Al-Nahrain Journal of Science
ANJS, Vol.26 (1), March, 2023, pp. 41-49

Approximate Solution of Linear Interval Fuzzy Ordinary Differential Equations

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Article's Information

Received:
08.10.2022
Accepted:
06.12.2022
Published:
31.03.2023

Keywords:
Variational iteration method
Interval fuzzy initial value problem
Hukuhara derivative
Ordinary differential equation
General Lagrange multiplier

Abstract

Approximate solution of fuzzy initial value problem composed of interval fuzzy ordinary differential equation subject to initial conditions given as fuzzy intervals is the main objective of this work. The considered fuzzy initial value problem was given with Hukuhara definition of the difference and derivatives for interval-valued functions, in which two cases are considered depending on the comparison between the lower and upper functions of the solution derivative. The followed definition for presenting the fuzzy interval is generalized using trapezoidal fuzzy number, while the followed approximated method used in this work is the variational iteration method and its modification to increase the accuracy and the rate of convergence to the exact solution.

DOI: 10.22401/ANJS.26.1.07

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1. Introduction

More than 50 years ago, Zadeh invented and introduced the notion of fuzzy sets for the first time in 1965 [16]. Following then, fuzzy theory found its way into many domains of life, including logic management, artificial intelligence, and many others [10]. The theory of differential equations, which was initially introduced and investigated by Kandel and Byatt in 1978 and 1980, is one of the most essential concepts that have become of great importance in fuzzy set theory.

Systems of fuzzy differential equations can be investigated in a wide range of real-world applications, including medical physics, chemistry, biology, economics, and engineering [1,2,4,5,8,9]. Deterministic or crisp models and their methodologies may not always be able to produce results with the requisite conditions in the absence of relevant data. As a result, in recent years, fuzzy set theory and interval analysis have emerged as useful methods for investigating obstacles originating from confused models of imprecise conceptions.

In this work, we will study and modify fuzzy interval ordinary differential equations (FIODE's) using the trapezoidal fuzzy number and solve such type of equations approximately using the variational iteration method.

2. Basic Concepts

The goal of this section is to provide basic concepts associated with this work in order to make this article as self-contained as possible for interpretation level of necessary concepts given in literatures, which also are required for readers to study these topics in greater detail, and those more primitive concepts will not be provided.

In this paper, the following terms $\mathbb{R}, \mathbb{R}^+$, X and $\tilde{A}$ are used to denote the set of all real numbers, the set of all nonnegative real numbers, the universal set and the fuzzy subset of the universal set, respectively. Furthermore, it is well-known in fuzzy set theory that every fuzzy set $\tilde{A}$ must be characterized by a function which is called the characteristic or the membership function, denoted by $\mu_{\tilde{A}}$, in which $\mu_{\tilde{A}} : X \to [0,1]$ is used rather than $\{0,1\}$ in crisp sets (additional concepts related to fuzzy set theory can be found in several literatures, such as [6,10,16]).

Definition 2.1 [13]. A normal trapezoidal fuzzy number $\tilde{M}$ has the following function (see Figure 1):

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{a-x}{a-b}, & \text{for } a \le x \le b \\ 1, & \text{for } b \le x \le c \\ \dfrac{d-x}{d-c}, & \text{for } c \le x \le d \\ 0, & \text{otherwise} \end{cases}$$

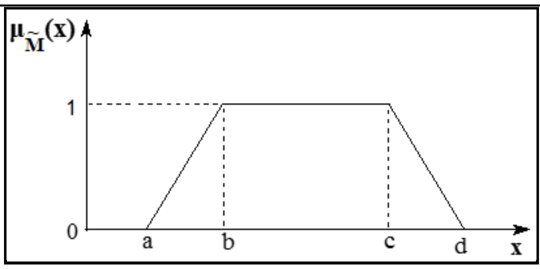

Figure 1. The membership function of the trapezoidal fuzzy number.

Definition 2.2 [7]. The set of all elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set

$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$

where $\mu_{\tilde{A}}$ refers to the characteristic function related to the fuzzy set $\tilde{A}$.

Fuzzy numbers may be characterized or parameterized in terms of α-level sets as it is given in the next definition:

Definition 2.3. For a trapezoidal fuzzy number $\tilde{M}$ in parametric form is a pair $[\underline{M}(\alpha), \overline{M}(\alpha)]$, $0 \leq \alpha \leq 1$ which satisfy the following requirements:

1. $\underline{M}(\alpha)$ is a bounded monotonic increasing right continuous function.
2. $\overline{M}(\alpha)$ is a bounded monotonic decreasing left continuous function.
3. $\underline{M}(\alpha) \leq \overline{M}(\alpha)$, for all $0 \leq \alpha < 1$.
4. $[\underline{M}(\alpha), \overline{M}(\alpha)] = [b, c]$, if $\alpha = 1$.

Remark 2.1. For an arbitrary two fuzzy numbers $\tilde{M} = [\underline{M}(\alpha), \overline{M}(\alpha)]$ and $\tilde{N} = [\underline{N}(\alpha), \overline{N}(\alpha)]$, the following algebraic operations may be fulfilled:

1. If k is any real number, then:
$$k\tilde{M} = \begin{cases} [k\underline{M}(\alpha), k\overline{M}(\alpha)], \text{ if } k \geq 0 \\ [k\overline{M}(\alpha), k\underline{M}(\alpha)], \text{ if } k < 0 \end{cases}$$
2. $\tilde{M} + \tilde{N} = [\underline{M}(\alpha) + \underline{N}(\alpha), \overline{M}(\alpha) + \overline{N}(\alpha)]$.
3. $\tilde{M} - \tilde{N} = [\underline{M}(\alpha) - \overline{N}(\alpha), \overline{M}(\alpha) - \underline{N}(\alpha)]$
4. $\tilde{M}\tilde{N} = [\min s, \max s]$, where:
$s = \{\underline{M}\,\underline{N}, \underline{M}\,\overline{N}, \overline{M}\,\underline{N}, \overline{M}\,\overline{N}\}$

3. Modified Fuzzy Interval Ordinary Differential Equations

Moore [11] in 1966 was the first whom introduced the notion of interval analysis and its algebraic properties. Following then, several authors successfully used this notion to uncertainty analysis. Numerous books [1,12,13] written by other authors have also been produced to represent the scope and various elements of interval analysis. These publications provide a thorough tests of interval calculations, which may aid readers in understanding the fundamental principles of interval analysis.

Several researchers have suggested various mathematical strategies to handle differential equations with interval investigation. As a result, Chalco-Cano et al. [3] investigate differential calculus for interval-valued functions using generalized the differentiability of Hukuhara, which is commonly referred to as the most general concept of differentiability for interval-valued functions. Stefanini and Bede [14] used the Hukuhara notion in a more extended form for interval differential equations (IDEs) and interval-valued functions. Ghazanfaria and Ebrahimia [12] used the differential methodology. Chakraverty et al. recently established new methodologies and approaches for dealing with differential equations with undetermined parameters.

Let us consider a linear FIODE as [14,15]:

$$\tilde{y}' = \tilde{f}(x, y), \tilde{y}(x_0) = \tilde{y}_0 \quad \ldots(1)$$

where:

$$\tilde{f}(x, y) = [\underline{f}(x, y), \overline{f}(x, y)], \tilde{y} = [\underline{y}, \overline{y}], \tilde{y}_0 = [\underline{y}_0, \overline{y}_0]$$

Thus, using the differential form to Eq. (1), one can obtain [12,13]:

$$\left.\begin{array}{l} \min\left\{\underline{y'}(x), \overline{y'}(x)\right\} = \underline{f}(x, y) \\ \max\left\{\underline{y'}(x), \overline{y'}(x)\right\} = \overline{f}(x, y) \end{array}\right\} \quad \ldots(2)$$

with respect to the initial conditions $y(x_0) = \underline{y}_0, \overline{y}(x_0) = \overline{y}_0$.

We obtain two cases from Eq. (2) which are:

Case (i): If $\underline{y'}(x) \leq \overline{y'}(x)$, then the result of the possible differential equations is:

$$\left.\begin{array}{l} \underline{y'}(x) = \underline{f}(x, y) \\ \overline{y'}(x) = \overline{f}(x, y) \end{array}\right\} \quad \ldots(3)$$

with initial conditions $y(x_0) = \underline{y}_0, \overline{y}(x_0) = \overline{y}_0$.

Case (ii): If $\underline{y'}(x) \geq \overline{y'}(x)$, then the differential equations reduce to:

$$\left.\begin{array}{l} \underline{y'}(x) = \overline{f}(x, y) \\ \overline{y'}(x) = \underline{f}(x, y) \end{array}\right\} \quad \ldots(4)$$

with respect to the initial conditions $y(x_0) = \underline{y}_0, \overline{y}(x_0) = \overline{y}_0$.

Now, in order to solve the linear interval-valued differential equation (1), the method of solution will be improved by introducing normal trapezoidal fuzzy number rather than just using the interval fuzzy number discussed in literatures in the linear coefficients and/or in the initial conditions. This is done in connection with the α-level sets and using the properties given in Remark 2.1, which are concerned with interval arithmetic operations, as follows:

The interval fuzzy solution of Eq. (1) will be assumed using α-level as $\tilde{y} = [\underline{y}, \overline{y}]$, then with the reference to the trapezoidal fuzzy number given in Definition 2.1 (see also Figure 1) and in order to find the lower interval solution $\underline{y}$, let for $a \le x \le b$, the α-cut will be $\mu_{\tilde{y}}(x) = \alpha$, $0 \le \alpha \le 1$, then $\frac{a-x}{a-b} = \alpha$ and hence $x = a - (a-b)\alpha$. So:

$$\underline{y}(\alpha) = a - (a-b)\alpha,\ 0 \le \alpha \le 1 \qquad \ldots(5)$$

Similarly, we can find the upper interval solution:

$$\bar{y}(\alpha) = d - (d-c)\alpha,\ 0 \le \alpha \le 1 \qquad \ldots(6)$$

Hence, the fuzzy solution function in terms of it α-level interval is given by:

$$\tilde{y} = [a - (a-b)\alpha, d - (d-c)\alpha],\ 0 \le \alpha \le 1 \qquad \ldots(7)$$

In this work, we will consider linear FIODE of the first order with fuzziness occurs in the differential equation, as well as, in the initial conditions.

Now, as an illustration we will consider the following linear FIODE of the first order with fuzziness occurs in the differential equation, as well as, in the initial conditions:

Example 3.1. Consider the FIODE:

$$\tilde{y}' = -2\tilde{y} + [1,3]x,\ x \ge 0 \qquad \ldots(8)$$

with initial condition $\tilde{y}(0) = [1,2]$.

Using the differentiation method described in Eqs. (3) and (4), implies upon applying the same approach in the derivation of Eq. (7) to find the α-level of the fuzzy intervals [1,3] and [1,2], As previously stated, the following two cases are obtained:

Case (i): If $\underline{y}'(x) \le \overline{y'}(x)$, the using Eq. (3), we have:

$$\left.\begin{aligned} \underline{y'} &= -2\overline{y} + \alpha x \\ \overline{y'} &= -2\underline{y} + (4-\alpha)x \end{aligned}\right\} \qquad \ldots(9)$$

with initial conditions:

$$\underline{y}(0) = \alpha, \bar{y}(0) = 3 - \alpha \qquad \ldots(10)$$

It is notable that system (9) will be solved using the classical methods for solving systems of non-homogeneous of ODEs in matrix form. For this purpose, first rewrite system (9) in matrix form as:

$$\begin{bmatrix} \underline{y'} \\ \overline{y'} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \underline{y} \\ \overline{y} \end{bmatrix} + \begin{bmatrix} \alpha x \\ (4-\alpha)x \end{bmatrix}$$

or

$$y' = Ay + B, \text{ where } y' = \begin{bmatrix} \underline{y'} \\ \overline{y'} \end{bmatrix}, A = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \alpha x \\ (4-\alpha)x \end{bmatrix} \qquad \ldots(11)$$

and in order to solve system (11), we must first find the fundamental matrix of solution related to the homogeneous linear system $y' = Ay$. For this purpose, the eigenvalues of the coefficient matrix are found to be $\lambda_1 = 2$ and $\lambda_2 = -2$ with their corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively. Hence the fundamental matrix of solution is:

$$M(x) = [v_1 e^{\lambda_1 x} \quad v_2 e^{\lambda_2 x}] = \begin{bmatrix} e^{2x} & e^{-2x} \\ -e^{2x} & e^{-2x} \end{bmatrix} \qquad \ldots(12)$$

and so:

$$M^{-1}(x) = \begin{bmatrix} \frac{1}{2}e^{-2x} & -\frac{1}{2}e^{-2x} \\ \frac{1}{2}e^{2x} & \frac{1}{2}e^{2x} \end{bmatrix} \qquad \ldots(13)$$

with initial condition:

$$y(0) = \begin{bmatrix} \alpha \\ 3-\alpha \end{bmatrix} \qquad \ldots(14)$$

The solution of the nonhomogeneous system (11) is

$$y(x) = \begin{bmatrix} \underline{y}(x) \\ \overline{y}(x) \end{bmatrix} = M(x)M^{-1}(0)y(0) + M(x)\int_0^x M^{-1}(s)B(s)\,ds$$

$$= \begin{bmatrix} e^{2x} & e^{-2x} \\ -e^{2x} & e^{-2x} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ 3-\alpha \end{bmatrix} + \begin{bmatrix} e^{2x} & e^{-2x} \\ -e^{2x} & e^{-2x} \end{bmatrix} \int_0^x \begin{bmatrix} \frac{1}{2}e^{-2s} & -\frac{1}{2}e^{-2s} \\ \frac{1}{2}e^{2s} & \frac{1}{2}e^{2s} \end{bmatrix} \begin{bmatrix} \alpha s \\ (4-\alpha)s \end{bmatrix} ds$$

$$= \begin{bmatrix} \left(\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right)\alpha + \left(\frac{-1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right)(3-\alpha) \\ \left(\frac{-1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right)\alpha + \left(\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right)(3-\alpha) \end{bmatrix} + \begin{bmatrix} e^{2x} & e^{-2x} \\ -e^{2x} & e^{-2x} \end{bmatrix} \begin{bmatrix} \left(\frac{-x}{4}e^{-2x} - \frac{1}{8}e^{-2x} + \frac{1}{8}\right)\alpha + \left(\frac{x}{4}e^{-2x} + \frac{1}{8}e^{-2x} - \frac{1}{8}\right)(4-\alpha) \\ \left(\frac{x}{4}e^{2x} - \frac{1}{8}e^{2x} - \frac{1}{8}\right)\alpha + \left(\frac{x}{4}e^{2x} - \frac{1}{8}e^{2x} - \frac{1}{8}\right)(4-\alpha) \end{bmatrix}$$

Upon carrying out some matrix computations, we get the following lower and upper solutions:

$$\underline{y}(x) = \alpha\left(\frac{e^{-2x}}{2} + \frac{e^{2x}}{2}\right) + e^{-2x}\left(\frac{-e^{2x}}{2} + xe^{2x} + \frac{1}{2}\right) - (\alpha-3)\left(\frac{e^{-2x}}{2} - \frac{e^{2x}}{2}\right) + e^{2x}\left(\frac{1}{2} - \frac{\alpha}{4}\right)(e^{-2x} + 2xe^{-2x} - 1)$$

$$\bar{y}(x) = \alpha\left(\frac{e^{-2x}}{2} - \frac{e^{2x}}{2}\right) + e^{-2x}\left(\frac{-e^{2x}}{2} + xe^{2x} + \frac{1}{2}\right) - (\alpha-3)\left(\frac{e^{-2x}}{2} + \frac{e^{2x}}{2}\right) - e^{2x}\left(\frac{1}{2} - \frac{\alpha}{4}\right)(e^{-2x} + 2xe^{-2x} - 1)$$

Accordingly, the interval fuzzy solution $\tilde{y} = [\underline{y}, \bar{y}]$ is sketch for different values of α, which are 0, 0.2, 0.4, 0.6, 0.8 and 1 as they are given in Figure 2 (a-f), respectively.

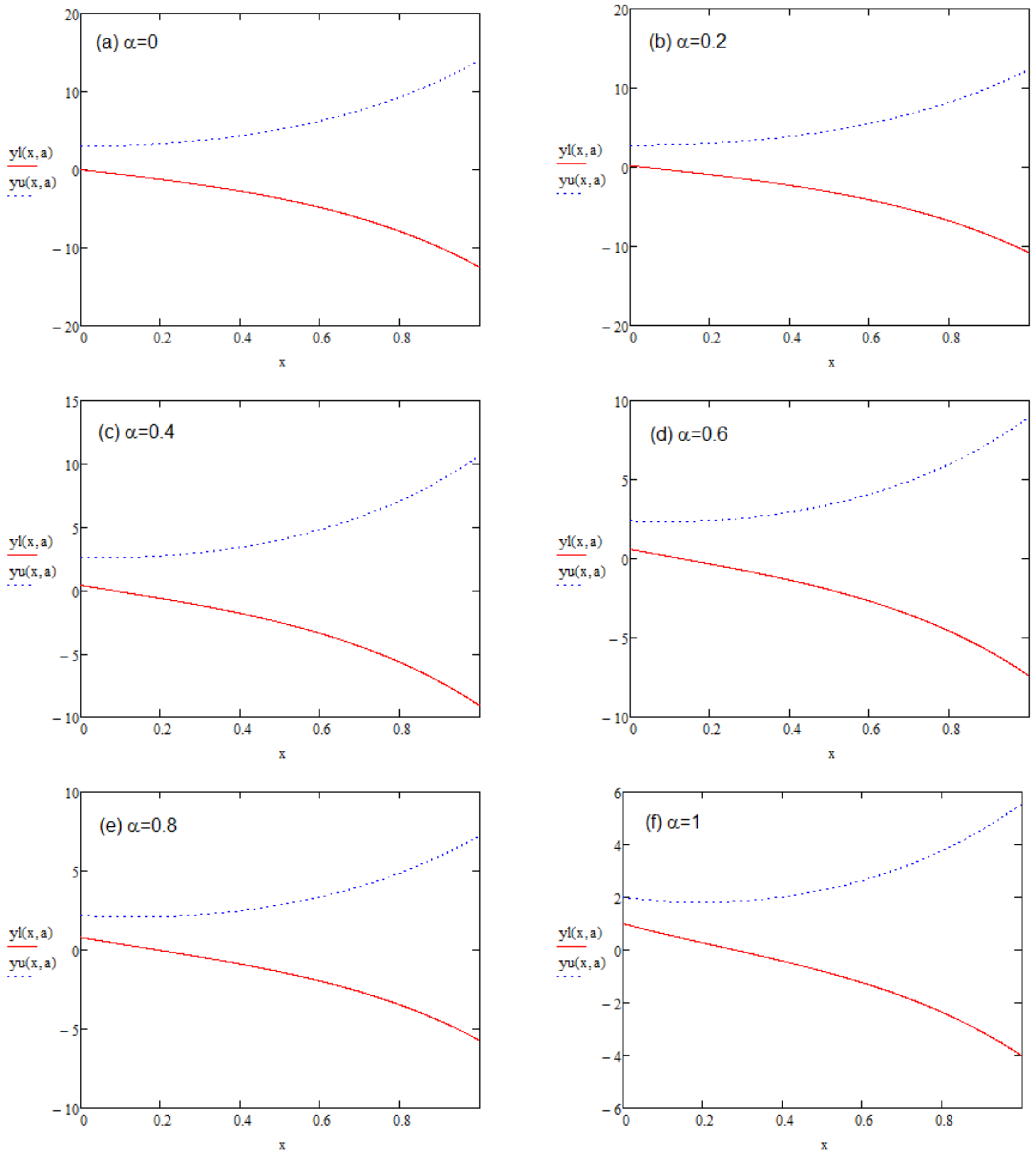

Figure 2. Comparison of the lower and upper bounds solutions of $\tilde{y}$ of Example 3.1 case (i) obtained for different values of α.

Case (ii): If $\underline{y}'(x) \geq \overline{y'}(x)$, then using Eq. (4), we have:

$$\left.\begin{aligned} \underline{y}' &= -2\underline{y} + (4-\alpha)x \\ \overline{y}' &= -2\overline{y} + \alpha x \end{aligned}\right\} \qquad \ldots(15)$$

with the same initial conditions given in Eq. (10), namely $\underline{y}(0) = \alpha, \overline{y}(0) = 3 - \alpha$.

Equations (15) may be solved to find $\underline{y}$ and $\overline{y}$, which are of the first order crisp linear ODE's, as follows:

Consider first the crisp ODE:

$$\underline{y}' = -2\underline{y} + (4-\alpha)x,\ \underline{y}(0) = \alpha,\ \alpha \in [0,1]$$

which is a first order linear ODE that can be rewritten as:

$$\frac{\mathrm{d}\underline{\mathrm{y}}}{\mathrm{dx}} + 2\underline{\mathrm{y}} = (4-\alpha)\mathrm{x},\ \underline{y}(0) = \alpha$$

and so the integration factor is $u(x) = e^{\int 2dx} = e^{2x}$. Thus, the solution is given by:

$$\begin{aligned} \underline{y}(x) &= \frac{1}{e^{2x}}\{\textstyle\int e^{2x}(4-\alpha)x\,dx + c\} \\ &= (4-\alpha)e^{-2x}\textstyle\int xe^{2x}\,dx + ce^{-2x} \\ &= (4-\alpha)e^{-2x}\left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right) + ce^{-2x} \\ &= (4-\alpha)\left(\frac{x}{2} - \frac{1}{4}\right) + ce^{-2x} \end{aligned}$$

using the initial condition to find c, which is found to be $c = \frac{3\alpha+4}{4}$. Hence, the lower solution is:

$$\underline{y}(x) = (4-\alpha)\left(\frac{x}{2} - \frac{1}{4}\right) + \left(\frac{3\alpha+4}{4}\right)e^{-2x}$$

Similarly, we can find the upper solution, which the solution of the crisp ODE:

$$\overline{y}' = -2\overline{y} + \alpha x, \overline{y}(0) = 3 - \alpha, \alpha \in [0,1]$$

which is found to be:

$$\overline{y}(x) = \alpha\left(\frac{x}{2} - \frac{1}{4}\right) + \left(\frac{12-3\alpha}{4}\right)e^{-2x}$$

Accordingly, the interval fuzzy solution $\tilde{y} = [\underline{y}, \overline{y}]$ is sketch for different values of α, which are 0, 0.2, 0.4, 0.6, 0.8 and 1 as they are given in Figure 3 (a-f), respectively.

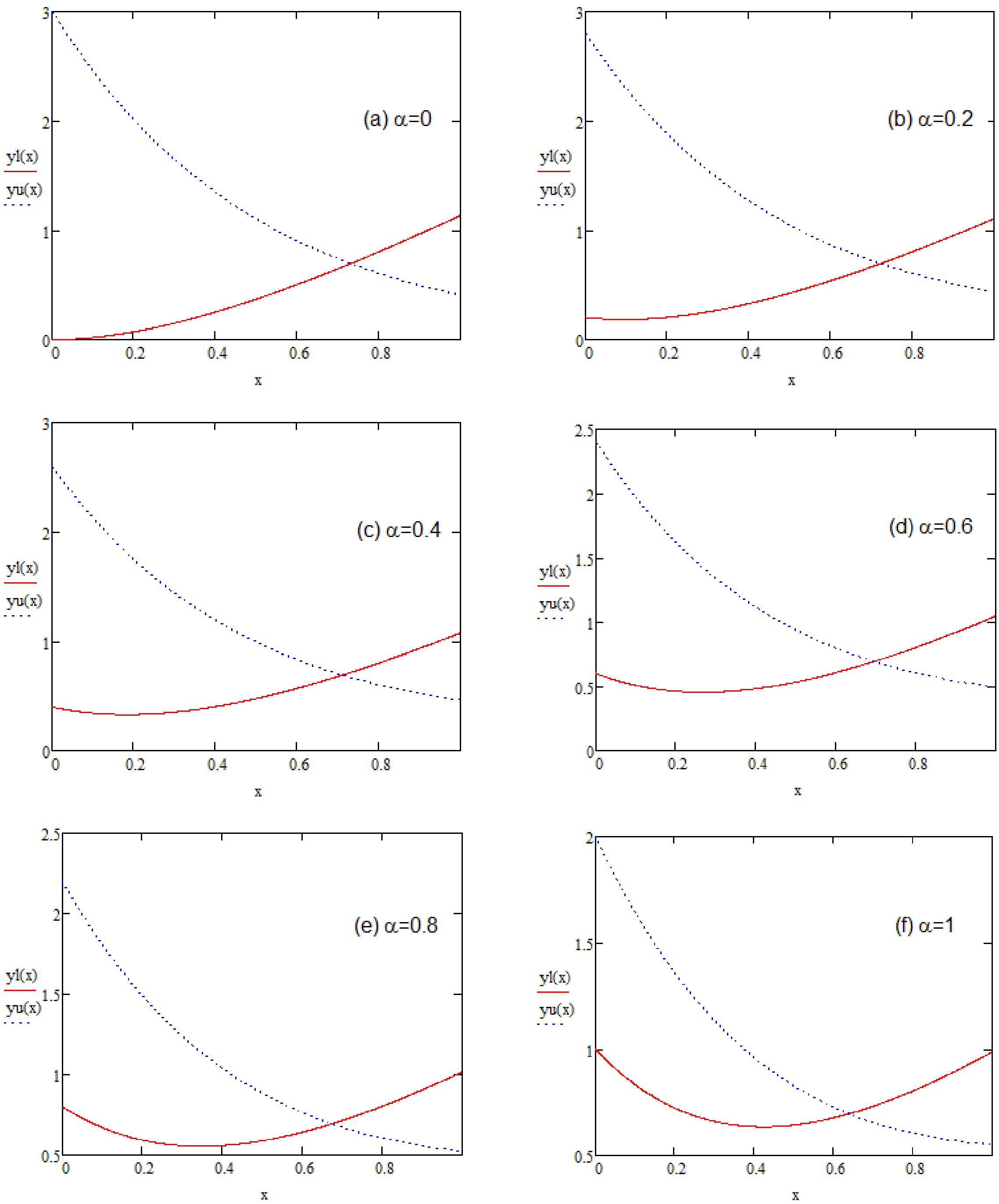

Figure 3. Comparison of the lower and upper bounds solutions of $\tilde{y}$ of Example 3.1 case (ii) obtained for different values of α.

4. Variational Iteration Method for Solving Fuzzy Interval Ordinary Differential Equations

Consider the FIODE (1)

$\tilde{y}' = \tilde{f}(x,y), \tilde{y}(x_0) = \tilde{y}_0$

with initial condition $\tilde{y}(0) = [\underline{y}_0, \overline{y}_0]$.

Using the VIM, the correction functional of the FIODE (1) for the upper and lower solutions of $\tilde{y}$ (namely $\tilde{y} = [\underline{y}, \overline{y}]$) will be read for all $n = 0, 1, \ldots$ as:

$$\underline{y}_{n+1}(x,\alpha) = \underline{y}_n(x,\alpha) + \int_0^x \underline{\lambda}(s,x)\left[\frac{d}{ds}\underline{y}_n(s,\alpha) + f(s,\underline{y}_n(s,\alpha),\overline{y}_n(s,\alpha)\right]ds \qquad \ldots(16)$$

$$\overline{y}_{n+1}(x,\alpha) = \overline{y}_n(x,\alpha) + \int_0^x \overline{\lambda}(s,x)\left[\frac{d}{ds}\overline{y}_n(s,\alpha) + f(s,\underline{y}_n(s,\alpha),\overline{y}_n(s,\alpha)\right]ds \qquad \ldots(17)$$

We obtain the Lagrange multiplier in the form:

$$\underline{\lambda}(s,x) = \overline{\lambda}(s,x) = -1 \qquad \ldots(18)$$

Hence starting with initial approximate solutions $\underline{y}_0(x,\alpha) = \underline{y}_0$, $\overline{y}_0(x,\alpha) = \overline{y}_0$ and when substituting Eqs. (18) back into Eqs. (16) and (17), we get respectively the following correction functionals using the VIM:

$$\underline{y}_{n+1}(x,\alpha) = \underline{y}_n(x,\alpha) - \int_0^x \left[\frac{d}{ds}\underline{y}_n(s,\alpha) + f(s,\underline{y}_n(s,\alpha),\overline{y}_n(s,\alpha)\right]ds \qquad \ldots(19)$$

$$\overline{y}_{n+1}(x,\alpha) = \overline{y}_n(x,\alpha) - \int_0^x \left[\frac{d}{ds}\overline{y}_n(s,\alpha) + f(s,\underline{y}_n(s,\alpha),\overline{y}_n(s,\alpha)\right]ds \qquad \ldots(20)$$

Example 4.1. Consider the FIODE:

$\tilde{y}' = -2\tilde{y} + [1,3]x, x \geq 0$

with initial condition $\tilde{y}(0) = [1,2]$.

Using the differentiation method described in Eqs. (3) and (4), implies upon applying the same approach in the derivation of Eq. (7) to find the α-level of the fuzzy intervals [1,3] and [1,2], and hence as previously stated, the following two cases can be obtained:

Case (i): If $\underline{y'}(x) \leq \overline{y'}(x)$, the using Eq. (3), we have:

$$\left.\begin{aligned} \underline{y'} &= -2\overline{y} + \alpha x \\ \overline{y'} &= -2\underline{y} + (4-\alpha)x \end{aligned}\right\} \qquad \ldots(21)$$

with initial conditions:

$$\underline{y}(0) = \alpha, \overline{y}(0) = 3 - \alpha \qquad \ldots(22)$$

Case (ii): If $\underline{y'}(x) \geq \overline{y'}(x)$, the using Eq. (4), we have:

$$\left.\begin{aligned} \underline{y'} &= -2\underline{y} + (4-\alpha)x \\ \overline{y'} &= -2\overline{y} + \alpha x \end{aligned}\right\} \qquad \ldots(23)$$

with the same initial conditions given in Eq. (18).

Now, applying the VIM to find the sequence of iterative solutions of the above FIODE:

Case (i): Let $\underline{y}_0(\alpha) = \alpha$ and $\overline{y}_0 = 3 - \alpha$, then using Eqs. (19) and (20) to find $\underline{y}_1$ and $\overline{y}_1$, as follows:

$$\begin{aligned} \underline{y}_1 &= \underline{y}_0 - \int_0^x \left[\frac{d}{ds}\underline{y}_0(s,\alpha) + f(s,\underline{y}_0(s,\alpha),\overline{y}_0(s,\alpha))\right]ds \\ &= \underline{y}_0 - \int_0^x \left[\frac{d}{ds}\underline{y}_0 + 2\overline{y}_0 - \alpha s\right]ds \\ &= \alpha - 6x + \frac{\alpha}{2}x^2 + 2\alpha x \end{aligned}$$

and

$$\begin{aligned} \overline{y}_1 &= \overline{y}_0 - \int_0^x \left[\frac{d}{ds}\overline{y}_0(s,\alpha) + f(s,\underline{y}_0(s,\alpha),\overline{y}_0(s,\alpha))\right]ds \\ &= \overline{y}_0 - \int_0^x \left[\frac{d}{ds}\overline{y}_0 + 2\underline{y}_0 - (4-\alpha)s\right]ds \\ &= 3 - \alpha - \int_0^x [0 + 2\alpha - (4-\alpha)s]ds \\ &= 2x^2 - \frac{\alpha}{2}x^2 - \alpha - 2\alpha x + 3 \end{aligned}$$

Similarly:

$$\begin{aligned} \underline{y}_2 &= \underline{y}_1 - \int_0^x \left[\frac{d}{ds}\underline{y}_1 + 2\overline{y}_1 - \alpha s\right]ds \\ &= \alpha + (2\alpha - 6)x + \frac{5\alpha}{2}x^2 + \frac{\alpha - 4}{3}x^3 \end{aligned}$$

$$\begin{aligned} \overline{y}_2 &= \overline{y}_1 - \int_0^x \left[\frac{d}{ds}\overline{y}_1 + 2\underline{y}_1 - (4-\alpha)s\right]ds \\ &= 3 - \alpha - 2\alpha x + \left(\frac{16 - 5\alpha}{2}\right)x^2 - \frac{\alpha}{3}x^3 \end{aligned}$$

$$\begin{aligned} \underline{y}_3 &= \underline{y}_2 - \int_0^x \left[\frac{d}{ds}\underline{y}_2 + 2\overline{y}_2 - \alpha s\right]ds \\ &= \alpha + (2\alpha - 6)x + \frac{5\alpha}{2}x^2 + \left(\frac{5\alpha - 16}{3}\right)x^3 + \frac{\alpha}{6}x^4 \end{aligned}$$

$$\begin{aligned} \overline{y}_3 &= \overline{y}_2 - \int_0^x \left[\frac{d}{ds}\overline{y}_2 + 2\underline{y}_2 - (4-\alpha)s\right]ds \\ &= 3 - \alpha - 2\alpha x + \left(\frac{16 - 5\alpha}{2}\right)x^2 - \frac{5\alpha}{3}x^3 + \left(\frac{4 - \alpha}{6}\right)x^4 \end{aligned}$$

$$\vdots$$

Accordingly, for $n = 4$ the lower and upper approximate interval fuzzy solutions are sketch for different values of α, which are 0, 0.2, 0.4, 0.6, 0.8 and 1 as they are given in Figure 4 (a-f), respectively.

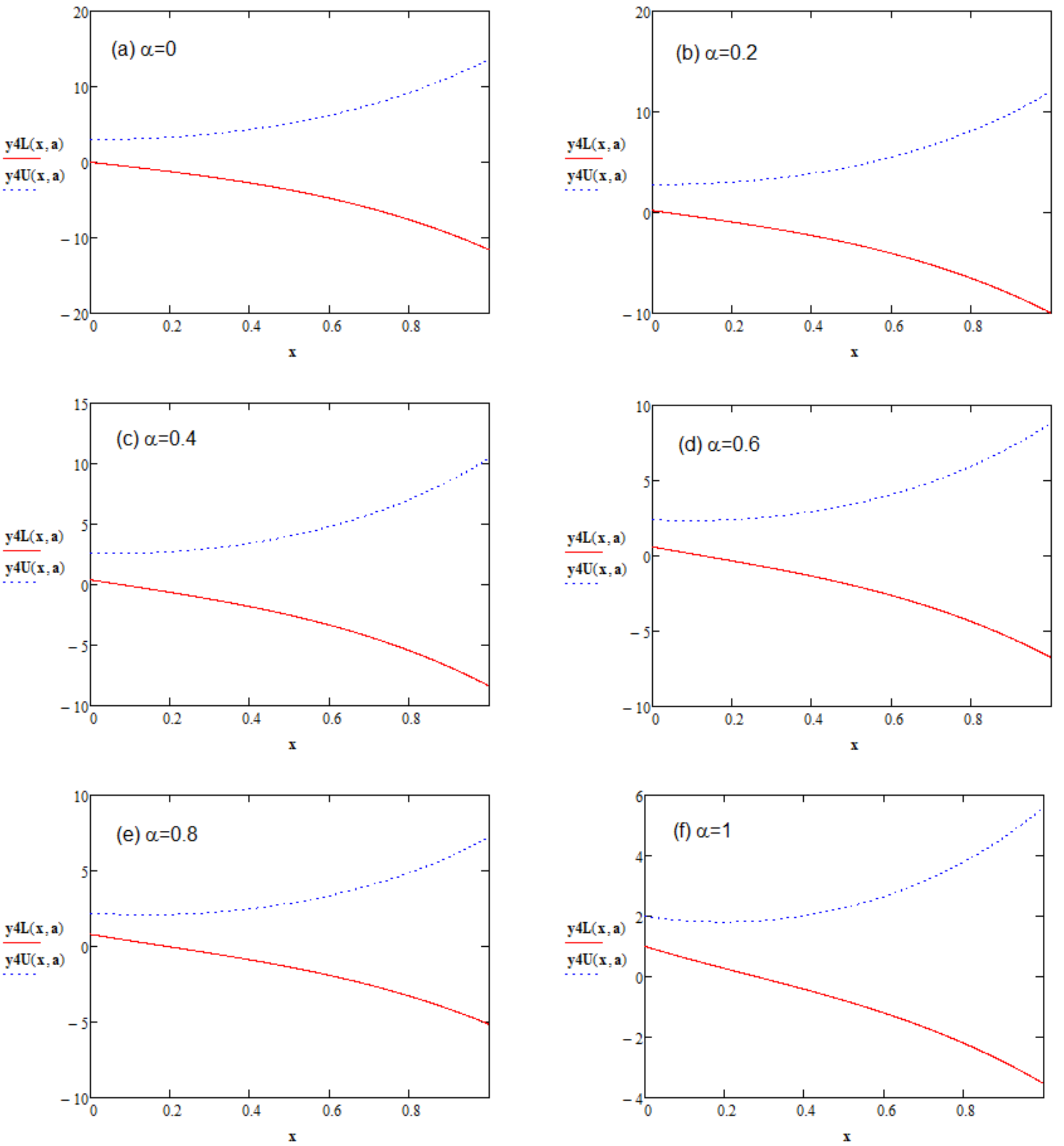

Figure 4. Approximate lower and upper solutions $\tilde{y}_4 = [\underline{y}_4, \bar{y}_4]$ of the Example 4.1 case (i) obtained for different values of α.

Case (ii): Let $\underline{y}_0(\alpha) = \alpha$ and $\bar{y}_0 = 3 - \alpha$, then using Eqs. (19) and (20), we get for $n = 0, 1, \ldots$:

$$\underline{y}_{n+1}(x,\alpha) = \underline{y}_n(x,\alpha) - \int_0^x \left[\frac{d}{ds}\underline{y}_n(s,\alpha) + 2\underline{y}_n(s,\alpha) - (a-\alpha)s \right] ds$$

$$\bar{y}_{n+1}(x,\alpha) = \bar{y}_n(x,\alpha) - \int_0^x \left[\frac{d}{ds}\bar{y}_n(s,\alpha) + 2\bar{y}_n(s,\alpha) - \alpha s \right] ds$$

and similarly as in case (i) above, for $n = 4$ the lower and upper approximate interval fuzzy solutions are sketch for different values of α, which are 0, 0.2, 0.4, 0.6, 0.8 and 1 as they are given in Figure 5 (a-f), respectively.

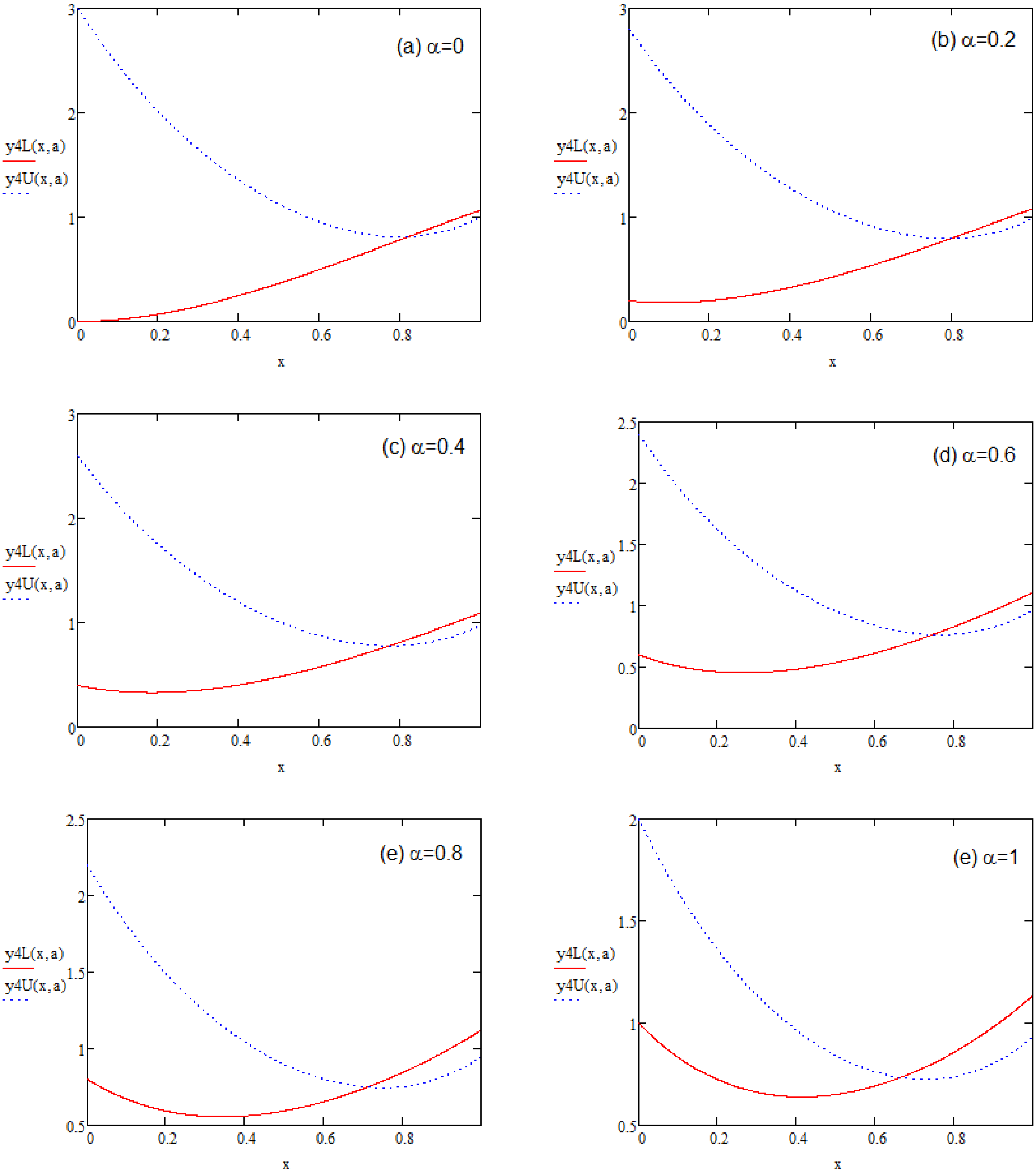

Figure 5. Approximate lower and upper solutions $\tilde{y}_4 = [\underline{y}_4, \bar{y}_4]$ of the Example 4.1 case (ii) obtained for different values of α.

5. Conclusions

Comparing the obtained results for each case, one can see the accuracy and the efficiency of the proposed approaches of solutions, either analytical or approximately which may be modified for solving nonlinear problems approximately using the VIM. But still the VIM may be difficult to be applied for nonlinear FIODE's and therefore a modified approach of the VIM may be needed.

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