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Variational Iteration Approach for Solving Two-Points Fuzzy Boundary Value Problems

Hussein R. Hasan[1,*], Fadhel S. Fadhel[2]

[1,2]Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Jadiriya, Baghdad, Iraq

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Abstract

The main objective of this paper is to introduce interval two-point fuzzy boundary value problems, in which the fuzziness course when the coefficients of the governing ordinary differential equation and/or the boundary conditions include fuzzy numbers of either triangular or trapezoidal types. Such equations will be solved by introducing the concept of $\alpha -$ level sets, $\alpha \in [0,1]$ to treat the fuzzy ordinary differential equation into two nonfuzzy ordinary differential equations, which correspond to the lower and upper solutions of the interval fuzzy solutions. The well-known variational iteration method has been used to solve two-point fuzzy boundary value problems and linear equations have been examined.

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*Corresponding author: hussein.alshebli93@gmail.com

1. Introduction

In the modern area of science and technology, initial and boundary value problems, often known as differential equations with initial or boundary conditions, are used to represent a number of problems in engineering and applied sciences [1]. Zadeh in 1965 [2], followed by Dubois, D. and Prade in 1978,1980, [3,4], the first researchers who initially presented the idea of fuzzy numbers and fuzzy arithmetic. On the other hand, several scientific fields are typically given as Fuzzy DEs with requirements enforced at more than one point, which are mathematically referred to as two-point boundary value problems (BVP). Fuzzy boundary value problems have been studied since the early 2000s. O'Regan et al. investigated Fuzzy BVPs based on Hukuhara derivatives by solving fuzzy integral equations given in terms of Aumann integrals [5]. Fuzzy differential equations can be introduced by several approaches, such as by using Hukuhara differentiability, generalized differentiability, the concept of differential inclusion, etc. Barnabás Bede in 2006 proved that a large class of two-point boundary value problems have no solutions at all under H-differentiability or Hukuhara differentiability, [6]. Murty and G. Suresh Kumar in 2007 studied the existence and uniqueness theorem for initial and boundary value problems associated with fuzzy differential equations of the first and second orders with the help of a modified Lipschitz condition, [7]. Khastan, A. and Juan J. Nieto 2010 studied two-point boundary value problems for the second-order fuzzy differential equation by using a generalized differentiability concept. They presented a new concept of solutions and, utilizing the generalized differentiability investigated the problem of finding new solutions, [8]. Nizami Gasilova et al. studied linear ordinary differential equations with boundary values expressed by fuzzy numbers in 2014, in which they showed that the fuzzy BVP has a unique solution if the corresponding crisp problem has a unique solution. It was proven that if the boundary values are triangular fuzzy numbers, then the value of the solution at a given time is also a triangular fuzzy number, [9]. Saadeh, Rania, et al. in 2016 implemented a relatively recent analytical technique called the iterative reproducing kernel method. The technique's methodology is based on constructing a solution in the form of a rabidly convergent series with a minimum size of calculations using symbolic computation software. The proposed technique is fully compatible with the complexity of such problems, while the obtained results are highly encouraging, [10].

Hussein M. Sagban in 2021 studied the solution of fuzzy initial value problem, using the modified variational iteration method (VIM), in which the fuzziness occurs in the initial conditions[11]. Rasheed in 2022 presented an approximate method for solving fuzzy differential and integral equations. The main objective of this study is to use the well-known VIM to solve interval linear fuzzy ODEs. He proved that the used VIM is accurate and reliable for solving interval fuzzy differential and integral equations, which is achieved theoretically through the convergence analysis and numerically [12]. In this paper, we will study the numerical solution of the second-order two-point fuzzy BVPs using VIM in which the fuzziness is based on two types of fuzzy numbers, namely the triangular and trapezoidal fuzzy numbers. The solution is approached based on the use of the α-level sets.

2. Basic Concepts

The materials and methods shall be described within this section, and in order to fill the gap that may appear with some readers when they try to be familiar with the present work of this paper. Some preliminary concepts and fundamental notions concerning this work will be given for completeness purposes.

Let X be used to represent the universe set of objects, where x stands for its generic elements. It is common to consider a classical subset A of X as a characteristic function from X onto the set {0,1}, i.e.,

$$\mu_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \notin A \end{cases} \quad \dots (1)$$

The set {0,1} is referred to as a valuation set. The symbol (~) will be used to distinguish between fuzzy and nonfuzzy or crisp sets, i.e., fuzzy subset will be symbolized as $\tilde{A}$. Fuzzy subset $\tilde{A}$ of X may be characterized using membership function $\mu_{\tilde{A}}$ from X onto the unit interval [0,1], i.e., $\mu_{\tilde{A}}(x): X \longrightarrow [0,1]$, and the fuzzy set $\tilde{A}$ may be written as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\} \quad \dots (2)$$

An essential concept in fuzzy set theory which will be discussed next is the α-level or (α-cuts), where α ∈ [0,1]. These sets relate to any fuzzy set $\tilde{A}$ and may be viewed as a connection set between fuzzy sets and crisp sets, which are used to demonstrate that majority of results may also be satisfied in fuzzy set. The α- level sets corresponding to a fuzzy set $\tilde{A}$ is defined as [2]:

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \ge \alpha, \forall\ \alpha \in [0,1]\} \quad \dots (3)$$

while the strong α-level set is defined by:

$$A_{\alpha^+} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha, \alpha \in [0,1]\} \quad \dots (4)$$

A fuzzy subset $\tilde{A}$ of R is said to be convex fuzzy set, if [13, 14]:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)\, x_2) \ge Min\,\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \quad \dots (5)$$

for all $x_1, x_2 \in \mathbb{R}$, and all $\lambda \in [0, 1]$, where $\mu_{\tilde{A}}(x)$ is standing for a suitable membership function.

Fuzzy number is a convex normalized fuzzy set of the real line and a fuzzy number $\breve{M}$ There is only one x_0Î R with the $\mu_{\tilde{M}}$ (x_0)=1, (x_0 is referred to as the mean value of $\tilde{M}$) and piecewise continuous $\mu_{\tilde{M}}$ (x). Now, in order to introduce fuzzy ODEs, two types of fuzzy numbers are considered in this work, namely the triangular and trapezoidal fuzzy numbers. Triangular fuzzy number is any symmetric with triangular shape function, which is defined using two parameters a and s, where a, s ∈ ℝ (s ≠ 0) and represented using the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{|x-a|}{s}, \text{when } a - s \le x \le a + s \\ 0, \quad \text{otherwise} \end{cases} \quad \dots (6)$$

The related interval of the α-level sets corresponding to triangular fuzzy number (1) is:

$$A_\alpha = [a + (b-a)\alpha, c - (c-b)\alpha], 0 \le \alpha \le 1 \quad \dots (7)$$

Another kind of fuzzy numbers may be defined, which is used frequently, is the trapezoidal fuzzy number and captures the trapezoidal-shaped class of membership functions. Each number in this class has four parameters a, b, c and d ∈ ℝ to characterize such number, which is expressed in the generic form:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{a-x}{a-b}, \text{when } a \le x \le b \\ 1, \text{when } b \le x \le c \\ \frac{d-x}{d-c}, \text{when } c \le x \le d \\ 0, \quad \text{otherwise} \end{cases} \quad \dots (8)$$

The related interval of the α- level sets corresponding to trapezoidal fuzzy number (8) as:

$$A_\alpha = [a - (a-b)\alpha, d - (d-c)\alpha], 0 \le \alpha \le 1 \quad \dots (9)$$

3. Variational Iteration Method

Mahdi S.R (reference [15]) and Sagban H.M (reference [11]) introduced the solution of fuzzy ODEs with initial conditions using the well-known VIM. In this section, the second order two-point fuzzy BVPs will be solved using the VIM. Consider the following second order fuzzy ODE:

$$\tilde{y}''(x) = f(x, \tilde{y}(x), \tilde{y}'(x)), \quad x \in [0,1] \quad \dots (10)$$

with boundary condition $\tilde{y}(a) = \tilde{\alpha}$ and $\tilde{y}(b) = \tilde{\beta}$.
The correction function for equation (10) for the upper and lower solutions $\underline{y}$ and $\bar{y}$ are given for all m = 0, 1, … as follows:

$$\underline{y}_{m+1}(x) = \underline{y}_m(x) + \int_0^x \underline{\lambda}(s,x)\left[\underline{y}_m''(s) - f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right)\right] ds \quad \dots (11)$$

$$\bar{y}_{m+1}(x) = \bar{y}_m(x) + \int_0^x \bar{\lambda}(s,x)\left[\bar{y}_m''(s) - f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right)\right] ds \quad \dots (12)$$

Then it can be proved they the general Lagrange multiplier $\tilde{y}(x) = \left[\underline{y}(x), \bar{y}(x)\right]$ is provided by:

$$\underline{\lambda}(s,x) = \bar{\lambda}(s,x) = (-1)^n \frac{(s-x)^{n-1}}{(n-1)!} = (s-x) \quad \dots (13)$$

As a result, we obtain the following iteration formulae for the second order (i.e., with order n=2) fuzzy BVPs

$$\underline{y}_{m+1}(x) = \underline{y}_m(x) + \int_0^x (s-x)\left[\underline{y}_m''(s) - f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right)\right] ds \quad \dots (14)$$

$$\bar{y}_{m+1}(x) = \bar{y}_m(x) + \int_0^x (s-x)\left[\bar{y}_m''(s) - f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right)\right] ds \quad \dots (15)$$

3.1. Convergence Analysis

The convergence of the iteration solutions (14) and (15) to the exact solution of the two-points fuzzy BVPs will be stated and proved in this chapter as in the next theorem.

3.2. Theorem

Let $\tilde{y}, \tilde{y}_n \in C^2[a, b]$ to be the two-point fuzzy BVP's exact and approximate solutions and suppose that $\|\tilde{y}'_m(x)\| \le c\|\tilde{y}_m(x)\|$ where $c > 0$ and $\|.\|$ is the supremum norm. If $\tilde{E}_m(x) = \tilde{y}_m(x) - \tilde{y}(x)$, $x \in [0, 1]$ and f has a constant L that satisfies the Lipschitz condition, such that $L < \frac{1}{(1+c)}$, then the approximate order of solutions $\{\tilde{y}_n\}$, n = 0, 1, …; converge to the exact solution $\tilde{y}$, i.e., the approximate sequences of crisp solutions $\{\underline{y}_n\}$ and $\{\bar{y}_n\}$, for all n = 0, 1, …; converge to crisp exact solutions $\bar{y}$ and respectively. $\underline{Y}$

Proof:
The lower case solution iteration solution is given as in the next equation, where the upper case iteration solution is treated:

$$\underline{y}_{m+1}(x) = \underline{y}_m(x) + \int_0^x (s-x)\left[\underline{y}_m''(s) - f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right)\right] ds \quad \dots (16)$$

Also, since the lower-case exact solution satisfies also the formulation of VIM, hence

$$\underline{y}(x) = \underline{y}(x) + \int_0^x (s-x)\left[\underline{y}''(s) - f\left(s, \underline{y}(s), \bar{y}(s), \underline{y}'(s), \bar{y}'(s)\right)\right] ds \quad \dots (17)$$

Subtracting equation (16) from equation (17):

$$\underline{y}_{m+1}(x) - \underline{y}(x) = \underline{y}_m(x) - \underline{y}(x) + \int_0^x (s-x)\left[\underline{y}_m''(s) - \underline{y}''(s) - f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right) + f\left(s, \underline{y}(s), \bar{y}(s), \underline{y}'(s), \bar{y}'(s)\right)\right] ds \quad \dots (18)$$

Since $\underline{E}_m(x) = \underline{y}_m(x) - \underline{y}(x)$, $\forall\, x \in [a, b]$, then:

$$\underline{E}_{m+1}(x) = \underline{E}_m(x) + \int_0^x (s-x)\left[\underline{E}_m''(s) - \left\{f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right) + f\left(s, \underline{y}(s), \bar{y}(s), \underline{y}'(s), \bar{y}'(s)\right)\right\}\right] ds$$
$$= \underline{E}_m(x) - \int_0^x (x-s)\, \underline{E}_m''(s) ds + \int_0^x (x-s)\left[f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right) + f\left(s, \underline{y}(s), \bar{y}(s), \underline{y}'(s), \bar{y}'(s)\right)\right] ds \quad \dots (19)$$

Since for all x, s $\in$ [0,1], we have

$\sup|x - s| < 1$, suppose that $\underline{E}_m'(0) = 0$

$$\underline{E}_{m+1}(x) \le \underline{E}_m(x) - \frac{1}{(n-1)!}\int_0^x \left[(n-1)!\, \underline{E}_m''(s) ds\right] + \int_0^x \left(f\left(s, \underline{y}_m(s), \bar{y}_m(s), \underline{y}_m'(s), \bar{y}_m'(s)\right) + f\left(s, \underline{y}(s), \bar{y}(s), \underline{y}'(s), \bar{y}'(s)\right)\right) ds \quad \dots (20)$$

Since the order of the ODEs is the second order, i.e., then n=2

$$\underline{E}_{m+1}(x) \le \underline{E}_m(x) - \int_0^x\int_0^x \underline{E}_m''(s)ds\,ds - \int_0^x \left(f\left(s,\underline{y}_m(s),\overline{y}_m(s),\underline{y}_m'(s),\overline{y}_m'(s)\right) + f\left(s,\underline{y}(s),\overline{y}(s),\underline{y}'(s),\overline{y}'(s)\right)\right)ds$$
$$= \underline{E}_m(x) - \int_0^x[\underline{E}_m'(x) - \underline{E}_m'(0)]ds - \int_0^x \left(f\left(s,\underline{y}_m(s),\overline{y}_m(s),\underline{y}_m'(s),\overline{y}_m'(s)\right) + f\left(s,\underline{y}(s),\overline{y}(s),\underline{y}'(s),\overline{y}'(s)\right)\right)ds$$
$$= \underline{E}_m(x) - \int_0^x \underline{E}_m'(s)ds - \int_0^x \left(f\left(s,\underline{y}_m(s),\overline{y}_m(s),\underline{y}_m'(s),\overline{y}_m'(s)\right) + f\left(s,\underline{y}(s),\overline{y}(s),\underline{y}'(s),\overline{y}'(s)\right)\right)ds$$
$$= \underline{E}_m(x) - [\underline{E}_m(x) - \underline{E}_m(0)] - \int_0^x \left(f\left(s,\underline{y}_m(s),\overline{y}_m(s),\underline{y}_m'(s),\overline{y}_m'(s)\right) + f\left(s,\underline{y}(s),\overline{y}(s),\underline{y}'(s),\overline{y}'(s)\right)\right)ds \quad \dots(21)$$

Taking the supremum norm on both sides of equation (21), give:

$$\|\underline{E}_{m+1}(x)\| \le \left\|\int_0^x \left(f\left(s,\underline{y}_m(s),\overline{y}_m(s),\underline{y}_m'(s),\overline{y}_m'(s)\right) + f\left(s,\underline{y}(s),\overline{y}(s),\underline{y}'(s),\overline{y}'(s)\right)\right)ds\right\|$$
$$\le \int_0^x \left\|\left(f\left(s,\underline{y}_m(s),\bar{y}_m(s),\underline{y}_m'(s),\bar{y}_m'(s)\right) + f\left(s,\underline{y}(s),\bar{y}(s),\underline{y}'(s),\bar{y}'(s)\right)\right)ds\right\|$$
$$\le L\int_0^x \left(\|\underline{y}_m(s) - \underline{y}(s)\| + \|\bar{y}_m(s) - \bar{y}(s)\| + \|\underline{y}_m'(s) - \underline{y}'(s)\| + \|\bar{y}_m'(s) - \bar{y}'(s)\|\right)ds$$
$$= L\int_0^t\left(\|\underline{E}_m(s)\| + \|\bar{E}_m(s)\| + \|\underline{E}_m'(s)\| + \|\bar{E}_m'(s)\|\right)ds \quad \dots(22)$$

and so:

$$\|\underline{E}_{m+1}(x)\| \le L\int_0^x(\|\underline{E}_m(s)\| + \|\bar{E}_m(s)\| + c\|\underline{E}_m(s)\| + c\|\bar{E}_m(s)\|)ds$$
$$= L(1+c)\int_0^x(\|\underline{E}_m(s)\| + \|\bar{E}_m(s)\|)ds \quad \dots(23)$$

Now, upon using mathematical induction on inequality (23), we have:

If m = 0, then:

$$\|\underline{E}_1(x)\| \le L(1+c)\int_0^x(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|)ds$$
$$\le L(1+c)(x)(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|) \quad \dots(24)$$

Also, if m = 1, then:

$$\|\underline{E}_2(x)\| \le L(1+c)\int_0^x\big(L(1+c)(x)(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|) + L(1+c)(x)(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|)\big)ds$$
$$\|\underline{E}_2(x)\| \le 2L^2(1+c)^2\int_0^x(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|)(x)ds$$
$$= 2L^2(1+c)^2(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|)\int_0^x(x)ds$$
$$\le L^2(1+c)^2x^2(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|) \quad \dots(25)$$

If m = 2, then

$$\|\underline{E}_3(x)\| \le L(1+c)\int_0^x(\|\underline{E}_2(s)\| + \|\bar{E}_2(s)\|)ds$$
$$\|\underline{E}_3(x)\| = L(1+c)\int_0^x\big(L^2(1+c)^2x^2(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|) + L^2(1+c)^2x^2(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|)\big)ds$$
$$\|\underline{E}_3(x)\| = \frac{2L^3(1+c)^3x^3}{3}(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|)$$
$$\|\underline{E}_3(x)\| \le L^3(1+c)^3x^3(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|) \quad \dots(26)$$

If m = 3, then

$$\|\underline{E}_4(x)\| \le L(1+c)\int_0^x\big(L^3(1+c)^3x^3(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|) + L^3(1+c)^3x^3(\|\underline{E}_0(s)\| + \|\bar{E}_0(s)\|)\big)ds$$
$$\|\underline{E}_4(x)\| = \frac{2L^4(1+c)^4x)^4}{4}(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|)$$
$$\|\underline{E}_4(x)\| \le (1+c)^4x^4(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|) \quad \dots(27)$$

Hence, in general for any m:

$$\|\underline{E}_m(x)\| \le L^m(1+c)^m x^m(\|\underline{E}_0(x)\| + \|\bar{E}_0(x)\|) \quad \dots(28)$$

and taking the supremum value over all x ∈ [0,1], then:

$$\|\underline{E}_m(x)\| \le L^m(1+c)^m \sup_{x\in[a,b]}(|\underline{E}_0(x)| + |\bar{E}_0(x)|)$$
$$= \big(L(1+c)\big)^m \sup_{x\in[a,b]}(|\underline{E}_0(x)| + |\bar{E}_0(x)|) \quad \dots(29)$$

where $0 \le x \le 1$.

Since L (1+ c) < 1, because $L < \frac{1}{(1+c)}$ and so $\big(L(1+c)\big)^m \longrightarrow 0$ as $m \longrightarrow \infty$

Thus $\|\underline{E}_m(x)\| \longrightarrow 0$ as $m \longrightarrow \infty$, i.e., $\underline{E}_m(x) \longrightarrow 0$ as $m \longrightarrow \infty$

Hence as $m \longrightarrow \infty$, There are $\underline{y}_m(x) \longrightarrow \underline{y}(x)$, ∀ x ∈ [a, b]

Similarly, as $m \longrightarrow \infty$, we have $\bar{y}_m(x) \longrightarrow \bar{y}(x)$, ∀ x ∈ [a, b]

Therefore, an approximation fuzzy solution sequence $\{\tilde{y}_m(x)\}$, m = 0, 1, …; converges to the exact fuzzy solution $\tilde{y}(x)$, ∀ x ∈ [0, 1].

3.3. Illustrative Examples

In the present section, two examples with triangular or trapezoidal fuzzy numbers are simulated, for linear fuzzy BVPs.

Example 1:
Consider the second order linear fuzzy BVP:

$$\tilde{y}'' + 5\tilde{y}' - 4\tilde{y} = 0,\ x \in [0,1] \quad \ldots(30)$$

with boundary conditions given in terms of triangular fuzzy numbers as:

$$y(0) = \tilde{1},\ y(1) = \tilde{2} \quad \ldots(31)$$

In order to the fuzzy boundary value problem be solved. (30) – (31), the interval α-level sets related to the boundary conditions are used, as well as, the fuzzy function $\tilde{y}$, as follows:

$$\tilde{y} = [\underline{y}, \overline{y}],\ [y(0)]_\alpha = \tilde{1} = [\alpha, 2-\alpha],\ [y(1)]_\alpha = \tilde{2} = [1+\alpha, 3-\alpha] \quad \ldots(32)$$

Then equation (30) may be characterized as:

$$[\underline{y}'', \overline{y}''] + 5[\underline{y}', \overline{y}'] - 4[\underline{y}, \overline{y}] = 0 \quad \ldots(32)$$

with BC:

$$[y(0)]_\alpha = [\alpha, 2-\alpha],\ [y(1)]_\alpha = [1+\alpha, 3-\alpha] \quad \ldots(33)$$

Equation (32) in terms of lower and upper solutions will take the form:

$$\underline{y}''(x) + 5\underline{y}'(x) - 4\overline{y}(x) = 0 \quad \ldots(34)$$

$$\overline{y}''(x) + 5\overline{y}'(x) - 4\underline{y}(x) = 0 \quad \ldots(35)$$

Using the VIM, the correction functional related to equations (34) and (35) for the upper and lower solutions of $\tilde{y}$ (namely $\tilde{y} = [\underline{y}, \overline{y}]$) will be read for all n = 0,1,… as follows:

$$\underline{y}_{m+1}(x) = \underline{y}_m(x) + \int_0^x \left\{(s-x)\left\{\underline{y}''_m(s) + 5\underline{y}'_m(s) - 4\overline{y}_m(s)\right\}\right\} ds \quad \ldots(36)$$

$$\overline{y}_{m+1}(x) = \overline{y}_m(x) + \int_0^x \left\{(s-x)\left\{\overline{y}''_m(s) + 5\overline{y}'_m(s) - 4\underline{y}_m(s)\right\}\right\} ds \quad \ldots(37)$$

Hence, the first approximate solution consists of combining the following lower and upper solutions

$$\underline{y}_1(x) = \underline{y}_0(x) + \int_0^x \left\{(s-x)\left\{\underline{y}''_0(s) + 5\underline{y}'_0(s) - 4\overline{y}_0(s)\right\}\right\} ds \quad \ldots(38)$$

$$\overline{y}_1(x) = \overline{y}_0(x) + \int_0^x \left\{(s-x)\left\{\overline{y}''_0(s) + 5\overline{y}'_0(s) - 4\underline{y}_0(s)\right\}\right\} ds \quad \ldots(39)$$

Now, we begin with initial approximation $\underline{y}_0(x) = A_1 + B_1 x$ and $\overline{y}_0(x) = A_2 + B_2 x$. By the above variational iteration formula (38) and (39), we can obtain following result:

$$\underline{y}_1(x) = A_1 + B_1 x + \int_0^x \{(s-x)\{5B_1 - 4A_2 - 4B_2 s\}\} ds$$
$$= A_1 + B_1 x + x^2\left(\frac{-5}{2}B_1 + 2A_2\right) + \frac{2}{3}B_2 x^3 \quad \ldots(40)$$

$$\overline{y}_1(x) = A_2 + B_2 x + \int_0^x \{(s-x)\{5B_2 - 4A_1 - 4B_1 s\}\} ds$$
$$= A_2 + B_2 x + x^2\left(\frac{-5}{2}B_2 + 2A_1\right) + \frac{2}{3}B_1 x^3 \quad \ldots(41)$$

By imposing the BC at x = 0 and x = 1 and solution the resulted system for A_1,A_2, $B_1$and B_2, yields

$$A_1 = \alpha,\ A_2 = 2 - \alpha \quad \ldots(42)$$

$$B_1 = \frac{138}{65} - \frac{12\alpha}{13} \text{ and } B_2 = \frac{12\alpha}{13} + \frac{18}{65}. \quad \ldots(43)$$

Thus

$$\underline{y}_1(x) = \alpha + \left(\frac{138}{65} - \frac{12\alpha}{13}\right)x + x^2\left(\frac{-5}{2}\left(\frac{138}{65} - \frac{12\alpha}{13}\right) + 4 - 2\alpha\right) + \frac{2}{3}\left(\frac{12\alpha}{13} + \frac{18}{65}\right)x^3 \quad \ldots(44)$$

$$\overline{y}_1(x) = 2 - \alpha + \left(\frac{12\alpha}{13} + \frac{18}{65}\right)x + x^2\left(\frac{-5}{2}\left(\frac{12\alpha}{13} + \frac{18}{65}\right) + 2\alpha\right) + \frac{2}{3}\left(\frac{138}{65} - \frac{12\alpha}{13}\right)x^3 \quad \ldots(45)$$

Figure 1 present the obtained results for the fuzzy solution with different levels, namely α = 0, 0.2, 0.4, 0.6, 0.8 and 1

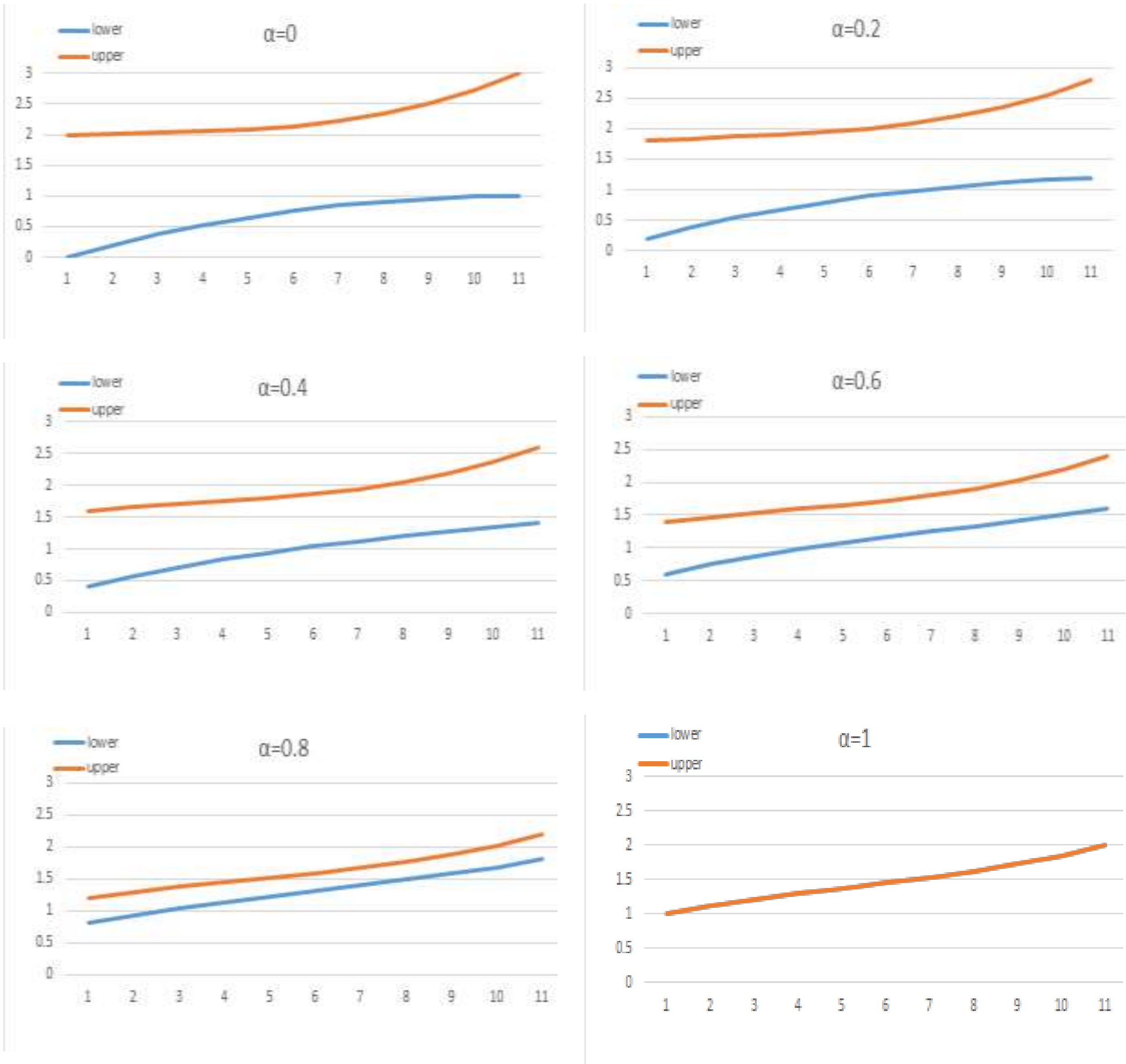

Figure 1. Lower and upper solutions of $\tilde{\mathbf{y}}$ of Example 1 for different values of α

Example 2:

Consider the second order linear fuzzy ODEs:

$$\tilde{y}'' + 5\tilde{y}' - 4\tilde{y} = 0,\ \forall\ x \in [0,1] \ \ldots (46)$$

with boundary conditions:

$$y(0) = [0.2, 0.8],\ y(1) = [0.8, 1.2] \ \ \ldots (47)$$

In order the fuzzy boundary value problem equations (46) and (47) must be solved. The interval α-level sets related to the boundary conditions are used, as well as, the fuzzy function ỹ, as follows:

$$\tilde{y} = \left[\underline{y}, \overline{y}\right], [y\ (0)]_\alpha = [0.2, 0.8] = [0.2\alpha, 1 - 0.2\alpha], [y\ (1)]_\alpha = [0.8, 1.2] = [0.6 + 0.2\alpha, 1.4 - 0.2\alpha] \quad \ldots (48)$$

Then equation (46) may be expressed as:

$$[\underline{y}''(x), \overline{y}''(x)] + 5[\underline{y}'(x), \overline{y}'(x)] - 4[\underline{y}(x), \overline{y}(x)] = 0 \quad \ldots(49)$$

with BC:

$$[y\ (0)]_\alpha = [\alpha, 2-\alpha],\ [y\ (1)]_\alpha = [1+\alpha, 3-\alpha] \quad \ldots(50)$$

Equation (46) in terms of lower and upper solution will take the form:

$$\underline{y}''(x) + 5\overline{y}'(x) - 4\underline{y}(x) = 0 \quad \ldots(51)$$

$$\overline{y}''(x) + 5\underline{y}'(x) - 4\overline{y}(x) = 0 \quad \ldots(52)$$

Using the VIM, the correction functional related to equation (51) and (52) for the upper and lower solutions of $\tilde{y}$ (namely $\tilde{y} = [\underline{y}, \overline{y}]$) will be read for all n = 0, 1,…as follows:

$$\underline{Y}_{m+1}(x) = \underline{y}_m(x) + \int_0^x \left\{(s-x)\left\{\underline{y}''_m(s) + 5\underline{y}'_m(s) - 4\overline{y}_m(s)\right\}\right\} ds \quad \ldots(53)$$

$$\overline{y}_{m+1}(x) = \overline{y}_m(x) + \int_0^x \left\{(s-x)\left\{\overline{y}''_m(s) + 5\overline{y}'_m(s) - 4\underline{y}_m(s)\right\}\right\} ds \quad \ldots(54)$$

Hence, the first approximate solution consists of combining the following lower and upper solutions

$$\underline{y}_1(x) = \underline{y}_0(x) + \int_0^x \left\{(s-x)\left\{\underline{y}''_0(s) + 5\underline{y}'_0(s) - 4\overline{y}_0(s)\right\}\right\} ds \quad \ldots(55)$$

$$\overline{y}_1(x) = \overline{y}_0(x) + \int_0^x \left\{(s-x)\left\{\overline{y}''_0(s) + 5\overline{y}'_0(s) - 4\underline{y}_0(s)\right\}\right\} ds \quad \ldots(56)$$

Now, we begin with initial approximation $\underline{y}_0(x) = A_1 + B_1 x$ and $\overline{y}_0(x) = A_2 + B_2 x$. By the above variational iteration formula (38) and (39), we can obtain following result:

$$\underline{y}_1(x) = A_1 + B_1 x + \int_0^x \{(s-x)\{5B_1 - 4A_2 - 4B_2 s\}\} ds = A_1 + B_1 x + x^2\left(\frac{-5}{2}B_1 + 2A_2\right) + \frac{2}{3}B_2 x^3 \quad \ldots(57)$$

$$\overline{y}_1(x) = A_2 + B_2 x + \int_0^x \{(s-x)\{5B_2 - 4A_1 - 4B_1 s\}\} ds$$

$$= A_2 + B_2 x + x^2\left(\frac{-5}{2}B_2 + 2A_1\right) + \frac{2}{3}B_1 x^3 \quad \ldots(58)$$

By imposing the BC at x = 0 and x = 1 and solution the resulted system for A_1, A_2, B_1 and B_2 yields to $A_1 = 0.2\alpha$, $A_2 = 1 - 0.2\alpha$,

$$B_1 = -0.184615384615384615\alpha + 1.015384615384615384 \quad \ldots(59)$$

$$B_2 = 0.184615384615384615\alpha + 0.184615384615384615 \quad \ldots(60)$$

thus

$$\underline{y}_1(x) = 0.2\alpha + (-0.184615384615384615\alpha + 1.015384615384615384)x + \left(-\frac{5}{2}(-0.184615384615384615\alpha + 1.015384615384615384) + 2 - 0.4\alpha\right)x^2 + \frac{2}{3}(0.184615384615384615\alpha + 0.184615384615384615)x^3 \quad \ldots(61)$$

$$\overline{y}_1(x) = 1 - 0.2\alpha + (0.184615384615384615\alpha + 0.184615384615384615)x + \left(-\frac{5}{2}(0.184615384615384615\alpha + 0.184615384615384615) + 0.4\alpha\right)x^2 + \frac{2}{3}(-0.184615384615384615\alpha + 1.015384615384615384)x^3 \quad \ldots(62)$$

Figure 2. present the obtained results for the fuzzy solution with different levels, namely α = 0, 0.2, 0.4, 0.6, 0.8 and 1.

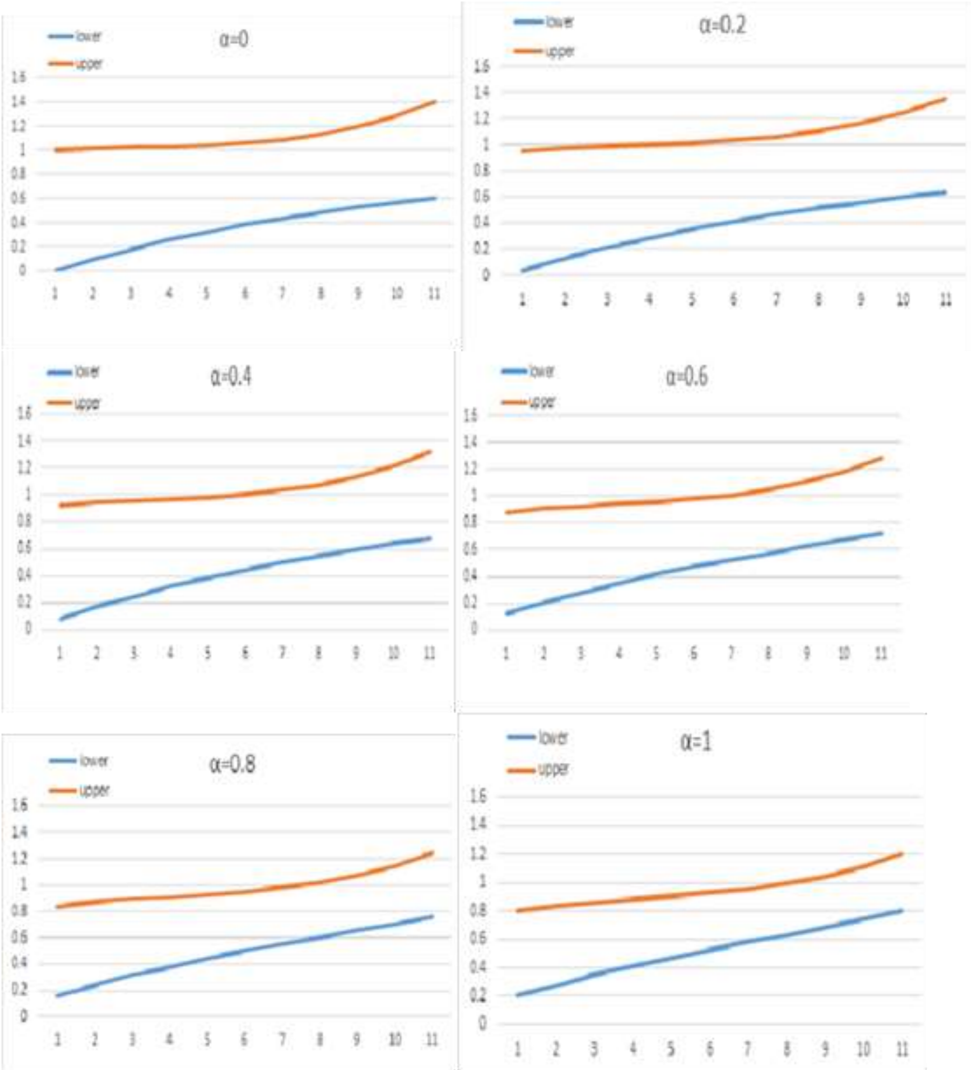

Figure 2. present the obtained results for the fuzzy solution with different levels, namely α =0, 0.2, 0.4, 0.6, 0.8 and 1.

4. Conclusions

The results presented of the sketches of the corresponding fuzzy solutions given in Figures 1-2 shows the efficiency and accuracy of the obtained results for the considered examples in this paper. The solution of the related crisp BVP is obtained when substituting the level α = 1. In addition, for fuzzy BVPs with triangular fuzzy numbers, and when α = 1, the lower and upper solutions are coinciding, as it is expected. (VIM)was easier and faster, and its formula is in one step. We get the solution, but we did not find a solution to the nonlinear equations.

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