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Category of Generalized Modules Over Two Different Rings

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Abstract

In this paper, a method of enriched category theory to form a closed symmetric monoidal structure on the category of generalized modules C_R = (mod R,Ab) have been implemented. Further, a new kind of generalized module category defined on two commutative rings R and S, denoted by C_{RS} = (mod R,mod S) is constructed. Finally, the authors studied a localization on this type of categories, both of them are Grothendieck.

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1. Introduction

An attempt to integrate disparate cohomology theories, Buchsbaum (1955) (under the name precise category) and Grothendieck (1957) proposed abelian categories. At that time, there existed a cohomology theory for sheaves and a cohomology theory for groups. However, these two had many characteristics in common, they were defined differently. In order to investigate these parallels, the category theory was created as a vocabulary to the investigation. The combination of these two theories were introduced by Grothendieck. Indeed, they both result from derived functors on abelian categories, namely the sheaves of abelian groups on a topological space and their abelian categories and of G-modules for a particular group G.

A Grothendieck category is an AB5 category with a generator, for definition see [13]. Both Gabriel's thesis and Grothendieck's Thoku article do not contain the term Grothendieck category, but writers like Roos, Stenstrm, Oberst and Pareigis used it in the second half of the 1960s. (Some writers employ a variant definition, one that does not need the presence of a generator).

An expanded categories approach is applied in this work. However, enhanced generalizes the concept of a category by substituting items from a broad monoidal category for homomorphism sets. It is inspired by the fact that the hom-set frequently contains extra structure in many practical applications such as a chain complex of morphisms or a vector space of morphisms. Numerous applications and uses for enriched categories make it useful to learn about their general theory. On the category of enriched functors, we in this work do a broad Grothendieck analysis.

Definition 1.1 [14]. The category $[C,V]$ of V-functors from C to V is made up of V-functors from C to V, then we say that the category $[C,V]$ is V-functors

Definition.1.2 [12]. Suppose that V is a closed symmetric monoidal Grothendieck category. Then $[C,V]$ is a Grothendieck category if V is a V-category.

Definition 1.3 [13]. A symmetric monoidal category V is said to be biclosed, if $B \in V$, both functors and

$$-\otimes B : V \to V,\ B \otimes - : V \to V$$

have a right adjoint. A biclosed symmetric monoidal category is called a symmetric monoidal closed category. The adjoint to the functor $-\otimes B$ will be denoted by Hom $(B,-)$.

Lemma 1.1 [14]. Let gi be a group of generators for Grothendieck category V, which is closed symmetric monoidal. If a V-category exists, and if so, then $[C,V]$ is category, that is a closed symmetric monoidal, with the set of generators $\{V(C,-) \oslash \text{gi}\}$.

Additionally, we established that $[C,V]$, which is the category of enriched functors, presents Grothendick as stated in the following theorem when the category is a small enriched over a closed symmetric monoidal

Grothendick category V. For more details, we refer the interested readers to [3].

Theorem 1.1 [14]. Suppose that V is a closed symmetric monoidal Grothendieck category. Then [C ,V] is a Grothendieck category if V is a V-category.
To investigate generalized modules, we use Grothendieck categories of enriched functors.

Let R = (mod R,Ab) is the category of generalized modules, that is composed of additive functors R-modules (mod R) to Ab which are the category of finitely to the category of abelian groups, respectively. Indeed, the natural transformations of functors present their morphisms. Since a totally faithful is existed, right exact functor $M \rightarrow -\otimes R$ M from the category of all R-modules to C_R, so it is termed the category of generalized R-modules C_R.

2. Grothendieck Categories

Here, we follow [17,19,22,23] to recall some basic concepts concerning the theory of abelian categories.

Definition 2.1 [14]. The category C_R := (mod R,Ab); whose objects are the additive functors F : mod $R \rightarrow Ab$ from the category of right finitely presented R-modules to the abelian group. Its morphisms are functors natural transformations. The additive functors from the category of left finitely generated functions make up the category $_RC$, provided Ab with R-modules.

Definition 2.2 [17]. Assume that R is a commutative ring. A module over R is made up of a set M, a binary operation denoted by the symbol + that converts the set to an abelian group with 0 as the identity element, and a multiplication rule.

$$R \times M \rightarrow M$$
$$(r,m) : \rightarrow r.m,$$

for the following to be hold,

1. $m = m$,
2. $(r.s).\ m = r.(s.m)$,
3. $(r + s).m = r.m + s - m$,
4. $r.(m + n) = r.m + r.\ n$,

for every $m, n \in M$ and $r, s \in R$.

Definition 2.3 [22]. If a category C has all small limits, then it is called complete.

Definition 2.4 [19]. A category C and an abelian group composition on each set of morphisms $\mathrm{Hom}_C(A;B)$ are referred to as preadditive categories. The composition mappings:

$$\alpha_{ABC}: \mathrm{Hom}_C(A,B) \times \mathrm{Hom}_C(B,C) \rightarrow \mathrm{Hom}_C(A,C),$$
$$(f,g) \rightarrow g{\circ}f$$

In each variable, g and f are group homomorphisms. We will add the group structure to the text. Of course, every category of modules over a ring is preadditive, including the category of abelian groups. Because of this, every entire subcategory of a category of modules is also preadditive.

Example 2.1 [19]. A preadditive category is Mod R. On any morphism set $\mathrm{Hom}_R(M;N)$, in which M and N are two right R-modules, an abelian group structure is defined as follows:

Two homomorphisms are given, $f, g \in \mathrm{Hom}_R(M;N)$, $(f + g)(m) := f(m) + g(m)$, for any element $m \in M$.

Remark 2.1 [19]. To indicate the coproduct (if it exists) we write $\bigoplus_I C_i$, whenever C is preadditive, and refer to it as the direct sum of the family $(C_i)_{i\in I}$.

Definition 2.5 [23]. The cokernel of an arrow $f : A \rightarrow B$, indicated by Coker f, is a pushout of the diagram if there is a preadditive category C with zero object 0,

$$0 \longleftarrow A \overset{f}{\rightarrow} B.$$

Coker f will also serve as a symbol for the canonical morphism $B \rightarrow$ Coker.

Similar to that, the pullback of the diagram, if it exists, is the kernel of $A \rightarrow B$, indicated as ker f.

$$0 \longrightarrow A \overset{f}{\longleftarrow} B$$

The standard morphism A will also be shown by the symbol ker $f \rightarrow A$.

Both kernels and cokernels are by definition unique up to canonical isomorphism since they both meet natural universal characteristics.

Definition 2.6 [19]. A category C is abelian if:

1. C is an additive.
2. There is a product in every finite family of items (and a coproduct).
3. In each morphism, there is a cokernel and a kernel.
4. Either $\bar{\alpha}$: Coker(ker α) $\rightarrow$ ker(coker α), there is an isomorphism for every morphism, or every morphism has a factorization with the formula = (cokernel, kernel), where is a cokernel. Let Im α := ker (coker α) represent the image of α, for every morphism α of an abelian category.

Definition 2.7 [4]. For any two objects in a category Hom(A,B) is the collection of all morphisms from A to B. If the morphisms are in the category C and we need to emphasize this, we will write C (A,B). In a small category, Hom(A,B) is a set. In some large categories Hom(A,B) will not be a set, but in the familiar ones it is a set, so we make that a definition and a convention.

Definition 2.8 [19]. Let C a category that is abelian. If Hom(C,−) is faithful, C is a generator and if Hom(−,C) is faithful, then it is a cogenerator.

Definition 2.9 [19]. Let $U = \{U_i\}_{i\in I}$ *be* a family that contains objects of C (a category). Then U is a family of generators of the category if there is an index $i \in I$ and a morphism for each object A and each sub object B of A different from $Au : U_i \longrightarrow A$ that is not factorizable using standard injection $i : B \longrightarrow A$ of B into A. If the family U is a family of generators of the category, then an item U of is a generator of the category.

Example 2.2 [19]. If R is a direct summand of some direct sum of copies of module M, then M is a generator for mod R.

The functor categories (B,Ab), where B is a preadditive category, are particularly interesting. It is composed, by definition, from B to Ab (additive functors). Morphisms are functors' natural transformations. The representable functor $h^B := \mathrm{Hom}_B(B, -)$ is a particular example of an object in (B, Ab) relating to the object $b \in B$.

The Yoneda Lemma shall be presented by the following assertion.

Proposition 2.1 (Yoneda Lemma) [4]. B should only be a minor preadditive category. There is an isomorphism for each B of B and each additive functor $T : B \to Ab$.

$$\mathrm{Hom}_{(B,Ab)}(h^B, T) \cong T(B)$$

that is natural in B and T.

Definition 2.10. If an abelian category contains arbitrary direct sums, it is an $Ab3$-category or cocomplete. If the connection holds for every $\{A_i\}_{i\in I}$; that is a directed family and contains sub-objects of A and for every sub object B of A, $Ab5$-category represents a cocomplete abelian category if $(\sum_{i\in I} A_i) \cap B = \sum_{i\in I}(A_i \cap B)$ holds.

Having arbitrary direct boundaries is identical to the condition $Ab3$. Additionally, $Ab5$ is identical to the assertion that inductive limits exist and that they are precise for directed families of indices, namely if we let I to be a directed set and $0 \to A_i \to B_i \to C_i \to 0$ is an exact sequence for any $i \in I$, then:

$$0 \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0$$

is an exact sequence. An abelian category which satisfies the condition $Ab5$ and which possesses a family of generators is called a Grothendieck category.

3. Enriched Category

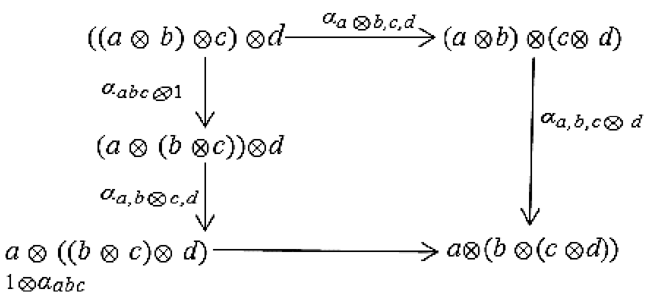
This section gathers the fundamental information about the enriched categories that we will require later. We cite [3,14,15,18,19,20,23] for more information.

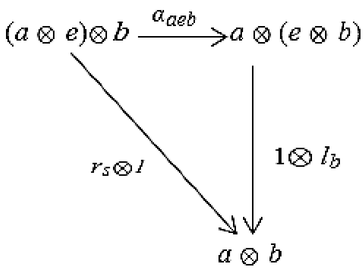
Definition 3.1 [4]. A category V, and make up a monoidal category V

1. A bifurcator known as the tensor product, $\otimes : V\times V \to V$.
2. A thing $e \in V$ referred to as the unit.
3. For any triple a, b, and c of objects, an isomorphism of associativity $\alpha_{abc} : (a\otimes b)\otimes c \to a\otimes (b\otimes c)$.
4. A left unit isomorphism for every item in a $l_a : e \otimes a \to a$.
5. For each object a, $r_a : a \otimes e \to a$ is an isomorphism of the right unit.

The facts needed to meet the criteria are:

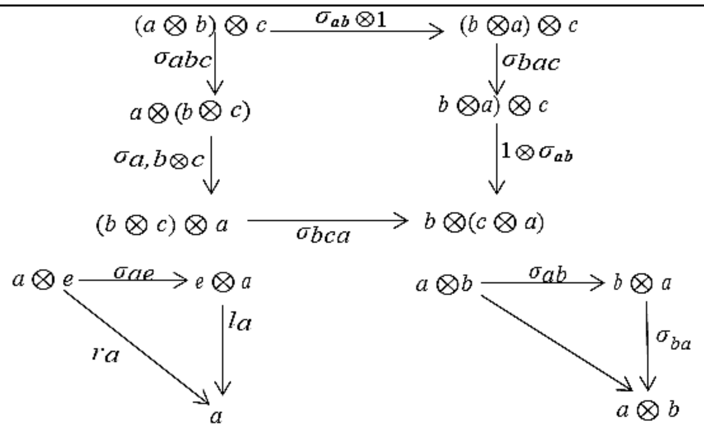
1. In a, b and c, the morphisms α_{abc} are natural, where the natural morphism means that it does not involve making any choices.
2. The natural morphisms l_a in a.
3. The naturalness of the morphisms r_a in a.
4. The diagrams below are commutative for every triple a, b and c.





Definition 3.2 [3]. If there are natural isomorphisms $\sigma_{a,b} : a \otimes b \to b \otimes a$ between $a \otimes b$ and $b \otimes a$, then the monoidal category is symmetric. Indeed, the following requirements have to be hold with these isomorphisms:

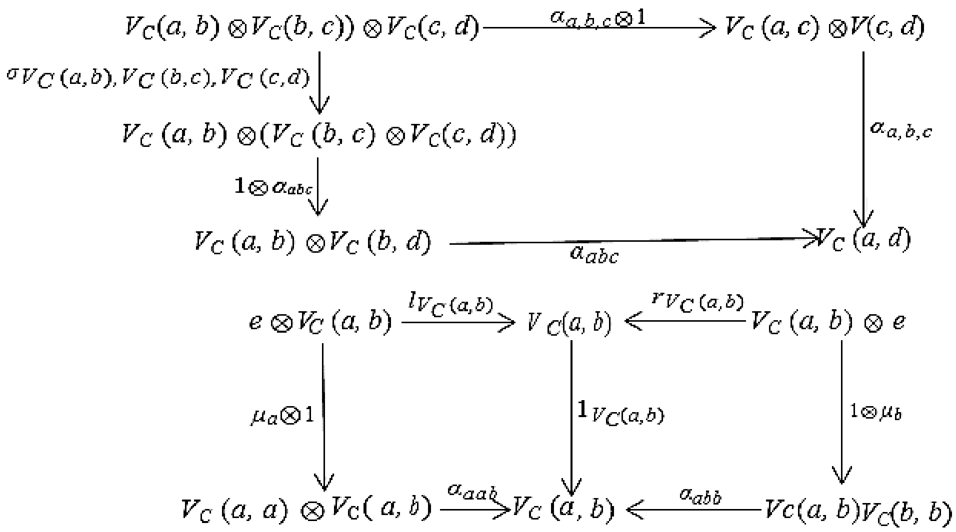
1. The morphisms in a and b are $\sigma_{a,b}$ natural.
2. The diagrams below are commutative for each a, b and c.



Definition 3.3 [3]. A monoidal category that is symmetric For each item, a ∊ V indicates that V is biclosed, both functors $-\otimes a : V \to V, a \otimes - : V \to V$ contain a right adjoint A symmetric monoidal closed category is defined as a biclosed symmetric monoidal category.

Definition 3.4 [3]. Assume that V is a closed symmetric monoidal category. Then a V-category, that is also known as a category enriched over V, is made up of the information:

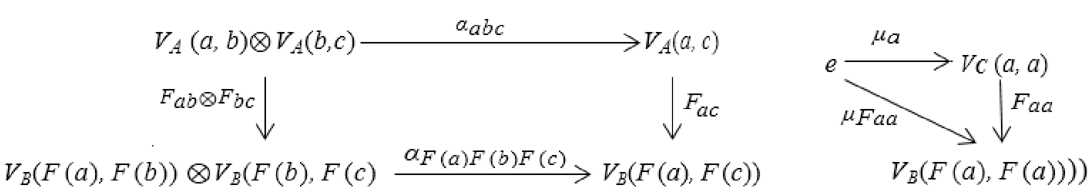
1. ob(C) is a class.
2. For any pair of objects $a,b \in$ ob(C), an object $VC(a, b)$ of V is termed the V-object mappings in C.
3. A composition morphism in V, abc: for any triple a, b and $c \in (C)$ of objects, $\alpha_{abc} : V_C(a, b) \otimes V_C(b, c) \to V_C(a, c)$
4. Every $a \in C$, $v_a : e \to V_C(a, a)$, a unit morphism in V exists.
5. The diagrams below are commutative.



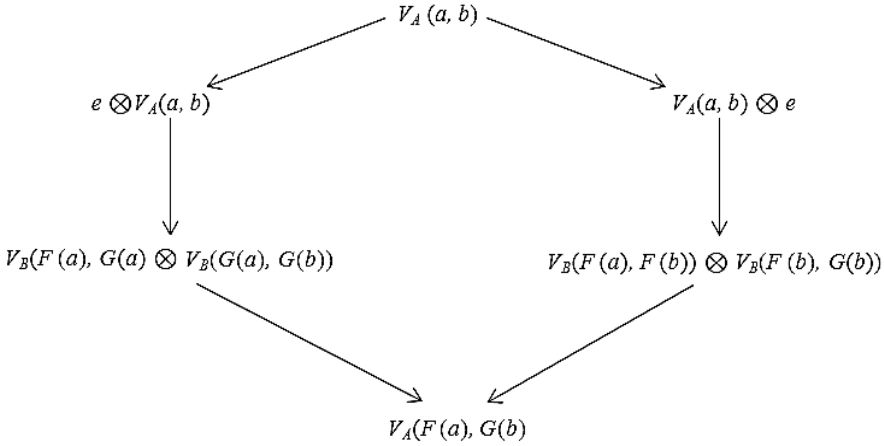
Definition 3.5 [19]. Suppose that V is a monoidal category. Let A and B be the two V-categories, that are given, a V-functor $F: A \to B$ composed of:

1. There is an object $F(a) \in B$ for any object aA,
2. A morphism in V exists for any pair of items $a, b \in A$, $F_{a,b} : V_A(a,b) \to V_B(F(a),F(b))$.

The diagrams below are commutative:



Definition 3.6 [14]. Suppose that V is a monoidal category, B and A are two V-categories, such that the V functors, F and G are connecting A and B. The purely natural transformation $t : F \rightarrow G$ consists in assigning a morphism to each item $t_a : e \rightarrow V_B(F(a),G(a))$ in V where the diagram bellow is commuting for the items $a, b \in R$.



Definition 3.7, [21]. The term ordinary refers to category U_C if C is a V-category and
1. $\text{ob}(U_C) = \text{ob}(C)$.
2. $\text{Hom}U_C(a, b) = \text{Hom}V(e, V_C(a, b)$.
3. the composite of $f : e \rightarrow V_C$ (a, b), $g : e \rightarrow V_C(b, c)$ is given by:

$$e \xrightarrow{r_e^{-1}} e \otimes e \xrightarrow{f \otimes g} VC(a,b) \otimes VC(b,c) \xrightarrow{\alpha_{abc}} VC(a,c)$$

Definition 3.8. There must be a V-functor for a V-category C to be a right V-module. $\oslash : C \otimes V \rightarrow C$ indicate $(c,a) \rightarrow c \oslash a$ with a V natural unit isomorphism $r_c : c \oslash e \rightarrow c$, subject to the terms:
1. The isomorphisms of natural associativity is existed
$$c \oslash (a \otimes b) \rightarrow (c \oslash a) \otimes b$$
2. There are coincided isomorphisms
$$c \oslash (e \otimes a) \rightarrow c \oslash a.$$

4. Enriched Functor Categories

The category $[C,V]$ of V-functors from to V is formed when is a tiny V-category together with its V-natural transformations. $[C,V]$ is a V-category as well if V is complete. The V-object morphism $V_{[c,V]}(X,Y)$ is the end $\int_{obC} V(X(c),Y(c))$.

The following lemma is called by the Enriched Yoneda Lemma:

Lemma 4.1 [9]. Suppose that V is a closed, full, symmetric monoidal category, and a tiny V-category. The natural isomorphism V exists $X(c) \cong V_{[C,V]}(V_C(c,-), X)$, for each V-functor $X : C \rightarrow V$ and every $c \in \text{ob}(C)$.

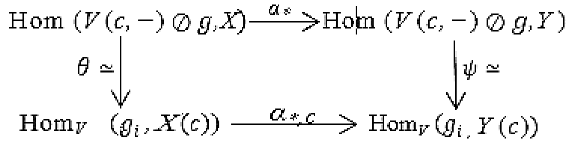
Definition 4.1 [9]. Let $\{U_i\}$ be a family of abelian category objects. If for any non-zero morphism $\alpha : b \rightarrow c$, there exists a morphism $\beta : U_i \rightarrow b$, as $i \in I$ and $\alpha\beta \neq 0$, then $\{U_i\}$ is a family of generators.

Lemma 4.2. Let V be a collection of generators in a closed symmetric monoidal Grothendieck category $\{g_i\}$ and suppose that C is a V-category. Therefore, $[C,V]$ is closed symmetric monoidal category with the generator set $\{V(c,-) \oslash g_i\}$.

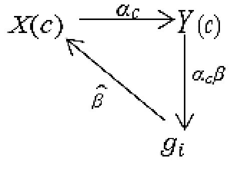
Proof. As $[C,V]$ is a right closed module over V,thus an action $\oslash : [C,V] \otimes V \rightarrow [C,V]$ exists with $X \in [C,V]$ a nonzero functor, we have:))

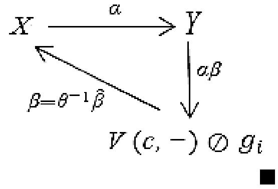
$$\text{Hom}_{[C,V]}(V(c,-) \oslash g_i, X)) \cong \text{Hom}_V(e, \text{Hom}_{[C,V]}(V(c,-) \text{O} g_i, X))$$
$$\cong \text{Hom}_V(e, \text{Hom}_V(g_i, \text{Hom}_V(V(c,-), X))$$
$$\cong \text{Hom}_V(e, \text{Hom}_V(g_i, X(c))$$
$$\cong \text{Hom}_V(g_i, X(c))$$

In fact, we will show there is $i \in I$, a map $\beta : V(c,-) \oslash g_i \rightarrow X$ with $\alpha\, \beta \neq 0$ and $\alpha : X \rightarrow Y$ *is a* map in $[C,V]$. Suppose there is no β satisfying this condition, i.e.; for all $i, c, \beta,\ \alpha\beta = 0,\ \alpha_c : X(c) \rightarrow Y(c)$ since g_i are generators for V, then there exist a map $\hat{\beta} : g_i \rightarrow X(c)$ for some i, such that $\alpha_c\hat{\beta} \neq 0$. Now we have the following commutative diagram.



$\alpha_{*,c}(\hat{\beta}) \neq 0$, for some i. Let $\beta := \theta^{-1}(\hat{\beta})$, then $\alpha\beta = \alpha_*(\beta) \neq 0$, because otherwise $\psi(\alpha_*(\beta)) = 0 = \alpha_{*,c} \circ \theta(\beta) = \alpha_{*,c}(\hat{\beta})$.





■

Theorem 4.1. Suppose that V is a Grothendieck category which is closed, symmetric, and monoidal. If $[C,V]$ is a Grothendieck category and $[C,V]$ is a V-category.

Proof. We start by showing that $[C,V]$ is a preadditive category. If $X, Y \in [C,V]$ are V- functors, then:

$$\mathrm{Hom}_{[C,V]}(X, Y) = \mathrm{Hom}_V(e, \int_{c\in \mathrm{Ob}C} V(X(c),Y(c))$$

Since V is preadditive, is an abelian group. The structure of the abelian group may also be formally described as follows. The V-functors from C to V are, by definition, the morphisms of $[C,V]$. ■

V-functors of any kind $\alpha, \beta : X \to Y$, with $\alpha + \beta$ is obtained by as follows:

$$\alpha_c + \beta_c : e \to V(X(c), Y(c))$$

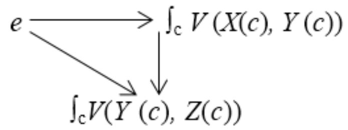
Since $\mathrm{Hom}_V(X(c), Y(c)) = \mathrm{Hom}_V(e, V(X(c), Y(c)))$ and $\alpha_c + \beta_c$, which is the sum of β_c *and* α_c in the group $\mathrm{Hom}_V(X(c), Y(c))$, that is an abelian.

For bilinear $\forall\ \alpha, \alpha' \in \mathrm{Hom}(X,Y)$ addition defined by $(\alpha + \alpha')_c : e \longrightarrow V(X(c), Y(c))$, if:

$$\beta \in \mathrm{Hom}_V(e, \int_c V(Y(c), Z(c)) = \int_c \mathrm{Hom}_V(e, V(Y(c), Z(c)) = \int_c \mathrm{Hom}_V((Y(c), Z(c))$$

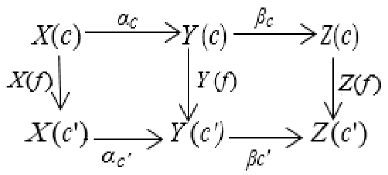
So, addition can be define as:

$$(\alpha + \alpha')_c := \alpha_c + \alpha'_c$$
$$(\alpha\beta)_c := \alpha_c \circ \beta_c$$



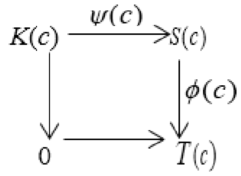
$$(\beta(\alpha + \alpha'))_c = \beta_c(\alpha_c + \alpha'_c) = \beta_c \circ \alpha_c + \beta_c \circ \alpha' c = (\beta\alpha + \beta\alpha')_c.$$

Let $f : C \to C'$, then the commutative diagram is as follows:



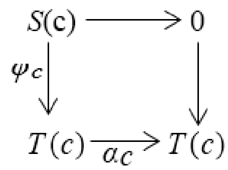
Coproduction and Product. Let the family of finite objects $\{T_1, T_2, \ldots, T_n\}$ in $[C,V]$. Assume the following to be a functor T. $T(X) = T_1(X) \times T_2(X) \times \ldots \times T_n(X)$ and $T(\lambda) = T_1(\lambda) \times T_2(\lambda) \times \ldots \times T_n(\lambda)$, for object X and morphism λ clearly $T \in [C,V]$ hence $[C,V]$ has a coproduct and product of a finite objects.

Kernel and Cokernel. Let $\phi : S \to T$ be a natural transformation, for each $c \in C$, let $\psi_c : K(c) \to S(c)$ be kernel of ϕ_c, then for each c we have the diagram of pullback



Let $\phi : c \to c'$ in C induced morphism exists $K(\phi) : K(c) \to K(c')$ among the kernels explain the functor. $K : C \to V$ together with natural transformation $\psi : K \to S$ as a pointwise limit. Clearly K is kernel of ψ.

The cokernel can be defined similarly, for each $c \in C$ let $\alpha_c : T(c) \to C(c)$ be thecokernel of ϕ_c Afterward, we have the pushout diagram shown below:



$Coker(ker\phi) \approx Ker(coker(\phi)$ can be calculated point-wise.

One can compute the complete and cocomplete limit and colimit pointwise. Let $F : I \to [C,V]$ be a functor, where I is any small category, and we define F(A) as the following for each $A \in C$: $F(A)_m = F(m)_A$ and $F(A)_i = F_i(A)$. Consider the morphism for any $A \in C$, $\rho_{i,A}$: lim $F(A) \to F_i(A)$ is the limit for the functor F, for each morphism $\alpha : A \to \grave{A}$ in C, $F_i(\alpha) : F(A) \to F(A`)$, which induces a morphism of the limits $L(\alpha)$ : lim $F(A) \to F(A`)$, this define a functor $L : C \to V$.We define natural transformation $\rho_i : L \to F_i$ by $(\rho_i)_A = \rho_{i,A}$, Consequently, it is evident that L is F's limit AB5. Let:

$$0 \longrightarrow F_i \xrightarrow{\alpha_i} G_i \xrightarrow{\beta_i} H_i \longrightarrow 0$$

be a short exact sequence in $[C,V]$, where $\alpha_i : e \to V(F_i(c), G_i(c))$ and $\beta_i : e \to V(G_i(c), H_i(c))$, now for all $c \in C$, we have:

$$0 \to \varinjlim F_i(c) \xrightarrow{\alpha_i} \varinjlim G_i(c) \xrightarrow{\beta_i} \varinjlim H_i(c) \to 0$$

is short exact sequences in V thus

$$0 \to \varinjlim F_i(c) \xrightarrow{\alpha_i} \varinjlim G_i(c) \xrightarrow{\beta_i} \varinjlim H_i(c) \to 0$$

is short exact sequence in $[C,V]$.

5. Localization in V-Functor Categories

Consider (A,Ab) the category of additive functors from A to Ab if A is a preadditive category (Abelian group category). Suppose that $p \in \mathrm{ob}(A)$, this implies that $\mathrm{Hom}_{(A,\mathrm{Ab})}((p, -), (p, -)) = \mathrm{End}_A p$. Assume that $S_p = \{F \in (A, Ab)\ |F(p) = 0\}$, thus S_p is localization full subset of (A,Ab).

(Serre and containing all coproduct) To see this, consider:

$$0 \to F \to G \to H \to 0$$

as a short exact sequence, then clearly $G \in S_p \Leftrightarrow F$, $H \in S_p$, for a family of V-functor, $(\coprod Fi)(p) = 0$. We claim that $\mathrm{Hom}_{(A,Ab)}/S_p \cong \mathrm{Mod}(\mathrm{End}_A\, p)$

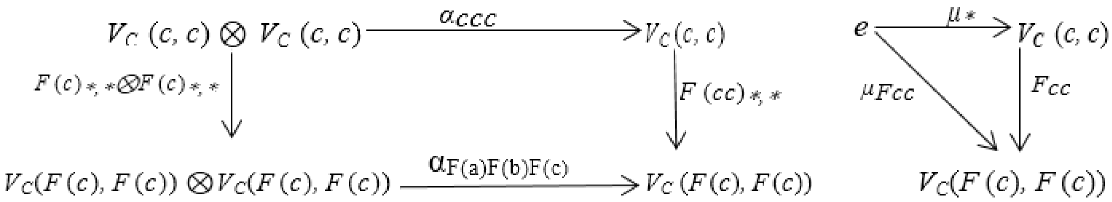
Let $G : (A, Ab) \to \mathrm{Mod}(\mathrm{End}_{Ap})$ be such that $G(F) := F(p)$, and let $H : \mathrm{Mod}(\mathrm{End}_{Ap}) \to (A, Ab)$ be such that $H(M) := (-\otimes \mathrm{End}_{A_p}\, M)_{S_p}$

Consider C is a V-category, V Grothendieck, $c \in \mathrm{ob}(C)$, let $S_c = \{F \in [C,V] \mid F(c) = 0\}$ is localizing. By 4.4 $[C,V]$ is Grothendieck, hence we can apply the localization theorem to $[C,V]$, according to our assertion $[C,V]/S_c \cong [End_c,V]$, where End_c is a V-category with one object $*$ and $V(*,*) := V(c,c) \in \mathrm{ob}(V)$. Next, we illustrate an example.

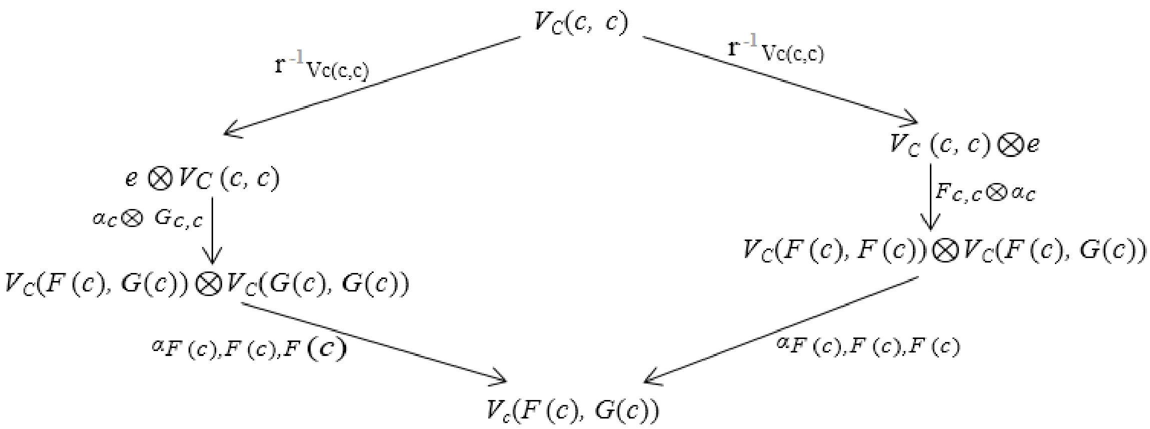
Example 5.1 [9]. If C is one object category $\mathrm{ob}(C) = \{*\}$, C is a V-category. Then $[C,V] = V$. We know $V(*,*) = e \in V$ suppose $F \in [C,V]$ then $* \to F(*) \in \mathrm{ob}(V)$, There is the functor. $F_{*,*} : V(*,*) = e \to V(F(*),F(*))$. Taking a morphism $f \in \mathrm{Hom}V$ (F same as taking an object $M \in V$ and $\mathrm{End}V\, M$) is equivalent to doing this. Now, let a V-natural transformation be $\alpha : F \to G$. Define $\alpha_* : e \to V(F(*), F(*)) \Leftrightarrow \alpha : F(*) \to G(*)$. Hence, we have $[C,V]$ is the same as V

Remark 5.1. $S_* = \{F \in [C,V] \mid F(*) = 0\} = 0$, therefore $[C,V]/0 = [C,V] = V$.

General once more specifying the evaluation function $Ev_c : [C,V] \to [Endc,V]$, such that $Ev_c(F) = F(c) \in V$. Let $F(c)_{*,*} : End_c \to V$ be such that $F_{*,*} := F_{c,c} : V(c,c) \to V(F(c),G(c))$, clearly $F_{*,*}$ is a V-functor since $* \to F(c) \in V$.



Therefore, $F(c)$ may be thought of as a V-functor from End_C to V. Let $\alpha : F \Rightarrow G$. Now be a morphism in $[C, V]$, and we get $\alpha_c : F(c) \to G(c)$ is a map in V. We need to show it is a map in $[\mathrm{End}_C, V]$. Let $\alpha \to \tilde{\alpha} : F(c) \Longrightarrow G(c)$ in $[\mathrm{End}_C, V]$ $\tilde{\alpha}_* : e \to V_C((F(c), G(c))$. Define $\tilde{\alpha}_* := \alpha_c$, we have:



Then $\tilde{\alpha}_* : F(c) \to G(c)$ is a V-natural transformation in $[\mathrm{End}_C, V]$

Proposition 5.1 [9]. Suppose that P to is a family of objects the map $I : P \to C$, where C is the V-category, then a V-functor. It generates 2 adjoint functors $i_* : [P,V] \leftrightarrows [C,V] : i^*$, with i_* is the enriched left can extension exactly; the functor i^* is essentially a constraint to P.

Proof. If $F \in [P,V]$, then $F \cong \int^{obP} V(P,-) \oslash F(p)$. The left Kan extension definition leads to:

$$i_*F \cong \int^{obP} V(i(p),-)\, 0F(p).$$

Our goal is to obtain that:

$$\mathrm{Hom}_{[C,V]}(i_*F, G) \cong \mathrm{Hom}_{[P,V]}(F, i^*G)$$

$$\mathrm{Hom}_{[P,V]}(F, i^*G) = \mathrm{Hom}_{[P,V]}(F, G \circ i)$$
$$= \int_{obP} V(F(c), G(i(c)) \qquad \dots(1)$$

From another hand

$$Hom_{\_[C,V]}(i * F, G) =$$
$$Hom_{\_[C,V]}\left(\int^{obp} V(i(c),-) \oslash F(c), G)\right)$$
$$V(i(c),-)0F(c), G)$$

$= \int_{obp} Hom\_{[C,V]} (F(c), V(V((i(c,-), G)$
$= \int_{obp} [\![V(F(c), G(i(c))]\!]$ ...(2)

Therefore, $i_*$and $i^*$are clearly adjoint functors from (1) and (2). ■

Theorem 5.1. Let $S_P := \{G \in [C,V] | G(p) = 0$ with $p \in P\}$. Therefore:

(1)A localizing subcategory of $[C,V]$ defines S_P.

(2)$[P,V]$ is equivalent to $[C,V]/S_P$.

Proof. (1) Clearly, from the detention of localization, the first component may be inferred.

(2) Let $\varkappa : [P,V] \to [C,V]/S_P$ be made up of the localization functor and the i∗ component.$(-)_{S_P} : [C,V] \to [C,V]/S_P$. In fact, $\varkappa$ is an equivalence of categories that what we want to demonstrate. We firstly have that for all $F \in [P,V]$, $i^*i_*F \cong F$. Moreover, we have that the adjunction map $\beta : i_*i^*G \to G$ for a given $G \in [C,V]$, such that Ker β, Coker $\beta \in S_P$. In fact, the exact functor i*is then applied to the exact sequence Ker$\beta \to i*i*G \to {}^{\perp}\beta\ G \to$ Cokerβ. We obtain that i∗(Ker)$\beta \to i\,i_*\,i\,G \to {}^{\perp}(i*\beta)i*G \to i*$(Coker$\beta$), which is an exact sequence. Since the composite map (that is the identity)

$i^*G \to i^*i_*i^*G \xrightarrow{i^*\beta} i^*G$

The left arrow is an in fact isomorphism. Therefore:

$i^*(Ker\beta) = i^*(Coker\beta) = 0$

Thus, we obtain that Cokerβ, Ker $\beta \in S_P$. Additionally, that gives:

$(i_*i^*G)_{S_P} \cong G_{S_P}$ ...(3)

Therefore:

$\mathrm{Hom}_{[C,V]/S_p}(\varkappa(F),\varkappa(F')) \cong \mathrm{Hom}_{[C,V]/S_p}((i^*(F)_{S_P,*}(F')_{S_P})$
$\cong \mathrm{Hom}_{[C,V]}(i_*F, (i_*F')_{S_P})$
$\cong \mathrm{Hom}_{[P,V]}(F, i^*((i_*F')_{S_P}))$
$\cong \mathrm{Hom}_{[P,V]}(F, i^*i_*F')$
$\cong \mathrm{Hom}_{[P,V]}(F, F')$

with $F, F' \in [P,V]$.

The isomorphism is being used for any $G \in [C,V]$, $i^*G \cong i^*(G_{S_P})$. The exact functor i^* to the exact sequence is being applied to obtain the isomorphism:

$$S \to G \xrightarrow{\lambda_F} G_{S_P} \to S', S, S' \in S\_p$$

That implies that $\varkappa$ is fully faithful. The isomorphism (3) leads to:

$\varkappa(i^*G) = (i_*i^*G)_{S_P} \cong G_{S_P} \cong G$

By having $G \in [C,V]/S_P$ is a S_P–closed object.

Indeed, $\varkappa(F) \cong G$ by setting $F := i^*G$. That proves $\varkappa$ is an equivalence of categories. ■

Lemma 5.1. The category of R-modules (R-Mod) is naturally identified with $[C,V]$, where R be a ring object of V with ob $C = \{*\}$.

Proof. Suppose that a functor $F : C \to V$ is defined with the following data (with the left R-module M being given):

$F(*) = M$

$F_{**}: V(*,*) \to V(F(*), F(*))$

$F_{**}: R \to V(M,N)$

We know:

$V(R \otimes M, M) \cong V(R, V(M,M))$

such a morphism results from the fact that M is a left R module $\phi : R \otimes M \to M$ exist, let $F_{**} = \phi$. Let N now be any R-left module that is distinct from M, let $f \in V(M,N) \Rightarrow f \in V(e, V(M,N))$. Consider $G : C \to V$ be a V-functor linked to N, and we have a natural transformation. $\alpha : F \Rightarrow G$. $\alpha_*: e \to V(F(*), G(*))$ this gives $\alpha_*: e \to V(M,N)$ take $\alpha_* = f$. ■

Corollary 5.5. Let c be any item of C, and let C be a V-category. Then there is a Grothendieck category equivalence R −Mod $\cong [C,V]/S_C$, with $S_c = \{G \in [C,V] | G(c) = 0\}$, and $R = V(c,c)$

Proof. It is a result of the previous Lemma 5.1 and Theorem 5.1. ■

6. The Categories of Generalized Modules C_R and CR_S

Similar to Herzog [13], C_R is here defined as:

$C_R := (\mathrm{mod}\,R, Ab)$

which have the objects from the additive functors F: mod $R \to Ab$ from the category of R-modules (right finitely) to Ab (the category of abelian groups), whose morphisms are functors' natural transformations. Similar to this, the category ${}_RC$ is made up of the additive functors that connect the R-modules to Ab, where R-modules represents the category of left finitely. It follows that C_R is a locally coherent Grothendieck category since the category mod R has cokernels. Additionally, the projective global dimension of the category of coherent objects coh R is no more than two.

As a result of the latter feature, each coh CC_R (that is the coherent object C) contains a resolution represented by the functors.

$0 \to (M,-) \to (N,-) \to (L,-) \to C \to 0$,

with the finitely M, L, N which present right R-modules.

Note that C_R is sometimes known as the category of generalized modules since a fully faithful existed, precise functor $M\ l \to -\otimes_R M$ from C_R to the group of all R-modules. Indeed, in representation theory, ring and module theory, and other fields, the category C_R features a number of exceptional traits that have important applications. (see [8, 12, 13]).

Theorem 6.1 [13]. Let's assume that R is a commutative ring. The category of enriched functors can thus be naturally associated with the category of generalized R-modules, or R. [mod R, mod R].

Example 6.1 [13]. Consider $C_R = (\mathrm{mod}\,R, \mathrm{mod}\,R)$, and let $P^R = \{F \in C_R | F(R) = 0\}$. Define functor $H : \mathrm{Mod}(\mathrm{End}_A\ p) \to C/p^R$, such that $H(M) = (-\otimes_R M)_{p^R}$ and define functor $G : C/p^R$ Mod R, such

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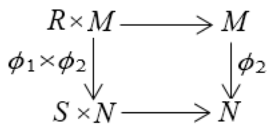
that $G(F) = F(R)$ then we have the following exact sequence $P \to F \xrightarrow{\lambda_F} F_{P^R} \to P'$, where $p, p' \in p^R$ and $p = t(F)$ the torsion subgroup, $F_P^R \cong (- \otimes_R F(R))_{p^R}$.

Definition 6.1. Let $\phi_1 : R \to S$ be a ring homomorphism, and let M, N be R, S-module respectively, let $\phi_2 : M \to N$ be an abelian groups homomorphism. A homorphism $\psi : M_R \to N_S$ is an R_S-module homorphism define as follows.

$\psi(rm) = \phi_1(r)\phi_2(m), \ \forall\, m \in M, r \in R$

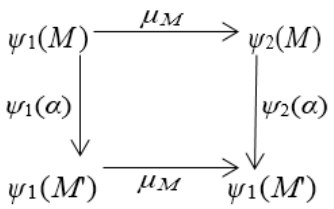
In this case we also say ψ is RS-linear. We denote the set of these as $\mathrm{Hom}_{RS}(M, N)$.



Definition 6.2. Suppose that R and S are rings (associative, containing identity), and designate the categories of right modules over R and S as Mod R and Mod S. (respectively). We specify the functor category for modules denoted by:

$C_{RS} = (\mathrm{mod}\, R, \mathrm{mod}\, S)$

whose objects are the morphisms $\psi : \mathrm{mod}\, R \to \mathrm{mod}\, S$ (as in Definition 6.3) and whose morphisms are the natural transformations as in the diagram below:



Where $\alpha : M \longrightarrow M'$ is an R-module homomorphism.

Proposition 6.1. Let Mod R and Mod S, respectively, be the right R-module and right S-module categories. In relation to the localizing subcategory, the category Mod R is therefore equal to the quotient category of C_{RS},

$M^R = \{F \in C_{RS} \mid F(M) = 0, \ \forall\, M \in \mathrm{mod}\, R\}$.

Proof. For an arbitrary functor $F \in C_{RS}$ by $F(R)$ denote a right S-module defined as follows. We have that $F(R)(M) = F(MS)$ if $M \in \mathrm{mod}\, R$. It is directly checked that $F(R) \in$ mod S. It follows that the functors $\phi : C_{RS} \to$ mod R, $F \to F(R)$, defines an equivalence of categories mod R and $C_{RS}/\ker\phi$. Clearly, $M^R = \ker\phi$. ■

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