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Some Models of the Finite Hyperbolic Geometry and the Finite Hyperbolic Plane

Jinan F. Al-Jobory

Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq

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Abstract

In this paper, two important models for the finite hyperbolic plane (finite Bolyai-Lobachevsky plane) $B_{n,m}$ will be given, the first model is when $n = 3$ and $m = 3$, while the second model is when $n = 3$ and $m = 4$.

Also, two important models for the finite hyperbolic geometry (finite Bolyai-Lobachevsky geometry) are given, the first model is when each line contains either 4 or 3 distinct points and each point is on 6 distinct lines, while the second model is when each line contains either 3 or 2 distinct points and each point is on either 7 or 8 lines. All models are represented in a simple form, which help the readers and researchers to understand the different facts about the finite Bolyai-Lobachevsky plane and the finite Bolyai-Lobachevsky geometry.

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Corresponding author: dr.jinan@uomustansiriyah.edu.iq

1. Introduction

In 1823 a Hungarian mathematician Janos Farakas Bolyai developed a new geometry, which he called it absolute geometry [3], but he didn't published the new geometry until 1832, during this period a Russian mathematician Nikolai Ivanovich Lobachevsky published the same work of Bolyai in 1829, the name of this geometry was changed by Felix Klein in 1871 to hyperbolic geometry. All the models and work of the hyperbolic geometry are infinite [1] till 1962 when L. M. Graves in his paper [4] gave a proposed axiomatic system for finite Bolyai-Lobachevsky plane.

In this paper, the finite Bolyai-Lobachevsky plane and the finite Bolyai-Lobachevsky geometry will be considered.

2. Preliminaries

In this section, basic concepts related to this work are presented, including definitions and axioms in hyperbolic geometry.

Definition 2.1. A finite hyperbolic geometry H consist of a finite collection of points arranged on a finite collection of lines in accordance to the following axioms:

Axiom 1. If ρ_1 and ρ_2 are two points, then there is one and only one line on both.

Axiom 2. There are at least n ($n \geq 2$) points on each line.

Axiom 3 (Bolyai-Lobachevsky postulate). If ρ is a point not on a line ℓ, then there are at least m ($m \geq 2$) distinct lines on ρ that do not intersect ℓ.

Axiom 4. There are four points in H no three of them on the same line.

Definition 2.2. A finite hyperbolic plane (or finite Bolyai-Lobachevsky plane) $B_{n,m}$ consist of a finite collection of points arranged on a finite collection of lines in accordance of the following axioms:

Axiom 1. If ρ_1 and ρ_2 are two points, then there is one and only one line on both.

Axiom 2. There are n ($n \geq 2$) points on each line.

Axiom 3 (Bolyai-Lobachevsky postulate). If ρ is a point not on a line ℓ, then there are m ($m \geq 2$) distinct lines on ρ that do not intersect ℓ.

Axiom 4. There are four points in $B_{n,m}$ no three of them on the same line.

Theorem 2.3 ([2], p.40). There are $m + n$ distinct lines on each point in $B_{n,m}$.

Theorem 2.4 ([2], p.40). There are $1+[m+n][n-1] = n^2 + mn - n - m + 1$ distinct points in $B_{n,m}$.

Theorem 2.5 ([2], p.40). There are $\frac{m+n}{n}[1+[m+n][n-1]]$ distinct lines in $B_{n,m}$.

3. Some Models of the Finite Hyperbolic Plane $B_{n,m}$

In the finite hyperbolic plane $B_{n,m}$, some models may be suggested, and among them are those presented next:

Model 3.1. If $\ell_1, \ell_2, \ldots, \ell_{21}$ are lines and $\rho_1, \rho_2, \ldots, \rho_7$ are points, then the following Table 1 which shows the points on the lines $\ell_1, \ell_2, \ldots, \ell_{21}$ will be a Model of $B_{2,4}$:

Table 1

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_1
ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_3

ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}
ρ_2	ρ_3	ρ_4	ρ_5	ρ_1	ρ_2	ρ_3
ρ_4	ρ_5	ρ_6	ρ_7	ρ_4	ρ_5	ρ_6

ℓ_{15}	ℓ_{16}	ℓ_{17}	ℓ_{18}	ℓ_{19}	ℓ_{20}	ℓ_{21}
ρ_4	ρ_1	ρ_2	ρ_3	ρ_1	ρ_2	ρ_1
ρ_7	ρ_5	ρ_6	ρ_7	ρ_6	ρ_7	ρ_7

Model 3.2. If $\ell_1, \ell_2, \ldots, \ell_{26}$ are lines and $\rho_1, \rho_2, \ldots, \rho_{13}$ are points, then the following Table 2 which shows the points on the lines $\ell_1, \ell_2, \ldots, \ell_{26}$ will be a Model of $B_{3,3}$:

Table 2

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_{13}	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8

ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}	ℓ_{15}	ℓ_{16}	ℓ_{17}	ℓ_{18}
ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_6	ρ_7	ρ_8	ρ_9	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}

ℓ_{19}	ℓ_{20}	ℓ_{21}	ℓ_{22}	ℓ_{23}	ℓ_{24}	ℓ_{25}	ℓ_{26}
ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}
ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_1	ρ_2
ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8

Model 3.3. If $\ell_1, \ell_2, \ldots, \ell_{35}$ are lines and $\rho_1, \rho_2, \ldots, \rho_{15}$ are points, then the following table 3 which shows the points on the lines $\ell_1, \ell_2, \ldots, \ell_{35}$ will be a Model of $B_{3,4}$:

Table 3

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}
ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}

ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}	ℓ_{15}	ℓ_{16}	ℓ_{17}	ℓ_{18}
ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_1	ρ_2	ρ_3
ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_1	ρ_7	ρ_8	ρ_9
ρ_{14}	ρ_{15}	ρ_1	ρ_2	ρ_3	ρ_4	ρ_{14}	ρ_{15}	ρ_1

ℓ_{19}	ℓ_{20}	ℓ_{21}	ℓ_{22}	ℓ_{23}	ℓ_{24}	ℓ_{25}	ℓ_{26}	ℓ_{27}
ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}
ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_1	ρ_2	ρ_3
ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}

ℓ_{28}	ℓ_{29}	ℓ_{30}	ℓ_{31}	ℓ_{32}	ℓ_{33}	ℓ_{34}	ℓ_{35}
ρ_{13}	ρ_{14}	ρ_{15}	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_4	ρ_5	ρ_6	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}
ρ_{11}	ρ_{12}	ρ_{13}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}

4. Some Models of the Finite Hyperbolic Geometry

Similarly, as in section 3, some models related to finite hyperbolic geometry are suggested in the next.

Model 4.1. If $\ell_1, \ell_2, \dots, \ell_{28}$ are lines and $\rho_1, \rho_2, \dots, \rho_{16}$ are points, then the following table 4 which shows the points on the lines $\ell_1, \ell_2, \dots, \ell_{28}$ will be a Model of the Finite Hyperbolic Geometry (finite Bolyai-Lobachevsky geometry):

Table 4

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
ρ_1	ρ_5	ρ_9	ρ_{13}	ρ_1	ρ_2	ρ_3	ρ_7	ρ_4
ρ_2	ρ_6	ρ_{10}	ρ_{14}	ρ_5	ρ_6	ρ_{10}	ρ_{11}	ρ_8
ρ_3	ρ_7	ρ_{11}	ρ_{15}	ρ_9	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{12}
ρ_4	ρ_8	ρ_{12}	ρ_{16}					ρ_{16}

ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}	ℓ_{15}	ℓ_{16}	ℓ_{17}	ℓ_{18}
ρ_2	ρ_3	ρ_8	ρ_4	ρ_1	ρ_3	ρ_5	ρ_4	ρ_1
ρ_{11}	ρ_7	ρ_9	ρ_5	ρ_6	ρ_8	ρ_{10}	ρ_6	ρ_{12}
ρ_{16}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{10}	ρ_{15}	ρ_{16}	ρ_{11}	ρ_{13}
				ρ_{15}				

ℓ_{19}	ℓ_{20}	ℓ_{21}	ℓ_{22}	ℓ_{23}	ℓ_{24}	ℓ_{25}	ℓ_{26}	ℓ_{27}	ℓ_{28}
ρ_2	ρ_6	ρ_4	ρ_1	ρ_2	ρ_3	ρ_4	ρ_1	ρ_2	ρ_3
ρ_7	ρ_{12}	ρ_9	ρ_7	ρ_8	ρ_5	ρ_7	ρ_8	ρ_5	ρ_6
ρ_9	ρ_{14}	ρ_{15}	ρ_{16}	ρ_{10}	ρ_{11}	ρ_{10}	ρ_{11}	ρ_{12}	ρ_9
ρ_{14}					ρ_{13}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{16}

Remark 4.2. In Model 4.1. above there are 6 distinct lines on each point. Thus any point ρ not on line ℓ containing four points, there are 4 distinct lines on ρ that intersect ℓ and 2 distinct lines on ρ parallel to ℓ (not intersecting ℓ). While any point ρ not on line ℓ containing three points, there are 3 distinct lines on ρ that intersect ℓ and 3 distinct lines on ρ parallel to ℓ (not intersecting ℓ).

Model 4.3. If $\ell_1, \ell_2, \dots, \ell_{43}$ are lines and $\rho_1, \rho_2, \dots, \rho_{15}$ are points, then the following table 5 which shows the points on the lines $\ell_1, \ell_2, \dots, \ell_{43}$ will be a Model of the Finite Hyperbolic Geometry (finite Bolyai-Lobachevsky geometry):

Table 5

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
ρ_1	ρ_4	ρ_7	ρ_{10}	ρ_{13}	ρ_1	ρ_4	ρ_7	ρ_{10}
ρ_2	ρ_5	ρ_8	ρ_{11}	ρ_{14}	ρ_5	ρ_8	ρ_{11}	ρ_{14}
ρ_3	ρ_6	ρ_9	ρ_{12}	ρ_{15}	ρ_9	ρ_{12}	ρ_{15}	ρ_3

ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}	ℓ_{15}	ℓ_{16}	ℓ_{17}	ℓ_{18}
ρ_{13}	ρ_1	ρ_4	ρ_7	ρ_{10}	ρ_{13}	ρ_1	ρ_4	ρ_7
ρ_2	ρ_8	ρ_{11}	ρ_{14}	ρ_2	ρ_5	ρ_{11}	ρ_{14}	ρ_2
ρ_6	ρ_{15}	ρ_3	ρ_6	ρ_9	ρ_{12}	ρ_6	ρ_9	ρ_{12}

ℓ_{19}	ℓ_{20}	ℓ_{21}	ℓ_{22}	ℓ_{23}	ℓ_{24}	ℓ_{25}	ℓ_{26}	ℓ_{27}
ρ_{10}	ρ_{13}	ρ_1	ρ_4	ρ_7	ρ_{10}	ρ_{13}	ρ_1	ρ_1
ρ_5	ρ_8	ρ_{14}	ρ_2	ρ_5	ρ_8	ρ_{11}	ρ_4	ρ_{10}
ρ_{15}	ρ_3	ρ_{12}	ρ_{15}	ρ_3	ρ_6	ρ_9	ρ_7	ρ_{13}

ℓ_{28}	ℓ_{29}	ℓ_{30}	ℓ_{31}	ℓ_{32}	ℓ_{33}	ℓ_{34}	ℓ_{35}	ℓ_{36}
ρ_2	ρ_2	ρ_3	ρ_3	ρ_4	ρ_4	ρ_7	ρ_7	ρ_8
ρ_8	ρ_5	ρ_{12}	ρ_9	ρ_{10}	ρ_{13}	ρ_{10}	ρ_{13}	ρ_5
ρ_{14}	ρ_{11}	ρ_6	ρ_{15}					

ℓ_{37}	ℓ_{38}	ℓ_{39}	ℓ_{40}	ℓ_{41}	ℓ_{42}	ℓ_{43}
ρ_8	ρ_{12}	ρ_{12}	ρ_{14}	ρ_{14}	ρ_6	ρ_6
ρ_{11}	ρ_9	ρ_{15}	ρ_5	ρ_{11}	ρ_9	ρ_{15}

Remark 4.4. In Model 4.3. We have the following:

1. There are exactly 7 distinct lines on each Point ρ_1, ρ_2, ρ_3.
2. There are exactly 8 distinct lines on each Point $\rho_4, \rho_5, \dots, \rho_{15}$.
3. There are 3 points on each of the lines $\ell_1, \ell_2, \dots, \ell_{31}$.
4. There are 2 points on each of the lines $\ell_{32}, \ell_{33}, \dots, \ell_{43}$.
5. If $\rho \in \{\rho_1, \rho_2, \rho_3\}$ and ρ is not on any of Lines $\ell_1, \ell_2, \dots, \ell_{31}$. Then there are 3 distinct lines on ρ that intersect ℓ and exactly 4 distinct lines on ρ parallel to ℓ (not intersecting ℓ).

6. If $\rho \in \{\rho_1, \rho_2, \rho_3\}$ and ρ is not on any of Lines $\ell_{32}, \ell_{33}, \ldots, \ell_{43}$. Then there are exactly 2 distinct lines on ρ that intersect ℓ and exactly 5 distinct lines on ρ parallel to ℓ (not intersecting ℓ).
7. If $\rho \in \{\rho_4, \rho_5, \ldots, \rho_{15}\}$ and ρ is not on any of Lines $\ell_1, \ell_2, \ldots, \ell_{31}$. Then there are 3 distinct lines on ρ that intersect ℓ and 5 distinct lines on ρ parallel to ℓ (not intersecting ℓ).
8. If $\rho \in \{\rho_4, \rho_5, \ldots, \rho_{15}\}$ and ρ is not on any of Lines $\ell_{32}, \ell_{33}, \ldots, \ell_{43}$. Then there are 2 distinct lines on ρ that intersect ℓ and 6 distinct lines on ρ parallel to ℓ (not intersecting ℓ).

5. Conclusions

From this work, the following conclusions may be drawn:

1. There is no finite hyperbolic plane $B_{n,m}$, if $\frac{m+n}{n}[1+[m+n][n-1]]$ is not an integer.
2. $\frac{m+n}{n}[1+[m+n][n-1]]$ is an integer doesn't mean that there is a model of $B_{n,m}$, and if there is a model then for $n \geq 3$ and $m \geq 3$ it is so difficult to find.
3. In a model of the finite hyperbolic geometry the number of distinct points on each line is not necessarily the same.
4. In a model of the finite hyperbolic geometry the number of distinct lines on each point is not necessarily the same.
5. In a model of the finite hyperbolic geometry the number of distinct parallel lines on each point not on a line is not necessarily the same.

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