 \int x\, e^{-\xi x}\left((\delta+1)e^{-\delta\xi x}-1\right)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^3}$$

$$\hat{\ell}_{\xi\delta\xi} = \left.\frac{\partial^3\tilde{\ell}}{\partial\xi\partial\delta\partial\xi}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \hat{\ell}_{\delta\xi\xi} = \left.\frac{\partial^3\tilde{\ell}}{\partial\delta\partial\xi\partial\xi}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} =$$

$$= \sum_{i=1}^{n} \frac{\int (\delta+1)\, x^2\, e^{-\xi x(\delta+1)}(\xi x(\delta+1)-2)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}$$

$$- \sum_{i=1}^{n} \frac{\int x^2\, e^{-\xi x}\left(1-(\delta+1)^2 e^{-\delta\xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx \int \xi x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2}$$

$$- 2\sum_{i=1}^{n} \frac{\int x\, e^{-\xi x}\left((\delta+1)e^{-\delta\xi x}-1\right)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}$$

$$\left[\frac{\int x\, e^{-\xi x(\delta+1)}(1-\xi(\delta+1)x)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx} - \frac{\int x\, e^{-\xi x}\left((\delta+1)e^{-\delta\xi x}-1\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx \int \xi x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2}\right]$$

$$\hat{\ell}_{\delta\delta\delta} = \left.\frac{\partial^3\tilde{\ell}}{\partial\delta^3}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \frac{2n[\delta(\delta+1)-(2\delta+1)^2]}{\delta^3(\delta+1)^3}$$

$$+ \sum_{i=1}^{n} \frac{\int \xi^3\, x^3\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}$$

$$+ \sum_{i=1}^{n} \frac{\int \xi^2 x^2\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx \int \xi\, x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\left(\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx\right)^2}$$

$$+2\sum_{i=1}^{n} \frac{\int \xi\, x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}\left[\frac{\int \xi^2 x^2\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx} + \left(\frac{\int \xi\, x\, e^{-\xi x(\delta+1)}\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, d\,x}\right)^2\right]$$

$$\hat{\ell}_{\xi\xi\xi} = \left.\frac{\partial^3\tilde{\ell}}{\partial\xi^3}\right|_{\substack{\delta=\hat{\delta}\\ \xi=\hat{\xi}}} = \frac{2n}{\xi^3}$$

$$+ \sum_{i=1}^{n} \frac{\int x^3\, e^{-\xi x}\left(\delta(\delta+1)^2\, e^{-\delta\xi x} + (\delta+1)^2\, e^{-\delta\xi x} - 1\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}$$

$$- 3\sum_{i=1}^{n} \frac{\int x^2\, e^{-\xi x}\left(1-(\delta+1)^2\, e^{-\delta\xi x}\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}$$

$$\left[\sum_{i=1}^{n} \frac{\int x\, e^{-\xi x}\left((\delta+1)\, e^{-\delta\xi x}-1\right) \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}\right]$$

$$+2\sum_{i=1}^{n}\left(\frac{\int x\, e^{-\xi x}\left((\delta+1)e^{-\delta\xi x}-1\right)\, \mathrm{M}_{\tilde{x}_i}(x)\, dx}{\int e^{-\xi x}\,(1-e^{-\delta\xi x})\, \mathrm{M}_{\tilde{x}_i}(x)\, d\,x}\right)^3$$

The approximate Bayesian estimates under the SELF will be:

- For parameter δ: $w(\delta,\xi) = \delta$ then,

$w_\delta = 1,\ w_{\delta\delta} = w_\xi = w_{\xi\xi} = w_{\delta\xi} = w_{\xi\delta} = 0.$

$$\hat{\delta} = E(\delta|\underline{\tilde{x}}) = \hat{\delta} + \hat{\psi}_\delta\hat{\varphi}_{\delta\delta} + \hat{\psi}_\xi\hat{\varphi}_{\delta\xi} + \frac{1}{2}[\hat{\varphi}_{\delta\delta}(\hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\xi\xi} + \hat{\ell}_{\xi\delta\delta}\hat{\varphi}_{\xi\delta} + \hat{\ell}_{\delta\xi\delta}\hat{\varphi}_{\delta\xi} + \hat{\ell}_{\delta\delta\delta}\hat{\varphi}_{\delta\delta}) + \hat{\varphi}_{\xi\delta}(\hat{\ell}_{\xi\xi\xi}\hat{\sigma}_{\xi\xi} + \hat{\ell}_{\xi\delta\xi}\hat{\varphi}_{\xi\delta} + \hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\delta\xi} + \hat{\ell}_{\delta\delta\xi}\hat{\varphi}_{\delta\delta})] \quad \dots(26)$$

- For parameter ξ : let $w(\delta,\xi) = \xi$ then,

$w_\xi = 1,\ w_{\xi\xi} = w_\delta = w_{\delta\delta} = w_{\delta\xi} = w_{\xi\delta} = 0.$

$$\hat{\xi} = E(\xi|\underline{\tilde{x}}) = \hat{\xi} + \hat{\psi}_\xi\hat{\varphi}_{\xi\xi} + \hat{\psi}_\delta\hat{\varphi}_{\xi\delta} + \frac{1}{2}[\hat{\psi}_{\xi\xi}(\hat{\ell}_{\xi\xi\xi}\hat{\varphi}_{\xi\xi} + \hat{\ell}_{\xi\delta\xi}\hat{\varphi}_{\xi\delta} + \hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\delta\xi} + \hat{\ell}_{\delta\delta\xi}\hat{\varphi}_{\delta\delta}) + \hat{\varphi}_{\delta\xi}(\hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\xi\xi} + \hat{\ell}_{\xi\delta\delta}\hat{\varphi}_{\xi\delta} + \hat{\ell}_{\delta\xi\delta}\hat{\varphi}_{\delta\xi} + \hat{\ell}_{\delta\delta\delta}\hat{\varphi}_{\delta\delta})] \quad \dots(27)$$

- For the reliability function $\mathcal{R}(t)$: $w(\delta,\xi) = \frac{1}{\delta}\, e^{-\xi t}\left(\delta + 1 - e^{-\delta\xi t}\right)$ then

$w_\delta = \frac{1}{\delta}e^{-\xi t}\left(\xi t e^{-\delta\xi t} - \frac{1}{\delta} + \frac{1}{\delta}e^{-\delta\xi t}\right)$;

$w_\xi = te^{-\xi t}\left(e^{-\delta\xi t} - \frac{1}{\delta} + \frac{1}{\delta}e^{-\delta\xi t} - 1\right)$;

$w_{\delta\delta} = \frac{2}{\delta^3}e^{-\xi t}(1-e^{-\delta\xi t}) - \frac{\xi t}{\delta}e^{-\xi t(\delta+1)}\left(\xi t + \frac{2}{\delta}\right)$;

$w_{\xi\xi} = t^2 e^{-\xi t}\left(1+\frac{1}{\delta}\right) - t^2 e^{-\xi t(\delta+1)}\left(\delta + \frac{1}{\delta} + 2\right)$;

$w_{\delta\xi} = w_{\xi\delta} = \frac{t}{\delta^2}e^{-\xi t}(1-e^{-\delta\xi t}) - \xi t^2 e^{-\xi t(\delta+1)}\left(1+\frac{1}{\delta}\right).$

$$\hat{\mathcal{R}}(t) = E(\mathcal{R}(t)|\underline{\tilde{x}}) = \frac{1}{\hat{\delta}}\, e^{-\hat{\xi}t}\left(\hat{\delta} + 1 - e^{-\hat{\delta}\hat{\xi}t}\right) + \frac{1}{2}[(\hat{w}_{\xi\xi} + 2\hat{w}_\xi\hat{\psi}_\xi)\hat{\varphi}_{\xi\xi} + (\hat{w}_{\delta\xi} + 2\hat{w}_\delta\hat{\psi}_\xi)\hat{\varphi}_{\delta\xi} + (\hat{w}_{\xi\delta} + 2\hat{w}_\xi\hat{\psi}_\delta)\hat{\varphi}_{\xi\delta} + (\hat{w}_{\delta\delta} + 2\hat{w}_\delta\hat{\psi}_\delta)\hat{\varphi}_{\delta\delta} + (\hat{w}_\xi\hat{\varphi}_{\xi\xi} + \hat{w}_\delta\hat{\varphi}_{\xi\delta})(\hat{\ell}_{\lambda\lambda\lambda}\hat{\varphi}_{\xi\xi} + \hat{\ell}_{\xi\delta\xi}\hat{\varphi}_{\xi\delta} + \hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\delta\xi} + \hat{\ell}_{\delta\delta\xi}\hat{\varphi}_{\delta\delta}) + (\hat{w}_\xi\hat{\varphi}_{\delta\xi} + \hat{w}_\delta\hat{\varphi}_{\delta\delta})(\hat{\ell}_{\delta\xi\xi}\hat{\varphi}_{\xi\xi} + \hat{\ell}_{\xi\delta\delta}\hat{\varphi}_{\xi\delta} + \hat{\ell}_{\delta\xi\delta}\hat{\varphi}_{\delta\xi} + \hat{\ell}_{\delta\delta\delta}\hat{\varphi}_{\delta\delta})] \quad \dots(28)$$

5. Simulation Algorithm and Outcomes

5.1 Simulation Algorithm

This section covers a summary of the five basic steps of a simulation study. Simulation is done using MATLAB (R2010b)

Step (1): This step is the most essential one:

Al-Nahrain Journal of Science
ANJS, Vol. 27(2), June, 2024, pp. 145-154

- Select different sample sizes $n = 25,\ 50$ and 100.
- Select default values for the shape parameter to be $\delta = 0.5, 1, 1.5$ and without loss of generality the scale parameter ξ is select to be equal to 1.
- Select the values of hyper parameters, $a = b = c = d = 2$
- Select the times to calculate the reliability function estimates: = 1,2,3,4 .
- Select the number of sample replicated = 100 .

Step (2): Because it is difficult to obtain a clear form of the inverse cdf of WE distribution. Random samples say x are generated by the summation of two independent random variables distributed WE with parameters ξ and ξ(δ+1).

Step (3): Based on the following fuzzy information system (Figure (4)), encryption of the simulated data such that every observation in the sample will be fuzzy constructed on a chosen membership function,

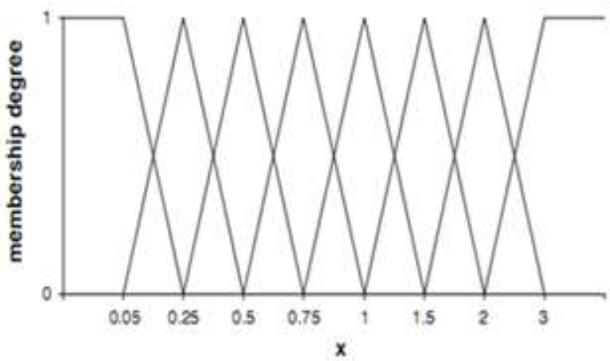

Figure 4: fuzzy information system used to encryption the simulated data [29]

- $$\mathrm{M}_{\tilde{x}_1}(x) = \begin{cases} 1 & ; \quad x \le 0.05\,, \\ \frac{0.25-x}{0.2} & ; \quad 0.05 \le x \le 0.25\,, \\ 0 & ; \quad o.w. \end{cases}$$

- $$\mathrm{M}_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.05}{0.2} & ; \quad 0.05 \le x \le 0.25\,, \\ \frac{0.5-x}{0.25} & ; \quad 0.25 \le x \le 0.5\,, \\ 0 & ; \quad o.w. \end{cases}$$

- $$\mathrm{M}_{\tilde{x}_3}(x) = \begin{cases} \frac{x-0.25}{0.25} & ; \quad 0.25 \le x \le 0.5\,, \\ \frac{0.75-x}{0.25} & ; \quad 0.5 \le x \le 0.75\,, \\ 0 & ; \quad o.w. \end{cases}$$

- $$\mathrm{M}_{\tilde{x}_4}(x) = \begin{cases} \frac{x-0.5}{0.25} & ; \quad 0.5 \le x \le 0.75\,, \\ \frac{1-x}{0.25} & ; \quad 0.75 \le x \le 1\,, \\ 0 & ; \quad o.w. \end{cases}$$

- $$\mathrm{M}_{\tilde{x}_5}(x) = \begin{cases} \frac{x-0.75}{0.25} & ; \quad 0.75 \le x \le 1\,, \\ \frac{1.5-x}{0.5} & ; \quad 1 \le x \le 1.5\,, \\ 0 & ; \quad o.w. \end{cases}$$

- $$\mathrm{M}_{\tilde{x}_6}(x) = \begin{cases} \frac{x-1}{0.5} & ; \quad 1 \le x \le 1.5\,, \\ \frac{2-x}{0.5} & ; \quad 1.5 \le x \le 2\,, \\ 0 & ; \quad o.w. \end{cases}$$

Step (4): Calculate the estimates of the unknown parameters and the reliability function. The iterative procedure ends when the absolute difference among two successive iterations reaches less than $\varepsilon = 0.0001$.

Step (5): After repeating the above steps (100) times, we compare the different estimates for the unknown parameters depending on the mean squared error $MSE(\hat{\delta})$ and $MSE(\hat{\xi})$ and compare the different estimators of reliability function with different times depending on the integrated mean squared error $IMSE(\hat{\mathcal{R}}(t))$ as,

$$MSE(\hat{\delta}) = \frac{\sum_{j=1}^{L}(\hat{\delta}_j - \delta)^2}{L} \qquad \ldots (29)$$

$$MSE(\hat{\xi}) = \frac{\sum_{j=1}^{L}(\hat{\xi}_j - \xi)^2}{L} \qquad \ldots (30)$$

$$IMSE(\hat{\mathcal{R}}(t)) = \frac{\sum_{j=1}^{L}\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\hat{\mathcal{R}}_j(t_i) - \mathcal{R}(t_i)\right)^2}{L} \qquad \ldots (31)$$

where,

$\hat{\delta}_j, \hat{\xi}_j$: Estimate of δ and ξ at the j^{th} repeat (run).

L: Number of samples repeated.

n_t: Number of times selected to be (4)

$\hat{\mathcal{R}}_j(t_i)$: Estimates of $\mathcal{R}(t)$ at the j^{th} repeat and i^{th} time.

5.2 Simulation Outcomes for Estimating the Parameters

Using $\boldsymbol{\delta = 0.5, 1, 1.5}$ and fixed $\boldsymbol{\xi = 1}$, so from the estimated $\boldsymbol{MSE}$ values we observed:

- The values of MSE related to all estimations decrease when the sample size increases.
- When the value of the shape parameter, increase the values of MSE related to all estimations increasing for total sample sizes with maximum likelihood and Bayes estimates.
- The performance of Bayes estimators is better than the maximum likelihood for total sample sizes.

5.3 Simulation Outcomes for Estimating the Reliability Function

Using $\boldsymbol{\delta = 0.5, 1, 1.5}$ and fixed $\boldsymbol{\xi = 1}$, so from the estimated $\boldsymbol{IMSE}$ values we observed the:

- The values of $IMSE$ related to all estimations decrease when the sample size increases.
- When the value of the shape parameter, increase the values of $IMSE$ associated with all estimations decrease for total sample sizes.
- The performance of obtained Bayes estimators is better than the maximum likelihood for total sample sizes.

Table 1. MSE Values Related ML Parameters estimates with Different Sample Sizes

n	δ, ξ	$MSE(\hat{\delta})$	$MSE(\hat{\xi})$
25	0.5,1	0.7573368	0.2148409
	1,1	0.8290270	0.2257785
	1.5,1	1.3632167	0.2744387
50	0.5,1	0.6878762	0.0458749
	1,1	0.8116826	0.1426626
	1.5,1	0.9961581	0.1595756
100	0.5,1	0.2510535	0.0121039
	1,1	0.5295170	0.0468772
	1.5,1	0.8299027	0.0614012

Table 2. MSE Values Related Bayes Parameters Estimates with Different Sample Sizes

$\boldsymbol{n}$	$\boldsymbol{\delta, \xi}$	$\boldsymbol{MSE(\hat{\delta})}$	$\boldsymbol{MSE(\hat{\xi})}$
25	0.5,1	0.4533799	0.0186437
	1,1	0.4938780	0.0188310
	1.5,1	0.6134889	0.0198231
50	0.5,1	0.3273896	0.0123893
	1,1	0.3658930	0.0155298
	1.5,1	0.3996709	0.0193793
100	0.5,1	0.0917286	0.0086008
	1,1	0.1686639	0.0091631
	1.5,1	0.1789889	0.0093009

Table 3. IMSE Values Related ML and Bayes Reliability Function Estimates with Different Sample Sizes

n	δ, ξ	ML $IMSE(\hat{\mathcal{R}}(t))$	Bayes $IMSE(\hat{\mathcal{R}}(t))$
25	0.5,1	0.0333747	0.0072386
	1,1	0.0028296	0.0019059
	1.5,1	0.0027134	0.0016223
50	0.5,1	0.0052202	0.0038890
	1,1	0.0019512	0.0011085
	1.5,1	0.0006616	0.0004556
100	0.5,1	0.0008538	0.0002418
	1,1	0.0002580	0.0002088
	1.5,1	0.0002333	0.0001086

6. Conclusions

1. From estimated mean squared error values associated with different estimations of the unknown shape and scale parameters of the weighted exponential distribution, we have observed:
   - For all sample sizes, increase the value of the shape parameter, increasing the values of mean squared error associated with maximum likelihood and Bayes estimates.
   - With all estimators, the values of mean squared error decrease as the sample size increases.
   - For all sample sizes, the performance of obtained Bayes estimators according to Lindley's approximation is better than that of maximum likelihood estimators.
2. From estimated integrated mean squared error values associated with different estimations of the reliability function of weighted exponential distribution we have observed:
   - For all sample sizes, increase the value of the shape parameter, decreasing the values of integrated mean squared error associated with all estimations.
   - The values of integrated mean squared error associated with all estimations decrease as the sample size increases.
   - The performance of obtained Bayes estimators according to Lindley's approximation is better than that of maximum likelihood estimators for all sample sizes.

7. Recommendations

For future studies, some recommendations can be given, including:

- Estimate the unknown parameters and reliability function of *WE* depending on fuzzy data with various changes such as estimation method, fuzzy information system, membership function, and approximation technique.
- Do similar work with a different continuous probability distribution.

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