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Mixed Transform Iterative Method for Solving Wave Like Equations

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Abstract

In this article, a hybrid method that combines Aboodh transform, variational iteration method, and the homotopy perturbation method is presented for approximate the solution of important partial differential equations that describe wave like differential equations in one, two and three dimensions. The suggested method utilizes only the initial conditions to provide an analytical solution, in contrast to the method of separation of variables, that also requires boundary conditions. This method makes use of the Aboodh's transform advantageous on the Lagrange multiplier computation, hence there is no need to use the convolution theorem or any integration in a recurrence relation for the process of finding it. The obtained exact solutions are found as a convergent series with components that are simple to compute. In addition, the method needs no any assumptions that change the problem's physical nature, such as those that involve discretization, linearization, or minor factors compared to certain other techniques. Some examples are given to show how efficient and useful the method is.

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1. Introduction

The Differential Equations (DEs), such as wave like type have recently gained great interest from scientists and researchers in a variety of scientific and technical domains including physics, fluid mechanics, and applied mathematics. Nonlinear partial differential equations (NLPDE) can be used to explain a wide variety of scientific and engineering facts [1]. The NLPDEs can be solved using a number of different numerical and analytical methods. Due to the limitations of numerical solutions, which cannot provide us with much information about the qualitative behavior of systems, the analytical solution for a given PDE is always preferable. However, the systems that PDEs describe are frequently so large or complex where a solution to the equations that can be solved analytically is impractical, or many be the modeled problems have resulted in NLPDEs for which exact solutions are challenging to find using analytical techniques. Despite the fact that the numerical methods have defects, many researchers have developed different numerical methods over a period in order to find accurate approximation solutions to these nonlinear equations [1-6]. On the other hand, research into localization in a restricted region of space, which is frequently modeled by nonlinear equations is crucial for a number of disciplines, including geophysics, hydrodynamics, plasma physics, nonlinear optics, etc. The closed-form solutions of these equations are essential.

A variety of iterative techniques such as the homotopy perturbation method (HPM) [7-8], the Adomian decomposition method (ADM) [9-10], and the variational iteration method (VIM) [11-13] have been suggested to solve PDEs. By using terms from an infinite series, these methods generate either an approximate or an exact solution that quickly converges to an accurate solution. However, these techniques have some drawbacks, including the computation of Lagrange multiplier (LM), the selection of Adomian polynomials, and the evaluation of small parameters. Many researchers are putting a lot of efforts into combining different iterative methods with transforms to solve various NLPDEQs and integral equations [14-29]. Usually, the integral transformations are used to convert DEs into algebraic equations that can be easily solved. It may not always be easy to solve nonlinear equations using just the known integral transforms, though, because nonlinearity does exist. However, it is noticeable that most of the iterative techniques in use today have defects, such as unnecessary linearization, variable discretization, transformation, or the use of constraining assumptions. By combining an appropriate integral transform with other iterative methods, a substantial amount of approximate exact solutions was consequently generated.

The wave equation is most likely the PDE that is used in practical applications. There are waves in various media, such as the ocean's surface and internal waves in geophysics, but there are also more complex nonlinear models than PDE. Wave processes include electromagnetic waves in various media, acoustic waves in the air and water, and seismic waves in the earth. Even though the initial equation varies for each situation, the wave equation can typically be modeled.

In this study, an efficient hybrid method termed Mixed Transform Iterative Method (MTIM) that combines Aboodh transform [30], VIM, and the HPM is proposed for solving DEs in one, two, and three dimensional that describe the wave like equations. The exact solution, if it exists, is always obtained using this technique, even though a small number of iterations can be utilized for numerical purposes with a high degree of precision in the form of a rapidly convergence series with simple mathematical terms for specific cases. The method also requires no assumptions that affects the physical nature of the problem, such as those that involve discretization, linearization, or minor factors, making it easy to understand. The achievements show that the suggested method is efficient, dependable, accurate, and adaptable.

2. Preliminaries

An overview of each component of the combined suggested MTIM will be provided in this section.

2.1 Aboodh Transform (AT):

Bulleted For an exponential order functions $\hbar(t)$ over the set $\mathfrak{B}$ given by [30]:

$$\mathfrak{B} = \{\hbar;\ |\hbar(t)| < Me^{|t|\beta_j}, t \in (-1)^j \times [0,\infty[, j = 1,2;\ \mathcal{M}, \beta_1, \beta_2 > 0\} \quad \ldots(1)$$

The constant $\mathcal{M}$ must be a finite number and β_1, β_2 may be finite or infinite numbers for a given function in $\mathfrak{B}$. The AT is denoted by the operator $\mathbb{A}$ and defined as:

$$\mathrm{A}[\hbar(t)] = \frac{1}{v}\int_0^\infty \hbar(t)e^{-vt}\,dt = \mathbb{A}(v), t \geq 0;\ -\beta_1 \leq v \leq \beta_2. \quad \ldots(2)$$

The Laplace transform of the function $\hbar(t) \in \mathfrak{B}$ is defined as:

$$\mathfrak{L}(t) = \int_0^\infty \hbar(t)e^{-st}\,dt \quad \ldots(3)$$

and it is written as follows:

$$\mathcal{L}[\hbar(t)] = \mathfrak{L}(s) \quad \ldots(4)$$

If v and s are equal to 1, then Eqs. (2) and (3) are equal, and hence the Aboodh transform and the Laplace transform are symmetrical. Furthermore, it is shown that $\mathbb{A}(v) = \frac{1}{v}\mathfrak{L}(v)$ [31].

If $\mathbb{A}(v)$ is the AT of $\hbar(t)$, the fundamental properties of AT are given below [30],

1. $\mathrm{A}[a\hbar_1(t) + b\hbar_2(t)] = a\mathrm{A}[\hbar_1(t)] + b\mathrm{A}[\hbar_2(t)]$ …(5)
2. $\mathrm{A}[1] = \frac{1}{v^2}$ …(6)
3. $\mathrm{A}[t] = \frac{1}{v^3}$ …(7)
4. $\mathrm{A}[t^n] = \frac{n!}{v^{n+2}}$ …(8)
5. $\mathrm{A}[e^{ct}] = \frac{1}{v^2 - cv}$ …(9)
6. $\mathrm{A}[t\hbar(t)] = \left(-\frac{d}{dv} - \frac{1}{v}\right)\mathbb{A}(v)$ …(10)
7. $\mathrm{A}[e^{ct}\hbar(t)] = \frac{v-c}{v}\,\mathbb{A}(v-c)$ …(11)
8. $\mathrm{A}[\sin(ct)] = \frac{c}{v(v^2+c^2)}$ …(12)
9. $\mathrm{A}[\cos(ct)] = \frac{1}{(v^2+c^2)}$ …(13)
10. $\mathrm{A}[\sinh(ct)] = \frac{c}{v(v^2-c^2)}$ …(14)
11. $\mathrm{A}[\cosh(ct)] = \frac{1}{(v^2-c^2)}$ …(15)
12. $\mathrm{A}[\hbar'(t)] = v\mathbb{A}(v) - \frac{\hbar(0)}{v}$ …(16)
13. $\mathrm{A}[\hbar''(t)] = v^2\,\mathbb{A}(v) - \frac{\hbar'(0)}{v} - \hbar(0)$ …(17)
14. $\mathrm{A}[\hbar^{(n)}(t)] = v^{(n)}\,\mathbb{A}(v) - \sum_{i=0}^{n-1}\frac{\hbar^{(i)}(0)}{v^{2-n+i}}$ …(18)
15. $\mathrm{A}\,[t\hbar'(t)] = -\frac{d}{dv}\left[v\mathbb{A}(v) - \frac{\hbar(0)}{v}\right] - \frac{1}{v}\left(v\mathbb{A}(v) - \frac{\hbar(0)}{v}\right)$ …(19)
16. $\mathrm{A}\,[t^2\hbar'(t)] = v\frac{d^2\mathbb{A}(v)}{dv^2} + 2\frac{d\mathbb{A}(v)}{dv} - 2\frac{\hbar(0)}{v^3}$ …(20)
17. $\mathrm{A}[t\,\hbar''(t)] = -\frac{d}{dv}\left[v^2\mathbb{A}(v) - \frac{\hbar'(0)}{v} - \hbar(0)\right] - \frac{1}{v}\left(v^2\mathbb{A}(v) - \frac{\hbar'(0)}{v} - \hbar(0)\right)$ …(21)
18. $\mathrm{A}[t^2\,\hbar''(t)] = v^2\frac{d^2\mathbb{A}(v)}{dv^2} + 4v\frac{d\mathbb{A}(v)}{dv} + 2\mathbb{A}(v) - 2\frac{\hbar'(0)}{v^3}$ …(22)

The AT for partial derivative is given by [32]:

19. $\mathrm{A}[u(x,t)] = \mathbb{A}(x,v)$ …(23)
20. $\mathrm{A}\left[\frac{\partial u(x,t)}{\partial t}\right] = v\mathbb{A}(x,v) - \frac{u(x,0)}{v}$ …(24)
21. $\mathrm{A}\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = v^2\mathbb{A}(x,v) - \frac{\frac{u(x,0)}{\partial t}}{v} - u(x,0)$ …(25)
22. $\mathrm{A}\left[\frac{\partial u(x,t)}{\partial x}\right] = \mathbb{A}'(x,v)$ …(26)
23. $\mathrm{A}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \mathbb{A}''(x,v)$ …(27)
24. $\mathrm{A}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right] = \mathbb{A}^{(\mathrm{k})}(x,v)$ …(28)

2.2 The Variational Iteration Method (VIM):

The One of the most commonly used methods for solving linear and nonlinear equations is the VIM. is one of the most popular and significant approaches for solving linear and nonlinear equations. The concept of VIM will be illustrated using the NLPDE [11]:

$$\mathbb{L}u + \mathcal{N}u = \mathcal{G}(x) \quad \ldots(29)$$

where $\mathbb{L}$ and $\mathcal{N}$ are a linear operator and a nonlinear operator respectively; and $\mathcal{G}(x)$ is an analytical function. The correction functional to VIM for Eq. (29) may be written as:

$$u_{m+1}(x) = u_m(x) + \int_0^x \lambda(\rho)\left[\mathbb{L}u_m(\rho) + \mathcal{N}_{\tilde{u}'_m}(\rho) - \mathcal{G}(\rho)\right]d\rho \quad \ldots(30)$$

where $\lambda(\rho)$ is a LM that the variational theory can accurately determine. $\tilde{u}'_m$ is considered as a restricted variation, i.e. $\delta\tilde{u}'_m = 0$, and the index m denotes the m^{th} approximation. By utilizing any chosen function u_0 and LM, the approximation $u_{m+1}, m \geq 0$ of u will be achieved. Integration by parts can be used to determine $\lambda(\rho)$; and the solution is provided by:

$$u = \lim_{m\to\infty} u_m \quad \ldots(31)$$

2.3 Homotopy Perturbation Method (HPM):

HPM [7] could be a reliable and effective technique for finding the exact or approximate solutions of DEs. To present the HPM, consider the following nonlinear system with the boundary conditions to clarify the fundamental idea of the HPM:

$$\mathbb{L}(u) + \mathcal{N}(u) - \mathcal{G}(s) = 0, s \in \Phi \qquad \ldots(32)$$

$$\mathfrak{B}\left(u, \frac{\partial u}{\partial n}\right) = 0, s \in \Pi \qquad \ldots(33)$$

where $\mathbb{L}$ and $\mathcal{N}$ are a linear and a nonlinear operator respectively; and $\mathcal{G}(s)$ is an analytical function; $\frac{\partial u}{\partial n}$ is the differentiation of u with respect to n and Π is the boundary of the domain Φ.

Applying the homotopy technique to Eq. (32), the following homotopy $v(r, p)$ may be constructed:

$$v(r, p): \Phi \times [0,1] \to \mathbb{R} \qquad \ldots(34)$$

which satisfies,

$$\mathcal{W}(v, p) = (1 - p)[\mathbb{L}(v) - \mathbb{L}(u_0)] + p[\mathbb{L}(v) + \mathcal{N}(v) - \mathcal{G}(s)] = 0, s \in \Phi \qquad \ldots(35)$$

where u_0 is an initial approximate solution of Eq (32), which satisfies the boundary conditions Eq. (33), $\mathbb{R}$ is the set of real numbers, and $p \in [0,1]$, which increases from 0 to 1. Obviously, from Eq. (35) we have:

$$\mathcal{W}(v, 0) = \mathbb{L}(v) - \mathbb{L}(u_0) = 0 \qquad \ldots(36)$$

and:

$$\mathcal{W}(v, 1) = \mathbb{L}(v) + \mathcal{N}(v) - \mathcal{G}(s) = 0 \qquad \ldots(37)$$

Consider $p \in [0,1]$ as a small parameter and assume that the solution to Eq. (35) can be expressed as the following power series in p:

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \qquad \ldots(38)$$

The solution u to Eq.(35) is given as, with $p = 1$,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \qquad \ldots(39)$$

The series in Eq. (39) converges in the majority of cases, but the convergence rate varies depending on the nonlinear operator $\mathcal{N}(v)$.

3. Methodology of the Proposed MTIM

The proposed MTIM is combined from Aboodh transform (AT), VIM, and the HPM. The AT is first applied to each side of a given DE, then the resulting equation will be multiplied by the Lagrange multiplier (LM) to produce the recurrence relation. The recurrence relation is then constrained to yield the LM. The method is important because it does not require the convolution theorem nor the integral part which are usually used in VIM.

Now, applying AT of Eq. (29) yields:

$$\mathrm{A}[\mathbb{L}u + \mathcal{N}u - \mathcal{G}(x)] = 0 \qquad \ldots(40)$$

Multiplying Eq. (40) by LM $\lambda(v)$, we get:

$$\lambda(v)\mathrm{A}[\mathbb{L}u + \mathcal{N}u - \mathcal{G}(x)] = 0. \ (41)$$

To compute the value of LM, the following recurrence relation is utilized:

$$u_{m+1}(x, v) = u_m(x, v) + \lambda(v)\mathrm{A}[\mathbb{L}u + \mathcal{N}u - \mathcal{G}(x)] \qquad \ldots(42)$$

The LM $\lambda(v)$ is determined using the optimality criterion, and:

$$\frac{\delta u_{m+1}(x, v)}{\delta u_m(x, v)} = 0 \qquad \ldots(43)$$

We obtain $\lambda(v) = -\frac{1}{v^2}$ by considering $\mathbb{L} = \frac{\partial}{\partial x^2}$ to be a linear differential operator. By using this value of LM and the inverse of AT of Eq. (42), the approximate solution will be obtained:

$$u_{m+1}(x, v) = u_m(x, v) + \mathrm{A}^{-1}\left[-\frac{1}{v^2}[\mathbb{L}u + \mathcal{N}u - \mathcal{G}(x)]\right], m = 0,1, \ldots \qquad \ldots(44)$$

or nonlinear terms, the HPM can be expressed as follows:

$$\mathcal{N}(u) = \sum_{j=0}^{\infty} p^j \mathrm{H} = \mathrm{H}_0 + p\mathrm{H}_1 + p^2\mathrm{H}_2 + \cdots \qquad \ldots(45)$$

The He's polynomials $\mathrm{H}_m{}'$ s may be computed by using the following formula:

$$\mathrm{H}_m(u_0 + u_1 + u_2 + \cdots + u_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[\mathcal{N}\left(\sum_{j=0}^{\infty} p^j u_j\right)\right]_{p=0}, m = 0,1, \cdots \qquad \ldots(46)$$

The following approximations can be computed by comparing the coefficients of like powers of p:

$$p^0 = u_0(x, t) \qquad \ldots(47)$$

$$p^1 = u_1(x, t) = -\mathrm{A}^{-1}\left[\frac{1}{v^2}\mathrm{A}\left[\mathcal{N}\left(u_0(x, t)\right) - \mathrm{H}\left(u_0(x, t)\right)\right]\right] \qquad \ldots(48)$$

$$p^2 = u_2(x, t) = -\mathrm{A}^{-1}\left[\frac{1}{v^2}\mathrm{A}\left[\mathcal{N}\left(u_1(x, t)\right) - \mathrm{H}\left(u_1(x, t)\right)\right]\right] \qquad \ldots(49)$$

$$p^3 = u_3(x, t) = -\mathrm{A}^{-1}\left[\frac{1}{v^2}\mathrm{A}\left[\mathcal{N}\left(u_2(x, t)\right) - \mathrm{H}\left(u_2(x, t)\right)\right]\right] \qquad \ldots(50)$$

and so on. Then:

$$u_m(x, t) = u_0 + u_1 + u_2 + u_3 + \cdots \qquad \ldots(51)$$

4. Results and Discussion

In this section, we will examine the performance of the suggested MTIM in solving 1, 2 and 3-dimensional equations that describe the wave like partial differential equations.

Example 4.1. Consider the following wave like equation:

$$u_{tt} = \frac{1}{2}x^2 u_{xx} \ 0 < x < 1, t > 0 \qquad \ldots(52)$$

with the initial conditions:

$$u(x, 0) = x, u_t(x, 0) = x^2$$

The exact solution:

$$u(x, t) = x + x^2 \sinh(t) \qquad \ldots(53)$$

We continue the discussion from Section 3. The AT is applied to both sides of Eq. (52) to obtain the recurrence relation shown below, which would then multiplied by $\lambda(v)$:

$$\mathrm{A}\left[u_{tt} - \frac{1}{2}x^2 u_{xx}\right] = 0 \qquad \ldots(54)$$

Multiplying Eq. (54) by $\lambda(v)$, yields:

$$\lambda(v)\ \left[\mathrm{A}\left[u_{tt} - \frac{1}{2}x^2 u_{xx}\right]\right] \qquad \ldots(55)$$

The recurrence relation is:

$$u_{n+1}(x, v) = u_n(x, v) + \lambda(v)\ \left[\mathrm{A}\left[u_{tt} - \frac{1}{2}x^2 u_{xx}\right]\right] \qquad \ldots(56)$$

Now, taking the variation with respect to the independent variable u_n on both sides of Eq. (56), and then applying Aboodh transformation, we get:

$$\delta u_{n+1}(x,v) = \delta u_n(x,v) + \lambda(v)\left[\delta\left(v^2 u_n(x,v) - \frac{1}{v}u_t(x,0) - u(x,0)\right) - A\left(\frac{1}{2}x^2 u_{xx}\right)\right] \quad \ldots(57)$$

$$\delta u_{n+1}(x,v) = \delta u_n(x,v) + \left(1 + v^2\lambda(v)\right)$$

When $\frac{\delta u_{n+1}}{\delta u_n} = 0$, $(v^2)\lambda(v) = -1$, then $\lambda(v) = -\frac{1}{v^2}$

$$u_{n+1}(x,v) = u_n(x,v) - \frac{1}{v^2}\left[A\left[u_{tt} - \frac{1}{2}x^2 u_{xx}\right]\right] \quad \ldots(58)$$

Taking the inverse of AT, implies that:

$$u_{n+1}(x,v) = u_n(x,v) - A^{-1}\left[\frac{1}{v^2}\left[A\left[u_{tt} - \frac{1}{2}x^2 u_{xx}\right]\right]\right] \quad \ldots(59)$$

Applying the HPM, we get:

$$u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots = u_n(x,t) + pA^{-1}\left[\frac{1}{v^2}\left[A\left\{\left(\frac{1}{2}x^2 u_{0xx}\right) + p\left(\frac{1}{2}x^2 u_{1xx}\right) + p^2\left(\frac{1}{2}x^2 u_{2xx}\right) + p^3\left(\frac{1}{2}x^2 u_{3xx}\right) + \cdots\right\}\right]\right] \quad \ldots(60)$$

The He's polynomials after equating p with same powers on both sides, are:

$$p^0: u_0 = u_0(x,t) = x + x^2 t$$

$$p^1: u_1 = A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}x^2 u_{0xx}\right]\right]\right] = \frac{t^3}{3!}x^2$$

$$p^2: u_2 = A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}x^2 u_{1xx}\right]\right]\right] = \frac{t^5}{5!}x^2$$

$$p^3: u_3 = A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}x^2 u_{2xx}\right]\right]\right] = \frac{t^7}{7!}x^2$$

and so on. Then:

$$u_n = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots$$

$$u_n(x,t) = x + t x^2 + \frac{t^3}{3!}x^2 + \frac{t^5}{5!}x^2 + \frac{t^7}{7!}x^2 \ldots = x + x^2\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right]$$

so that the exact solution is given by:

$$u(x,t) = x + x^2 \sinh(x)$$

Example 4.2. Consider the following wave like equation:

$$u_{tt} = \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right), 0 < x, y < 1, t > 0 \quad \ldots(61)$$

with the initial conditions

$$u(x,y,0) = x^4, u_t(x,y,0) = y^4$$

The exact solution:

$$u(x,t) = x^4\cosh(t) + y^4\sinh(t) \quad \ldots(62)$$

By implementing the procedure of the proposed MTIM, we get:

$$A\left[u_{tt} - \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right)\right] = 0 \quad \ldots(63)$$

Multiplying Eq. (63) by $\lambda(v)$:

$$\lambda(v)\left[A\left[u_{tt} - \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right)\right]\right] = 0$$

The recurrence relation takes the form:

$$u_{n+1}(x,v) = u_n(x,v) + \lambda(v)\left[A\left[u_{tt} - \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right)\right]\right] \quad \ldots(64)$$

Now, taking the variation with respect to the independent variable u_n on both sides of Eq. (64), and then applying Aboodh transform, we get:

$$\delta u_{n+1}(x,v) = \delta u_n(x,v) + \lambda(v)\left[\delta\left(v^2 u_n(x,v) - \frac{1}{v}u_t(x,0) - u(x,0)\right) - A\left(\frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right)\right)\right]$$

$$\delta u_{n+1}(x,v) = \delta u_n(x,v) + \left(1 + v^2\lambda(v)\right)$$

when $\frac{\delta u_{n+1}}{\delta u_n} = 0$, then $(v^2)\lambda(v) = -1$, and thus $\lambda(v) = -\frac{1}{v^2}$.

$$u_{n+1}(x,v) = u_n(x,v) - \frac{1}{v^2}\left[A\left[u_{tt} - \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right)\right]\right]$$

The inverse of the Aboodh transform implies that:

$$u_{n+1}(x,v) = u_n(x,v) - A^{-1}\left[\frac{1}{v^2}\left[A\left[u_{tt} - \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right)\right]\right]\right] \quad \ldots(65)$$

Applying the HPM, we get:

$$u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots = u_n(x,t) + pA^{-1}$$

He's polynomials can be determined by equating p with same powers on both sides of Eq. (66) as:

$$p^0: u_0 = u_0(x,t) = x^4 + y^4 t$$

$$p^1: u_1 = A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{12}\left(x^2 u_{0xx} + y^2 u_{0yy}\right)\right]\right]\right] = \frac{t^2}{2!}x^4 + \frac{t^3}{3!}y^4$$

$$p^2: u_2 = A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}\left(x^2 u_{1xx} + y^2 u_{1yy}\right)\right]\right]\right] = \frac{t^4}{4!}x^4 + \frac{t^5}{5!}y^4$$

$$p^3: u_3 = A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}\left(x^2 u_{2xx} + y^2 u_{2yy}\right)\right]\right]\right] = \frac{t^6}{6!}x^4 + \frac{t^7}{7!}y^4$$

and so on. Then:

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots$$

$$u_n u(x,t) = x^4\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + y^4\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)$$

so that the exact solution is given by:

$$u(x,t) = x^4\cosh(t) + y^4\sinh(t)$$

Example 4.3. Consider the following wave like equation:

$$u_{tt} = (x^2 + y^2 + z^2) + \frac{1}{2}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), 0 < x, y, z < 1, t > 0$$

with the initial conditions:

$$u(x,y,z,0) = 0, u_t(x,y,z,0) = x^2 + y^2 - z^2$$

By implementing the procedure of the proposed MTIM, we get:

$$A\left[u_{tt}-(x^2+y^2+z^2)-\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right)\right]=0 \quad \ldots(67)$$

Multiplying Eq. (67) by $\lambda(v)$:

$$\lambda(v)\left[A\left[u_{tt}-(x^2+y^2+z^2)-\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right)\right]\right]=0$$

and hence, the recurrence relation takes the form:

$$u_{n+1}(x,v)=u_n(x,v)+\lambda(v)\left[A\left[u_{tt}-(x^2+y^2+z^2)-\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right)\right]\right] \quad \ldots(68)$$

Now, taking the variation with respect to the independent variable u_n on both sides of Eq. (68), and then applying Aboodh transform, we get:

$$\delta u_{n+1}(x,v)=\delta u_n(x,v)+\lambda(v)\left[\delta\left(v^2u_n(x,v)-\frac{1}{v}u_t(x,0)-u(x,0)\right)-A\left((x^2+y^2+z^2)+\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right)\right)\right]$$

$$\delta u_{n+1}(x,v)=\delta u_n(x,v)+\left(1+v^2\lambda(v)\right)$$

when $\frac{\delta u_{n+1}}{\delta u_n}=0$, then $(v^2)\lambda(v)=-1$, and thus $\lambda(v)=-\frac{1}{v^2}$

$$u_{n+1}(x,v)=u_n(x,v)-\frac{1}{v^2}\left[A\left[u_{tt}-(x^2+y^2+z^2)-\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right)\right]\right]$$

The inverse of the Aboodh transform implies that:

$$u_{n+1}(x,v)=u_n(x,v)-A^{-1}\left[\frac{1}{v^2}\left[A\left[u_{tt}-(x^2+y^2+z^2)-\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right)\right]\right]\right] \quad \ldots(69)$$

Applying the HPM, we get:

$$u_0+pu_1+p^2u_2+p^3u_3+\cdots=u_n(x,t)+pA^{-1}\left[\frac{1}{v^2}\left[A\left\{\left((x^2+y^2+z^2)\right)+\frac{1}{2}\left(x^2u_{0xx}+y^2u_{0yy}+z^2u_{0zz}\right)+p\left(\frac{1}{2}\left(x^2u_{1xx}+y^2u_{1yy}+z^2u_{1zz}\right)\right)+p^2\left(\frac{1}{2}\left(x^2u_{2xx}+y^2u_{2yy}+z^2u_{2zz}\right)\right)+p^3\left(\frac{1}{2}\left(x^2u_{3xx}+y^2u_{3yy}+z^2u_{3zz}\right)\right)+\cdots\right\}\right]\right] \quad \ldots(70)$$

He's polynomials can be determined by equating p with same powers on both sides of Eq. (70) as:

$$p^0: u_0=u_0(x,t)=(x^2+y^2-z^2)t$$

$$p^1: u_1=A^{-1}\left[\frac{1}{v^2}\left[A\left[(x^2+y^2+z^2)+\frac{1}{2}\left(x^2u_{0xx}+y^2u_{0yy}+z^2u_{0zz}\right)\right]\right]\right]$$

$$=\frac{t^2}{2!}(x^2+y^2+z^2)+\frac{t^3}{3!}(x^2+y^2-z^2)$$

$$p^2: u_2=A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}\left(x^2u_{1xx}+y^2u_{1yy}+z^2u_{1zz}\right)\right]\right]\right]$$

$$=\frac{t^4}{4!}(x^2+y^2+z^2)+\frac{t^5}{5!}(x^2+y^2-z^2)$$

$$p^3: u_3=A^{-1}\left[\frac{1}{v^2}\left[A\left[\frac{1}{2}\left(x^2u_{2xx}+y^2u_{2yy}+z^2u_{2zz}\right)\right]\right]\right]$$

$$=\frac{t^6}{6!}(x^2+y^2+z^2)+\frac{t^7}{7!}(x^2+y^2-z^2)$$

and so on. Then:

$$u_n=u_0+pu_1+p^2u_2+p^3u_3+\cdots$$

$$u_n(x,t)=(x^2+y^2)\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\cdots\right)+z^2\left(-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}-\frac{t^7}{7!}+\cdots\right) \quad \ldots(71)$$

so that the exact solution is given by:

$$u(x,t)=(x^2+y^2)(e^t-1)+z^2(e^{-t}-1)$$

5. Conclusions

In this work, various differential equations in 1, 2, and 3-dimensional that describes wave like equation which are of great importance in a variety of scientific disciplines, were analytically solved using mixed transform iterative method (MTIM). The method combines Aboodh transformation with the semi-analytical VIM and the HPM. The exact solutions were obtained in the form of a rapidly convergent series with simple mathematical terms. Additionally, the suggested MTIM utilizes only the initial conditions to provide an analytical solution, in contrast to the method of separation of variables, that also requires boundary conditions. In contrast to popular VIM and modified VIM processes, this method makes use of the Aboodh's transform advantageous on the Lagrange multiplier (LM) computation. There was no need to use the convolution theorem or any integration in a recurrence relation for the process in order to obtain the LM. Another advantage of the proposed technique over the decomposition method is that it can solve nonlinear problems without the use of Adomian's polynomials. Moreover, the proposed method is more flexible, adaptable and simple to handle the nonlinear terms by utilizing the He's polynomials which are calculated using the HPM. Furthermore, the method needs no any assumptions that change the problem's physical nature, such as those that involve discretization, linearization, or minor factors compared to certain other techniques. The results show the efficiency, reliability, accuracy and flexibility of the proposed method for obtaining the exact solutions of the given problems.

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Conflicts of Interest
The authors declare that there is no conflict of interest.

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