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Modified Variational Iteration Method for Solving Sedimentary Ocean Basin Moving Boundary Value Problem

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Abstract

The motion of the shoreline in sedimentary ocean basin were investigated and solved using the modified variational iteration method since it provides an efficient approximate solution of the problem under consideration, which appears to be accurate when the consecutive error between iterative solutions of the problem is evaluated. A diffusion equation with specified beginning and boundary conditions, with a moving interface type reflecting sediment transportation, is used to simulate the physical phenomena of sand particle migration in sea water. This article also discusses the shifting boundary value problem that occurs from shoreline change in a sedimentary ocean basin.

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1. Introduction

It is commonly known that several nonlinear partial differential equations are employed in the study of many domains such as physics, mechanics, and so on. The answers to these equations can help you better comprehend the given procedure. However, due to the limitations of mathematical tools and the difficulty of nonlinear partial differential equations, accurate solutions to these problems are difficult to obtain. As a result of Because of this complexity, more applications of nonlinear partial differential equations are possible [1-4]. Backlund transformation and other analytical and numerical approaches were employed to solve these challenges [5], Hirota's bilinear technique [6,7], Darboux transformation [8,9], symmetry method [10,11], tanh method [10,11], Adomian decomposition method [12–16], improved Adomian decomposition method [17], and additional asymptotic approaches for extremely nonlinear equations [18,19].

In 1978, Inokuti et. al [20] suggested a general Lagrange multiplier approach for solving nonlinear problems, with the purpose of resolving quantum mechanics problems He [21] later converted the approach to an iterative method and termed it variational iteration method (VIM) in 1997 many writers have presented it as a strong mathematical tool for tackling many forms of nonlinear problems that span a wide range of current science areas [22-28]. In 2007, Abassy et. al introduced the modified variational iteration technique (MVIM) and utilized it to provide approximate power series solutions for a number of well-known nonlinear issues [29,30]. The modified variational iteration technique (MVIM) both simplifies and reduces computing labor. This strategy can significantly enhance convergence speed. Abassy et. al presented further treatments for MVIM data utilizing "Padé approximants" and the "Laplace transform". In some circumstances, the technique enhances convergence and provides a closed form solution see [31,32]. The goal of this endeavor is to launch a new MVIM application It is used to solve non-linear non-homogeneous differential equations. The MVIM may be used to provide numerical solutions to nonlinear differential equations, and writing computer codes is simple. The modified version modifies the formulation of the variational iteration relation and offers a qualitative improvement over the regular VIM. Furthermore, because MVIM does not require variable discretization, it does not take as much computer memory or time as VIM does. The methodology shares traits with many other methodologies, but upon closer examination, it is quite different, and seeming simplicity should not be utilized to draw hasty conclusions.

In this paper, we will study and find the solution of the moving boundary value problem that emerges as the shoreline that moves in a sedimentary ocean basin. The MVIM is used successfully to tackle the considered problem of this work.

2. Problem Formulation

The fluvio-deltaic sedimentary issue entails determining the propagation of shorelines in a sedimentary ocean basin as a result of sediment line flow and tectonic subsidence of the earth's crust, and sea level change with the assumptions of a fixed line flow and a constant ocean level at $z = 0$, also no tectonic subsidence of the earth's crust, and a constant sloping basement $b < a$ where a is the slope of off-shore sediment wedge and b is the slope of basement. This hypothesis is a reasonable approximation for some modern continental margins as shown in Figure 1.

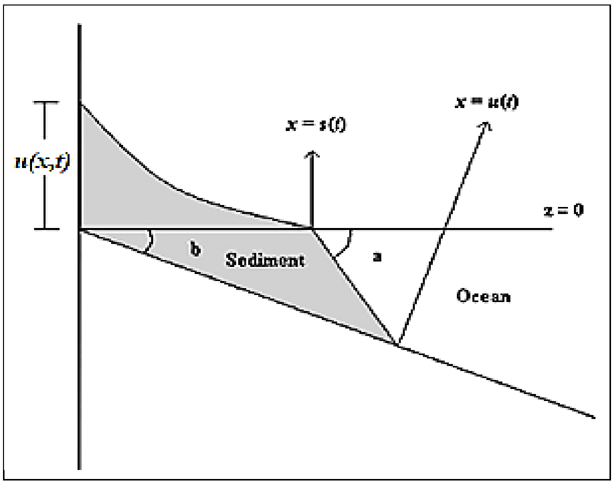

Figure 1. A schematic cross section of a basin with no tectonic subsidence and sea level change, [33].

The dynamics of the sedimentation process, in this case becomes a moving boundary value problem with variable latent heat equation as the governing equation [34], which may be modeled as follows:

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}, 0 < x < s(t), t \geq 0 \quad \ldots(1)$$

With initial and boundary conditions

$$v\frac{\partial u}{\partial x}\Big|_{x=0} = -q(t) \quad \ldots(2)$$

and

$$u(s,t) = 0 \quad \ldots(3)$$

where $u(s,t)$ is the height of sediment above datum, v is a diffusion coefficient, $q(t)$ is the time-dependent sediment line flow, and $s(t)$ is the moving contact boundary or moving interface. In order to be capable to solve this problem some additional conditions on the moving interface are introduced, namely

$$-v\frac{\partial u}{\partial x}\Big|_{x=s(t)} = \gamma s\frac{ds}{dt} \quad \ldots(4)$$

and

$$s(0) = 0 \quad \ldots(5)$$

3. The Modified Approach

According to the VIM, consider the following general nonlinear equation in operator form:

$$L[u(x,t)] + N[u(x,t)] = g(x,t) \quad \ldots(6)$$

where L is a linear operator, N is a nonlinear operator, $g(x,t)$ is a given continuous function. Depending on the VIM, we may construct a correction functional of the form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(w,t)\{L[u_n(w,t)] + N[\tilde{u}_n(w,t)] - g(w,t)\}dw, n = 0,1,\ldots \quad \ldots(7)$$

where $\tilde{u}_n$ is considered as a restricted variation, i.e.; $\delta\tilde{u}_n = 0$ and λ is the general Lagrange multiplier, which can be identified optimally via the variational theory.

Now, the new approach, which is termed in this paper as the modified approach, may be introduced by modifying the linear operator related to problem given by:

$$L\{u(t)\} + N\{u(t)\} = g(t), t \geq t_0 \quad \ldots(8)$$

with initial and boundary conditions given by eqs. (2)-(5), the correction functional (7) will results that the Lagrange multiplier will be modified, i.e.; introducing an arbitrary linear operator L_1 of $u(x,t)$, by adding and subtracting this linear operator L_1 from eq.(6), then eq.(6) may be rewritten as:

$$L[u(w,t)] + L_1[u(w,t)] - L_1[u(w,t)] + N[u(w,t)] = g(x,t) \quad \ldots(9)$$

Hence, the correction functional may be constructed based on the following new or renamed operators:

$$\bar{L}[u(w,t)] = L[u(w,t)] + L_1[u(w,t)]$$
$$\bar{N}[\tilde{u}(w,t)] = -L_1[\tilde{u}(w,t)] + N[\tilde{u}(w,t)]$$

and thus the new correction functional related to eq.(9) will be:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(w,t)\{L[u_n(w,t)] + L_1[u_n(w,t)] - L_1[\tilde{u}(w,t)] + N[\tilde{u}_n(w,t)] - g(w,t)\}\,dw$$
$$= u_n(x,t) + \int_0^x \lambda(w,t)\{\bar{L}[u_n(w,t)] + \bar{N}[\tilde{u}_n(w,t)] - g(w,t)\}dw, n = 0,1,\ldots \quad \ldots(10)$$

where $\tilde{u}_n$ is considered here as a restricted variation, i.e.; $\delta\tilde{u}_n = 0$. The general Lagrange multiplier λ obtained from the correction functional (10) differs from that obtained from eq. (9). Consequently, we can arbitrarily choose the auxiliary linear operator to improve the accuracy and the rate of convergence to the exact solution. This provides a great freedom in applying the VIM to the nonlinear problems. Similar strategy was also suggested by Ji-Huan He for nonlinear oscillators, [22].

The analysis of evaluating λ starts by considering first the following correction functional with respect to x:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(w,t)\left[\frac{\partial^2 u(w,t)}{\partial w^2} + u'(w,t) - u'(w,t) - \frac{1}{v}\frac{\partial\tilde{u}(w,t)}{\partial t}\right]ds \quad \ldots(11)$$

where $\tilde{u}_n$ is the restricted variational and thus by taking the first variation with respect to the independent variable u_n and noticing that $\delta u_n = 0$, one may get for $n = 0, 1, \ldots$:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^x \lambda(w,t)\left[u_n''(w,t) + u'(w,t) - u'(w,t) - \frac{1}{v}\frac{\partial\tilde{u}(w,t)}{\partial t}\right]dw \quad \ldots(12)$$

and consequently:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^x \lambda(w,t)[u_n''(w,t) + u'(w,t)]dw \quad \ldots(13)$$

and by using twice the method of integration by parts, then eq. (13) will be reduced to:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(w,t)\delta u_n'(w,t)|_{s=x} - \lambda'(w,t)\delta u_n(x,t)|_{w=x} + \int_0^x \lambda''(w,t)\delta u_n(w,t)ds + \lambda(w,t)\delta u_n(w,t)|_{w=x} - \int_0^x \lambda'(w,t)\delta u_n(w,t)dw$$

and hence:

$$\delta u_{n+1}(x,t) = \left(1-\lambda'(w,t)+\lambda(w,t)\right)\Big|_{w=x}\delta u_n(x,t) + \lambda(w,t)|_{w=x}\delta u_n'(w,t) + \int_0^x \lambda''(w,t)\delta u_n(w,t)ds + \int_0^x \lambda'(w,t)\delta u_n(w,t)\,dw = 0$$

which yields to:

$$\lambda''(w,t) - \lambda'(w,t) = 0 \qquad \ldots(14)$$

with the initial conditions:

$$1-\lambda'(w,t)+\lambda(w,t)|_{w=x} = 0, \lambda(w,t)|_{w=x} = 0 \qquad \ldots(15)$$

Solving the second order initial value problem (14) with initial conditions (15) will yields the following value of the general Lagrange multiplier:

$$\lambda(w,t) = e^{w-x} - 1$$

Hence, the modified form of the correction functional which will be used to solve problem is:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (e^{w-x}-1)\left[\frac{\partial^2 u_0(w,t)}{\partial w^2} - \frac{1}{v}\frac{\partial u_0(w,t)}{\partial t}\right]dw \qquad \ldots(16)$$

4. Numerical Simulation of the Problem

The findings for sediment height $u(x,t)$ and shoreline $s(t)$ are computed using MATHCAD15 computer program with the following parameters, namely $v = 2$, final time $t = 3$ and for different values of γ and q, which are $\gamma = 10, 15$ and $q = 0.5, 1$ respectively. Also, the results are scheduled in tables and are illustrated in figures.

By choosing the initial approximations of $u(x,t)$ and $s(t)$ to be as follows [33]:

$$u_0(x,t) = c(s_0 - x) \text{ and } s_0(t) = a_0 t^{\frac{1}{2}}$$

where $c = \frac{q(t)}{v}$ and $a_0 = \left(\frac{cv}{\gamma}\right)^{\frac{1}{2}}$

We can evaluate the first approximate solution as:

$$u_1(x,t) = u_0(x,t) + \int_0^x (e^{w-x}-1)\left[v\frac{\partial^2 u_0(w,t)}{\partial w^2} - \frac{\partial u_0(w,t)}{\partial t}\right]dw$$

$$= c(s_0 - x) + \int_0^x (e^{w-x}-1)\left[0 - \frac{q(t)\sqrt{\frac{cv}{\gamma}}}{2\sqrt{t}\,v}\right]dw$$

$$= c(s_0 - x) + \frac{q(t)\sqrt{cv}(x+e^{-x}-1)}{2\sqrt{\gamma}\sqrt{t}\,v}$$

While the second approximate solution is:

$$u_2(x,t) = u_1(x,t) + \int_0^x (e^{w-x}-1)\left[v\frac{\partial^2 u_1(w,t)}{\partial w^2} - \frac{\partial u_1(w,t)}{\partial t}\right]dw$$

$$= c(s_0 - x) + \frac{q(t)\sqrt{cv}(x+e^{-x}-1)}{2\sqrt{\gamma}\sqrt{t}\,v} + \int_0^x (e^{w-x}-1)\left[\frac{q(t)e^{-x}\sqrt{cv}}{2\sqrt{\gamma}\sqrt{t}} - \frac{q(t)\sqrt{cv}}{2\sqrt{\gamma}\sqrt{t}\,v} - \frac{q(t)e^{-x}\sqrt{cv}}{2\sqrt{\gamma}\sqrt{t}\,v}\right]dw$$

$$= c(s_0 - x) + \frac{q(t)\sqrt{cv}(x+e^{-x}-1)}{2\sqrt{\gamma}\sqrt{t}\,v} - \frac{q(t)\sqrt{cv}(e^{-x}+ve^{-x}-1)}{2\sqrt{\gamma}\sqrt{t}\,v} - \frac{xq(t)\sqrt{cv}(e^{-x}+ve^{-x}-1)}{2\sqrt{\gamma}\sqrt{t}\,v}$$

By the same manipulation, we have the third approximate solution

$$u_3(x,t) = u_2(x,t) + \int_0^x (e^{w-x}-1)\left[v\frac{\partial^2 u_2(w,t)}{\partial w^2} - \frac{\partial u_2(w,t)}{\partial t}\right]dw$$

$$= c(s_0 - x) + \frac{q(t)\sqrt{cv}(x+e^{-x}-1)}{2\sqrt{\gamma}\sqrt{t}\,v} + \int_0^x (e^{s-x}-1)\left[v\frac{\partial^2 u_2(w,t)}{\partial w^2} - \frac{\partial u_2(w,t)}{\partial t}\right]dw$$

$$= \frac{1}{2\sqrt{\gamma}\sqrt{t}\,v}[(q(t)\sqrt{cv}(x+e^{-x}-1)(2e^{-2x} - 3e^{-x} + 2ve^{-x}2ve^{-2x} + xe^{-x} + 3v^2e^{-x} - 4v^2e^{-2x} - v^2xe^{-x} + 1)]$$

Same procedure we can find u_3 and u_4, which are found to be $u_4(x,t)$.

The second important part of the problem is to find the moving interface $s(t)$, which fulfills condition (4), since this condition gives the relation between the moving interface $s(t)$ and the height of the sediment $u(x,t)$, we can find the moving interface by using VIM for evaluating as follows:

$$s_{n+1}(t) = s_n(t) - \int_0^t \left[\gamma s_n(t)\frac{ds_n(t)}{dt} + v\frac{\partial u}{\partial x}\Big|_{x=s_n(t)}\right]ds$$

Tables 1-4 depicts the dependence of sediment height $u(x,t)$ on space increment x for a standard moving boundary value problem at fixed value of diffusion coefficient $v = 2$ and different sediment line flux $q = 0.5, 1$ and time $t = 3$ for $\gamma = 10, 15$, respectively.

Table 1. Sediment height relative to the space increment, with $v = 2$, $q = 0.5$, $\gamma = 10$ and $t = 3$.

x	$u_0(x,t)$	$u_1(x,t)$	$u_2(x,t)$	$u_3(x,t)$	$u_4(x,t)$
0	0.09682458	0.09682458	0.09682458	0.09682458	0.09682458
0.1	0.07182458	0.07190265	0.07182969	0.07189779	0.0718343
0.2	0.04682458	0.04712685	0.04686339	0.04709183	0.04689492
0.3	0.02182458	0.02248328	0.02194906	0.02237683	0.02203958
0.4	–0.00317542	–0.00204063	–0.00289488	–0.00226758	–0.00271322
0.5	–0.02817542	–0.02645629	–0.02765424	–0.02685447	–0.02735539
0.6	–0.05317542	–0.05077398	–0.0523184	–0.05139116	–0.05188609
0.7	–0.07817542	–0.07500303	–0.07687977	–0.0758807	–0.07630925
0.8	–0.10317542	–0.09915189	–0.10133332	–0.10032297	–0.10063159
0.9	–0.12817542	–0.12322817	–0.12567614	–0.12471562	–0.12486118

Table 2. Sediment height relative to the space increment, with $v = 2$, $q = 0.5$, $\gamma = 15$ and $t = 3$.

x	$u_0(x,t)$	$u_1(x,t)$	$u_2(x,t)$	$u_3(x,t)$	$u_4(x,t)$
0	0.07905694	0.07905694	0.07905649	0.07905694	0.07905694
0.1	0.05405694	0.05412068	0.05406111	0.05411672	0.05406488
0.2	0.02905694	0.02930374	0.02908863	0.02927515	0.02911437
0.3	0.04056942	0.00459477	0.00415858	0.00450786	0.00423248
0.4	–0.02094306	–0.02001651	–0.020714	–0.02020181	–0.02056566
0.5	–0.04594306	–0.04453939	–0.04551752	–0.04486451	–0.0452735
0.6	–0.07094306	–0.06898229	–0.07024331	–0.06948622	–0.06989034
0.7	–0.09594306	–0.09335282	–0.09488517	–0.09406943	–0.09441935

Table 3. Sediment height relative to the space increment, with $v = 2$, $q = 1$, $\gamma = 10$ and $t = 3$.

x	$u_0(x,t)$	$u_1(x,t)$	$u_2(x,t)$	$u_3(x,t)$	$u_4(x,t)$
0	0.27386128	0.27386128	0.27386127	0.27386127	0.27386127
0.1	0.22386128	0.22408208	0.22387575	0.22406831	0.22388879
0.2	0.17386128	0.17471622	0.17397152	0.17461674	0.17406061
0.3	0.12386128	0.12572437	0.12421564	0.12542129	0.12447114
0.4	0.07386128	0.07707094	0.07466169	0.07642314	0.0751736
0.5	0.02386128	0.02872372	0.02535169	0.02758421	0.02619141
0.6	–0.02613872	–0.01934643	–0.0268217	–0.02111748	–0.02247239
0.7	–0.07613872	–0.06716587	–0.07241601	–0.06969168	–0.07082855
0.8	–0.12613872	–0.11475846	–0.12083302	–0.11813907	–0.1188959
0.9	–0.17613872	–0.16214579	–0.1689223	–0.16645386	–0.16669755
1	–0.22613872	–0.2093474	–0.21667788	–0.21462581	–0.21425822
1.1	–0.27613872	–0.25638095	–0.26409792	–0.26264195	–0.2616017
1.2	–0.32613872	–0.30326244	–0.31118394	–0.31048784	–0.30874985

Table 4. Sediment height relative to the space increment, with $v = 2$, $q = 1$, $\gamma = 15$ and $t = 3$.

x	$u_0(x,t)$	$u_1(x,t)$	$u_2(x,t)$	$u_3(x,t)$	$u_4(x,t)$
0	0.2236068	0.22360679	0.22360679	0.22360679	0.22360679
0.1	0.1736068	0.17378707	0.17361858	0.17377586	0.17362924
0.2	0.1236068	0.12430484	0.12369642	0.12422397	0.12376922
0.3	0.0736068	0.07512799	0.07389427	0.07488219	0.07410329
0.4	0.0236068	0.02622746	0.02425466	0.02570335	0.02467422
0.5	−0.0263932	−0.02242305	−0.0251896	−0.02334262	−0.02449942
0.6	−0.0763932	−0.07084733	−0.07441401	−0.07227266	−0.07341566
0.7	−0.1263932	−0.11906691	−0.12340104	−0.12109379	−0.12208352
0.8	−0.1763932	−0.16710127	−0.17213907	−0.16980576	−0.1705185
0.9	−0.2263932	−0.21496803	−0.22062139	−0.21840315	−0.21873938
1	−0.2763932	−0.26268315	−0.26884538	−0.26687711	−0.26676588
1.1	−0.3263932	−0.31026106	−0.31681181	−031521669	−0.31461723

Figures 2, 4, 6 and 8 depicts the dependence of sediment height $u(x,t)$ on space increment x for a standard moving boundary value problem at fixed value of diffusion coefficient $v = 2$ and different sediment line flux $q = 0.5, 1,$ and time $t = 3$ for $\gamma = 10, 15$, whereas Figures 3, 5, 7, and 9 depicts the dependence of shoreline position on time at the fixed value of an approximate solution to a moving boundary problem.

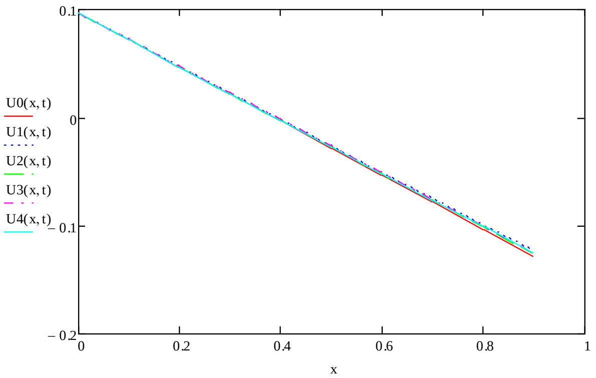

Figure 2. Plot of u(x, t) versus x for $q = 0.5$ and $\gamma = 10$.

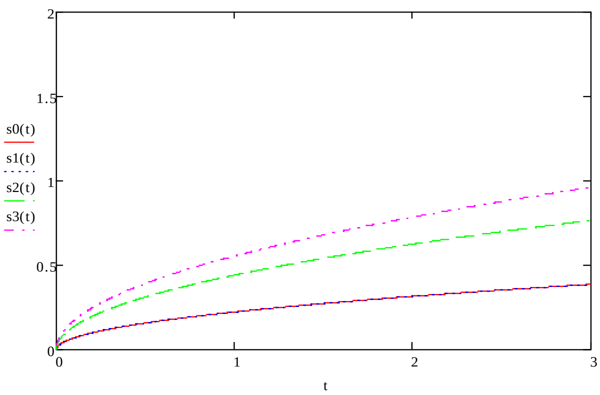

Figure 3. Plot of $s(t)$ with respect to t for $q = 0.5$ and $\gamma = 10$.

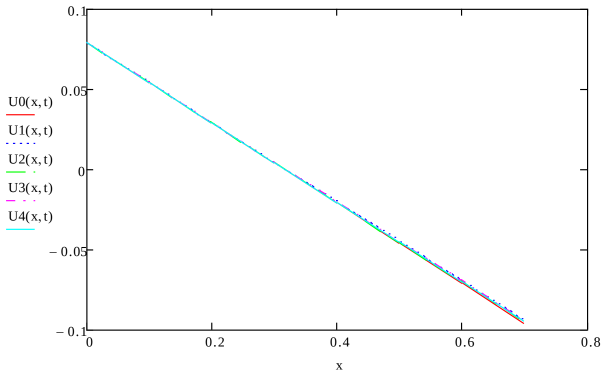

Figure 4. Plot of $u(x,t)$ versus x for $q = 0.5$ and $\gamma = 15$.

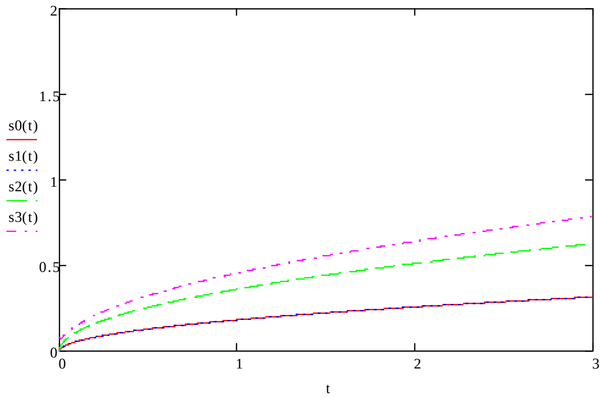

Figure 5. Plot of $s(t)$ with respect to t for $q = 0.5$ and $\gamma = 15$.

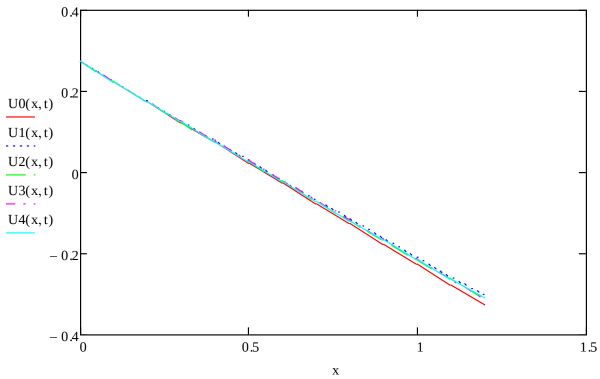

Figure 6. Plot of $u(x,t)$ versus x for $q=1$ and $\gamma\ =\ 10$.

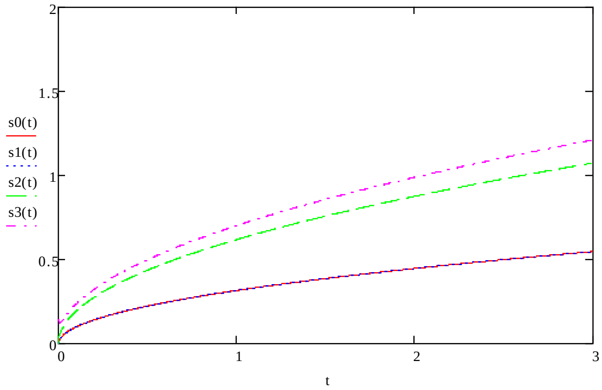

Figure 7. Plot of $s(t)$ with respect to t for $q=1$ and $\gamma\ =\ 10$.

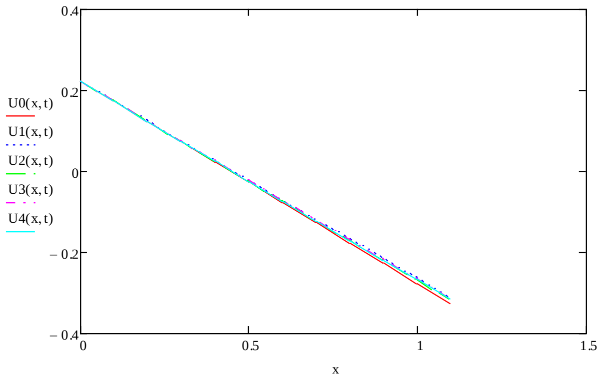

Figure 8. Plot of $u(x,t)$ versus x for $q=1$ and $\gamma\ =\ 15$.

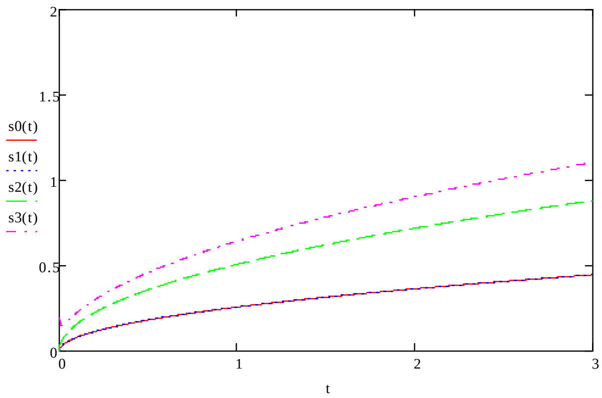

Figure 9. Plot of $s(t)$ with respect to t for $q=1$ and $\gamma\ =\ 15$.

5. Conclusions

In this article, we looked at a mathematical formulation for the moving boundary problem that occurs on the earth's surface during the fluvio deltaic sedimentation process. To obtain the model's analytic approximation solution, the modified VIM is applied. It is clear that modified VIM is a reliable and effective method for solving the moving boundary problem. It is an appropriate technique for calculating the answer to other problems since we believe the outcomes will benefit employees in the field of expertise.

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Conflicts of Interest

The authors declare that there is no conflict of interest between them.

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