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On $m_{\mathcal{X}}$-ω$_b$-Open Sets

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Abstract

In this paper, we shall use the *m*-structure topological spaces to introduce new concept, which is the $m_{\mathcal{X}}$-ω-open set. Several facts, results and examples are given to illustrate our work.

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1. Introduction

In this work, *m*-space always mean m-structure topological spaces on which no separation axioms are assumed unless explicitly stated. In this paper, the structure topological space on a nonempty set $\mathcal{X}$ is denoted by $(\mathcal{X}, m_{\mathcal{X}})$ or simply by $\mathcal{X}$.

For a subset A of $\mathcal{X}$, the $m_{\mathcal{X}}$-closure and the $m_{\mathcal{X}}$-interior of A in $\mathcal{X}$ are denoted by $m_{\mathcal{X}}$-Cl(A) and $m_{\mathcal{X}}$-Int(A) respectively. The researchers Maki in 1996 [6], Popa in 2000 [8], Ali in 2018 [2] and Humadi in 2019 [4] introduced and study the properties of *m*-structure space also introduce some types of spaces via *m*-structure. A subset A of a space $\mathcal{X}$ is said to be ω-open set if for every point x in A, there exists an open set U containing x, such that U-A = countable, [1,3]. If U-A = finite, then A is called *N*-open set. In the same context Humadi in 2020 defined supra ω-open set, [5]. A subset A of a space $\mathcal{X}$ is said to be *b*-open if $A \subseteq$ Int(Cl(A)) $\cup$ Cl(Int(A)) [9] and a set A is called ω$_b$-open if for every point in A, there exists *b*-open set U containing x, such that U-A = countable, [7].

2. $m_{\mathcal{X}}$-ω-Open Sets

In this section, we shall study the ω-open set by using *m*-structure spaces, we presented some of results and examples regarding this concept. Also, we introduced other kinds and concepts.

Definition 2.1, [2,6,8]. Suppose that $\mathcal{X}$ is a non-empty set and $P(\mathcal{X})$ the family of subsets of $\mathcal{X}$. Let $m_{\mathcal{X}}$ be any subfamily of $P(\mathcal{X})$.Then $m_{\mathcal{X}}$ is said to be minimal structure (shortly *m*-structure) on $\mathcal{X}$ if $\emptyset$, $\mathcal{X} \in m_{\mathcal{X}}$ and the pair $(\mathcal{X}, m_{\mathcal{X}})$ is used to denote the m-structure space. Any element of $m_{\mathcal{X}}$ is said to be $m_{\mathcal{X}}$ open ($m_{\mathcal{X}}$-O) and the complement of an $m_{\mathcal{X}}$-open set is said to be $m_{\mathcal{X}}$-closed ($m_{\mathcal{X}}$-C). Clearly each topological space is *m*-space.

Note. From now on, we will denote all $m_{\mathcal{X}}$-closed sets by F$m_{\mathcal{X}}$.

Definition 2.2, [2,6,8]. Suppose that $\mathcal{X}$ is a non-empty set and $m_{\mathcal{X}}$ is an *m*-structure on $\mathcal{X}$. For a subset V of $\mathcal{X}$, the $m_{\mathcal{X}}$-closure of V ($m_{\mathcal{X}} - \overline{V}$) and the $m_{\mathcal{X}}$-interior of V ($m_{\mathcal{X}}$-V°) are distinguishing by:

1. $m_{\mathcal{X}}$-$\overline{V} = \cap\{W : V \subseteq W, \mathcal{X} \setminus W \in m_{\mathcal{X}}\}$.
2. $m_{\mathcal{X}}$-$V^{\circ} = \cup\{E : E \subseteq V, E \in m_{\mathcal{X}}\}$.

Lemma 2.1, [2,8]. Assume that the pair $(\mathcal{X}, m_{\mathcal{X}})$ is a minimal structure space. For any to subset U and V of $\mathcal{X}$, the following properties are holds:

1. $m_{\mathcal{X}}$-$(Z \setminus U)^{\circ} = \mathcal{X} \setminus m_{\mathcal{X}}$–$\overline{U}$.
2. If $(\mathcal{X} \setminus U) \in m_{\mathcal{X}}$, then $m_{\mathcal{X}}$-$\overline{U} = U$ and if $U \in m_{\mathcal{X}}$, then $m_{\mathcal{X}} - U^{\circ}$ = U.
3. If $U \subseteq V$, then $m_{\mathcal{X}}$–$\overline{U} \subseteq m_{\mathcal{X}}$-$\overline{V}$ and $m_{\mathcal{X}}$-$U^{\circ} \subseteq m_{\mathcal{X}}$–$V^{\circ}$.
4. $m_{\mathcal{X}} - \overline{(m_{\mathcal{X}} - \overline{U})} = m_{\mathcal{X}}$ - $\overline{U}$ and $m_{\mathcal{X}} - (m_{\mathcal{X}} - U^{\circ})^{\circ} = m_{\mathcal{X}}$–$U^{\circ}$.

Lemma 2.2, [2]. Assume that V is a subset of m-space $\mathcal{X}$. Then $x \in m_{\mathcal{X}}$-$\overline{U}$ if and only if $K \cap U \neq \emptyset$, for every $K \in m_{\mathcal{X}}$ containing x.

Definition 2.3, [2]. An m-structure $m_{\mathcal{X}}$ on a non-empty set $\mathcal{X}$ is said to has property $\mathfrak{B}$ if the union of any family of $m_{\mathcal{X}}$-open subsets belong to $m_{\mathcal{X}}$.

Definition 2.4, [2]. The pair space $(\mathcal{X}, m_{\mathcal{X}})$ which is minimal space has the Property (I) if the any finite intersection of $m_{\mathcal{X}}$-open sets is $m_{\mathcal{X}}$-open.

Definition 2.5. A subset U of an *m*-space $(\mathcal{X}, m_{\mathcal{X}})$ is said to be**:**
1. $m_{\mathcal{X}}$-dense if $m_{\mathcal{X}} - \overline{U} = \mathcal{X}$.
2. $m_{\mathcal{X}}$-nowhere dense if $m_{\mathcal{X}} - (m_{\mathcal{X}} - \overline{U})^{\circ} = \varnothing$.

Definition 2.6, [2]. Suppose that *Y* is nonempty subset of *m*-space $(\mathcal{X}, m_{\mathcal{X}})$. Then the space (Y, m_Y) is called *m*-subspace of $(\mathcal{X}, m_{\mathcal{X}})$ if $m_Y = \{V \cap Y: V \in m_{\mathcal{X}}\}$.

Definition 2.7, [2]. We have $f: (\mathcal{X}, m_{\mathcal{X}}) \longrightarrow (Y, m_Y)$ is called *m*-open (*m*-closed) if *f*(*V*) is an m_Y-open (m_Y-closed) set in (Y, m_Y) for every $m_{\mathcal{X}}$-open ($m_{\mathcal{X}}$-closed) subset *V* in $(\mathcal{X}, m_{\mathcal{X}})$.

Definition 2.8. A subset *V* of *m*-space $(\mathcal{X}, m_{\mathcal{X}})$ is $m_{\mathcal{X}}$-*b*-open set if $V \subseteq m_{\mathcal{X}}\ \overline{(m_{\mathcal{X}} - U^{\circ})} \cup m_{\mathcal{X}} - (m_{\mathcal{X}} - \overline{U})^{\circ}$.

Definition 2.9. A subset *V* of an *m*-space $\mathcal{X}$ is called $m_{\mathcal{X}}$-ω-open set if every single $x \in V$, there exists an $m_{\mathcal{X}}$-open set *U* containing *x*, such that $U \backslash V$ is a countable set and the complement of an $m_{\mathcal{X}}$-ω-open set is called $m_{\mathcal{X}}$-ω-closed set. Clearly every $m_{\mathcal{X}}$-open set is $m_{\mathcal{X}}$-ω-open set, but the converse might be not valid as a rule.

Example 2.1. Suppose $(\mathcal{X}, m_{\mathcal{X}})$ is indiscrete *m*-structure space then any nonempty proper subset of it is $m_{\mathcal{X}}$-ω-open set, but not $m_{\mathcal{X}}$-open, where $\mathcal{X}$ is in finite countable.

Definition 2.10. Assume $(\mathcal{X}, m_{\mathcal{X}})$ is a minimal space and *V* is a subset of it then $x \in \mathcal{X}$ is called $m_{\mathcal{X}}$-ω-int-point to *V* if there exists an $m_{\mathcal{X}}$-ω-open set *U*, such that $x \in U \subseteq V$. The set of all $m_{\mathcal{X}}$-ω-int-points for *V* is denoted by $m_{\mathcal{X}}$-ω-U°.

Remark 2.1. Every $m_{\mathcal{X}}$ interior point is $m_{\mathcal{X}}$-ω-interior point but the convers is not true.

Example 2.2. Let $\mathcal{X} = \{a,b,c\}$, $m_{\mathcal{X}} = \{\varnothing, \mathcal{X}, \{a\}, \{b\}, \{c\}\}$. Since each point is $m_{\mathcal{X}}$-ω-interior, then the points of each set are its $m_{\mathcal{X}}$-ω-interior points.

Example 2.3. In Example 2.1, each point in $\mathcal{X}$ is $m_{\mathcal{X}}$-ω-interior points to $\mathcal{X}$.

Definition 2.11. Suppose $\mathcal{X}$ is a minimal space and $V \subseteq X$, then $x \in \mathcal{X}$ is called $m_{\mathcal{X}}$-ω-limit point to *V* if all $m_{\mathcal{X}}$-ω-open set *U* containing *x*, we have $(U \backslash \{x\}) \cap V \neq \varnothing$, the set of all $m_{\mathcal{X}}$-ω-limit points for *V* is denoted by $m_{\mathcal{X}}$-ω-d(*V*).

Remark 2.2. Every $m_{\mathcal{X}}$-ω limit point is $m_{\mathcal{X}}$-limit point but the converse is not true.

Example 2.4. Let $\mathcal{X} = \{1, 2, 3\}$ τ = indiscrete topology $A = \{1, 2\}$, then 3 is a $m_{\mathcal{X}}$-limit point to A, since {3} is ω-open set containing 3 and $\{3\} \cap A = \varnothing$.

Definition 2.12. Suppose $\mathcal{X}$ is a minimal space and $V \subseteq \mathcal{X}$ then $x \in \mathcal{X}$ is called $m_{\mathcal{X}}$-ω-adherent point to *V* if every single $m_{\mathcal{X}}$-ω-open set *U* containing x intersected with *V*. (i.e., $U \cap V \neq \varnothing$), the set of all $m_{\mathcal{X}}$-ω-adherent points for *V* is denoted by $m_{\mathcal{X}}$-ω-adh(*V*) or $m_{\mathcal{X}}$-ω-$\overline{V}$.

Example 2.5. Assume $\mathcal{X} = \{1, 2, 3\}$ and $m_{\mathcal{X}}$ be any minimal structure. Obviously $1, 2 \in \mathcal{X}$ are $m_{\mathcal{X}}$-ω-adherent points for the set {1, 2}.

Definition 2.13. Suppose $\mathcal{X} \neq \varnothing$ set and $m_{\mathcal{X}}$ a minimal structure on $\mathcal{X}$. For a subset *V* of $\mathcal{X}$, the $m_{\mathcal{X}}$-ω-Cl(*U*) of *V* and the $m_{\mathcal{X}}$-ω –int(*U*) of *V* are characterized by:
1. $m_{\mathcal{X}}$-ω-$U^{\circ} = \cup\{W: W \subseteq U$, ω is an $\mathcal{X}$-ω-open$\}$.
2. $m_{\mathcal{X}}$-ω-$\overline{U} = \cap\{E: U \subseteq E, E$ is an $m_{\mathcal{X}}$-ω-closed$\}$.

Example 2.6. Suppose $\mathcal{X} = \mathbb{R}$ and $m_{\mathcal{X}} = \{\varnothing, \mathbb{R}\backslash\{1\}, \mathbb{R}\backslash\{2\}, \mathbb{R}\}$. So $m_{\mathcal{X}}$-ω-Int($\mathbb{R}\backslash\{1\}$) = $\mathbb{R}\backslash\{1\}$ and $m_{\mathcal{X}}$-ω-Cl($\mathbb{R}\backslash\{1\}$) = $\mathbb{R}$.

Definition 2.14. A function $f: (\mathcal{X}, m_{\mathcal{X}}) \rightarrow (Y, m_Y)$ is called *m*-ω-open (*m*-ω-closed) function if *f*(*V*) is an m_Y-ω-open (m_Y-ω-closed) set in (Y, m_Y) for every $m_{\mathcal{X}}$-open ($m_{\mathcal{X}}$-closed) subset *V* in $(\mathcal{X}, m_{\mathcal{X}})$.

Example 2.7. Let $\mathcal{X} = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $m_{\mathcal{X}} = \{\varnothing, \{b,c\}, \{a,c\}, \{a,b\}, \mathcal{X}\}$, $m_Y = \{\varnothing, \{2,3\}, \{1,3\}\ Y\}$. We define $f: (\mathcal{X}, m_{\mathcal{X}}) \longrightarrow (Y, m_Y)$ such that $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. It is clear that f is an *m*-ω-open (*m*-ω-closed) function but neither $m_{\mathcal{X}}$-open nor $m_{\mathcal{X}}$-closed function.

Proposition 2.1. Suppose $(\mathcal{X}, m_{\mathcal{X}})$ be a minimal space, the following are equivalent.
1. The union of any family of $m_{\mathcal{X}}$-ω-O sets is an $m_{\mathcal{X}}$-ω-O set.
2. The intersection of any family of $m_{\mathcal{X}}$-ω-C sets is an $m_{\mathcal{X}}$-ω-C set.

Proof. 1. Assume V_α is a $m_{\mathcal{X}}$-ω-O set for each $\alpha \in \Lambda$. To prove that $\cup\{V_\alpha, \alpha \in \Lambda\}$ is an $m_{\mathcal{X}}$-ω-O. Let $x \in \cup\{V_\alpha, \alpha \in \Lambda\}$, then $x \in V_{\alpha i}$ for some $\alpha_i \in \Lambda$. Since $V_{\alpha i}$ is an $m_{\mathcal{X}}$-ω-O, then there exists U be an $m_{\mathcal{X}}$-O set such that $x \in U$ and $U \setminus V_{\alpha i}$ is a countable set. Since $V_{\alpha i} \subseteq \cup\{V_\alpha, \alpha \in \Lambda\}$, then $(\cup V_\alpha, \alpha \in \Lambda\})^c \subseteq (V_{\alpha i})^c$. So $U \cap (\cup V_\alpha, \alpha \in \Lambda\})^c \subseteq U \cap (V_{\alpha i})^c$. Hence $U \setminus \cup\{V_\alpha, \alpha \in \Lambda\} \subseteq U \setminus V_{\alpha i}$. Since $U \setminus V_{\alpha i}$ is a countable set, then $U \setminus \cup\{V_\alpha, \alpha \in \Lambda\}$ is also countable. Hence $\cup\{V_\alpha, \alpha \in \Lambda\}$ is an $m_{\mathcal{X}}$-ω-O set.
2. Clear by (1). ■

Remark 2.3. Suppose the pair $(\mathcal{X}, m_{\mathcal{X}})$ is a minimal space. For subsets *A* and *B* of $\mathcal{X}$, the following properties hold:

1. The intersection of any collection of $m_{\mathcal{X}}$-ω-O set may not be an $m_{\mathcal{X}}$-ω-O set.
2. The union of any collection of $m_{\mathcal{X}}$-ω-C set may not be an $m_{\mathcal{X}}$-ω-C set.

Example 2.8. Let $\mathcal{X} = \mathbb{R}$ and $m_{\mathcal{X}} = \{\emptyset, \mathbb{R}, (-1, 1], [1, 3]\}$, then $m_{\mathcal{X}}$-ω-O sets are {∅, (–1, 1], [1, 3], (–1, 1), (1, 3), (1, 3], [1, 3), ℝ \{finite set}, ℝ }, we note that the sets {(–1,1]}, {[1,3]} are an $m_{\mathcal{X}}$-ω-open set and {(–1,1]} ∩ {[1, 3]} = {1} not be an $m_{\mathcal{X}}$-ω-open set.

Proposition 2.2. Suppose V is a subset of minimal space $\mathcal{X}$, then:
1. V is an $m_{\mathcal{X}}$-ω-O set -if and only if and $m_{\mathcal{X}}$-ω-$V^{\circ} = V$.
2. V is an $m_{\mathcal{X}}$-ω –C set -if and only if $m_{\mathcal{X}}$-ω-$\bar{V} = V$.

Proof. 1. As the union of $m_{\mathcal{X}}$-ω-O set is $m_{\mathcal{X}}$-ω-O set, then $m_{\mathcal{X}}$-ω-V° is the biggest $m_{\mathcal{X}}$-ω-O set contained in V. But V is m_{W}-ω-O set, then $m_{\mathcal{X}}$-ω-$V^{\circ} = V$. Conversely, whenever $m_{\mathcal{X}}$-ω-$V^{\circ} = V$, then V is $m_{\mathcal{X}}$-ω-O set, (since $m_{\mathcal{X}}$-ω-V° is an $m_{\mathcal{X}}$-ω-O set).

2. As the intersection of $m_{\mathcal{X}}$-ω-C sets is $m_{\mathcal{X}}$-ω-C set, then $m_{\mathcal{X}}$-ω-$\bar{V}$ is the smallest $m_{\mathcal{X}}$-ω-C set containing V. Since V is an $m_{\mathcal{X}}$-ω-C set, then $m_{\mathcal{X}}$-ω-$\bar{V} = V$.

Conversely, whenever $m_{\mathcal{X}}$-ω-$\bar{V} = V$, then V is an $m_{\mathcal{X}}$-ω-C set (since $m_{\mathcal{X}}$-ω-$\bar{V}$ is an $m_{\mathcal{X}}$-ω-C set). ■

Proposition 2.3. Suppose V and U be two subsets of minimal space $\mathcal{X}$ and $V \subseteq U$, then:
1. $m_{\mathcal{X}}$-$V^{\circ} \subseteq m_{\mathcal{X}}$-ω-$V^{\circ}$.
2. $m_{\mathcal{X}}$-ω-$\bar{V} \subseteq m_{\mathcal{X}}$-$\bar{V}$.
3. $m_{\mathcal{X}}$-ω-$\bar{V} \subseteq m_{\mathcal{X}}$-ω-$\bar{U}$.
4. $m_{\mathcal{X}}$-ω-$V^{\circ} \subseteq m_{\mathcal{X}}$-ω-$U^{\circ}$.
5. $m_{\mathcal{X}}$ω-$X^{\circ} = \mathcal{X}$ and $m_{\mathcal{X}}$-ω-$(\emptyset)^{\circ} = \emptyset$.
6. $m_{\mathcal{X}}$-ω-Cl($\mathcal{X}$) = $\mathcal{X}$ and $m_{\mathcal{X}}$-ω-Cl(∅) = ∅.
7. $m_{\mathcal{X}}$-ω-$V^{\circ} \subseteq$ V and V$\subseteq m_{\mathcal{X}}$-ω-Cl(V)
8. $m_{\mathcal{X}}$-ω-Int($m_{\mathcal{X}}$-ω-int(V)) = $m_{\mathcal{X}}$-ω-Int(V)
9. $m_{\mathcal{X}}$-ω-Cl($m_{\mathcal{X}}$-ω-cl(V)) = $m_{\mathcal{X}}$-ω-Cl(V)
10. $m_{\mathcal{X}}$-ω-Cl(V^c) = ($m_{\mathcal{X}}$-ω-Int(V))^c.
11. $m_{\mathcal{X}}$-ω-Int(V^c) = ($m_{\mathcal{X}}$-ω-Cl(V))^c.

Proof. 1. Suppose x ∈ $m_{\mathcal{X}}$- V°, then there exists an $m_{\mathcal{X}}$-O set $V_{\mathcal{X}}$ such that $x \in V_{\mathcal{X}} \subseteq V$, for this reason that every $m_{\mathcal{X}}$-O set is an $m_{\mathcal{X}}$-ω-O set, then $x \in m_{\mathcal{X}}$-ω-V°.

2. Suppose $x \in m_{\mathcal{X}}$-ω-$\bar{V}$ and suppose $x \notin m_{\mathcal{X}}$-$\bar{V}$, then there exists M be an $m_{\mathcal{X}}$-O set such that $x \in M$ and $M \cap V = \emptyset$. For this reason that every $m_{\mathcal{X}}$-O set is an $m_{\mathcal{X}}$-ω-O set, then $x \notin m_{\mathcal{X}}$-ω-$\bar{V}$ and so $m_{\mathcal{X}}$-ω-$\bar{V} \subseteq m_{\mathcal{X}}$-$\bar{V}$.
3. Suppose $x \in m_{\mathcal{X}}$ -ω- $\bar{V}$, then each $m_{\mathcal{X}}$ -ω-O set K containing x intersect V, since $V \subseteq U$, then the set K intersect U. Hence $x \in m_{\mathcal{X}}$-ω-$\bar{U}$.
4. Suppose $x \in m_{\mathcal{X}}$-ω-V°, then there exists an $m_{\mathcal{X}}$-ω-O set $V_{\mathcal{X}}$, such that $x \in V_{\mathcal{X}} \subseteq V$. For this reason implies that $V \subseteq U$, then $x \in V_{\mathcal{X}} \subseteq U$. Hence $x \in m_{\mathcal{X}}$-ω-U°.
5. For this reason that $\mathcal{X}$ and ∅ are $m_{\mathcal{X}}$-ω-O sets, but$m_{\mathcal{X}}$-ω-Int($\mathcal{X}$) = ∪{V: V is an $m_{\mathcal{X}}$-ω-O, $V \subseteq \mathcal{X}$ } = $\mathcal{X}$ ∪ (all $m_{\mathcal{X}}$-ω-O sets) = $\mathcal{X}$. In this manner $m_{\mathcal{X}}$-ω-Int ($\mathcal{X}$) = $\mathcal{X}$. So far by the as ∅ is the only $m_{\mathcal{X}}$-ω-O set contained in ∅, then $m_{\mathcal{X}}$-ω-Int (∅) = ∅.
6. Since $m_{\mathcal{X}}$-ω-cl($\mathcal{X}$) = ∩{U: $\mathcal{X} \subseteq U$, U is $m_{\mathcal{X}}$-ω-C set}. But $\mathcal{X}$ is the only $m_{\mathcal{X}}$-ω-C set comprising X. In this way $m_{\mathcal{X}}$-ω-Cl($\mathcal{X}$) = $\mathcal{X}$. Thus $m_{\mathcal{X}}$-ω-Cl($\mathcal{X}$) = $\mathcal{X}$. Inasmuch as $m_{\mathcal{X}}$-ω $-\bar{\emptyset}$, $m_{\mathcal{X}}$-ω-$\bar{\emptyset}$ = ∩{U: ∅ ⊆ U, U is an $m_{\mathcal{X}}$-ω-C} = ∅ ∩ any $m_{\mathcal{X}}$-ω-C sets comprising ∅ = ∅. In this way $m_{\mathcal{X}}$-ω-$\bar{\emptyset}$ = ∅.
7. By definition.
8. By Proposition 2.1, we note that $m_{\mathcal{X}}$-ω-Int(V) is an $m_{\mathcal{X}}$-ω-O set. So, by Proposition 2.2, we conclude $m_{\mathcal{X}}$-ω-Int($m_{\mathcal{X}}$-ω-Int(V)) = $m_{\mathcal{X}}$-ω-Int(V).
9. By Proposition 2.1, we note that $m_{\mathcal{X}}$-ω-Cl(V) is an $m_{\mathcal{X}}$-ω-C set. So, by Proposition 2.2 we conclude that $m_{\mathcal{X}}$-ω-$\overline{m_{\mathcal{X}} - \omega - \bar{V}} = m_{\mathcal{X}}$-ω-$\bar{V}$.
10. Suppose $x \in m_{\mathcal{X}}$-ω-Cl(V^c) and suppose $x \notin$ ($m_{\mathcal{X}}$-ω-Int(V))^C, then $x \in m_{\mathcal{X}}$-ω-Int(V). Thus, there is an $m_{\mathcal{X}}$-ω-O set $V_{\mathcal{X}}$, such that x ∈ $V_{\mathcal{X}} \subseteq V$. In this way $x \in V_{\mathcal{X}}$ and $V_{\mathcal{X}} \cap V^c = \emptyset$. So x ∉ $m_{\mathcal{X}}$-ω-Cl(V^c). Thus we get $m_{\mathcal{X}}$-ω-Cl(V^c) ⊆ ($m_{\mathcal{X}}$-ω-Int(V))^C. Now assume $x \notin m_{\mathcal{X}}$-ω-Cl(V^c). Thus there is an $m_{\mathcal{X}}$-ω-O set $V_{\mathcal{X}}$, such that $x \in V_{\mathcal{X}}$ and $V_{\mathcal{X}} \cap V^c = \emptyset$. Thus $x \in V_{\mathcal{X}} \subseteq V$, and in this manner $x \in m_{\mathcal{X}}$-ω-Int(V). So x ∉ ($m_{\mathcal{X}}$-ω-Int(V))^c. Thus we get ($m_{\mathcal{X}}$-ω-Int(V))$^c \subseteq m_{\mathcal{X}}$-ω-cl($V^c$).
11. Assume $x \in m_{\mathcal{X}}$-ω-int(V^c), then there is an $m_{\mathcal{X}}$-ω-O set $V_{\mathcal{X}}$such that $x \in V_{\mathcal{X}} \subseteq V^c$. In this manner $x \in V_{\mathcal{X}}$ and $V_{\mathcal{X}} \cap V = \emptyset$. So $x \notin m_{\mathcal{X}}$-ω-Cl(V). Thus we get $x \in$ ($m_{\mathcal{X}}$-ω-cl(V))^c, therefore $m_{\mathcal{X}}$-ω-Int(V^c) ⊆ ($m_{\mathcal{X}}$-ω-Cl(V))^c. Now, let $x \in$ ($m_{\mathcal{X}}$-ω-Cl(V))^c, then $x \notin m_{\mathcal{X}}$-ω-Cl(V) and in this way there is an $m_{\mathcal{X}}$-ω-O set $V_{\mathcal{X}}$ such that $x \in V_{\mathcal{X}}$ and $V_{\mathcal{X}} \cap V = \emptyset$. So $x \in V_{\mathcal{X}}$and $V_{\mathcal{X}} \subseteq V^c$. Therefore $x \in m_{\mathcal{X}}$-ω-Int(V^c) and hence ($m_{\mathcal{X}}$-ω-Cl(V))$^c \subseteq m_{\mathcal{X}}$-ω-Int($V^c$). ■

Proposition 2.4. A subset A of an minimal space $\mathcal{X}$ is an $m_{\mathcal{X}}$-ω-O -if and only if for every $x \in A$, there exists an $m_{\mathcal{X}}$-open subset U containing x and a countable subset M, such that $U \setminus M \subseteq A$.

Proof. Let A be an $m_{\mathcal{X}}$-ω-O and $x \in A$, then there exists an $m_{\mathcal{X}}$-O subset V containing x, such that $V \setminus A$ is countable. Assume $M = V \setminus A = V \cap (X \setminus A)$. Then $V \setminus M \subseteq A$. Other side, assume $x \in A$. Then there exists an $m_{\mathcal{X}}$ -O subset V containing x and a countable subset M, such that $V \setminus M \subseteq A$. Thus $V \setminus A \subseteq M$ and $V \setminus A$ is countable set. ■

Theorem 2.1. Let $\mathcal{X}$ be an m-space and $M \subseteq \mathcal{X}$. If M be an $m_{\mathcal{X}}$-ω-closed, then $M \subseteq B \cup K$ for some $m_{\mathcal{X}}$-closed subset B and a countable subset K.

Proof. Let M be an $m_{\mathcal{X}}$-ω-closed, then $\mathcal{X} \setminus M$ is an $m_{\mathcal{X}}$-ω-O and hence for every x ∈ $\mathcal{X} \setminus M$, there exists $m_{\mathcal{X}}$-O set V containing x and a countable set K, such that $V \setminus K \subseteq \mathcal{X} \setminus$ M. Thus $M \subseteq (X \setminus V) \cup K$. Let $B = \mathcal{X} \setminus V$. Then B is an $m_{\mathcal{X}}$-closed such that $M \subseteq B \cup K$. ■

Theorem 2.2. If each non-empty $m_{\mathcal{X}}$-O set of an m-space $\mathcal{X}$ is an infinite and $\mathcal{X}$ have the Property (I), then $m_{\mathcal{X}}$-ω-Cl(V) = $m_{\mathcal{X}}$-Cl(V) for each $m_{\mathcal{X}}$-O set V of $\mathcal{X}$.

Proof. Clearly $m_{\mathcal{X}}$-ω-Cl(V) ⊆ $m_{\mathcal{X}}$-Cl(V) by proposition (2.3). Now, let $x \in m_{\mathcal{X}}$-Cl(V) and B be an $m_{\mathcal{X}}$-ω-O subset containing x. Then there exists an $m_{\mathcal{X}}$-O set U containing x and a countable set M, such that $U \setminus M \subseteq B$. Thus $(U \setminus M) \cap V \subseteq B \cap V$ and so $(U \cap V) \setminus M \subseteq B \cap V$. Since $x \in U$ and $x \in m_{\mathcal{X}}$-Cl(V), $U \cap V \neq \emptyset$. Since $\mathcal{X}$ have the Property (I), $U \cap V$ is an $m_{\mathcal{X}}$-O and by the hypothesis $U \cap V$ is an infinite set, so $(U \cap V) \setminus M$ is also infinite. Hence $B \cap V$ is infinite. Therefore, $B \cap V \neq \emptyset$, which means that $x \in \mathcal{X}$-ω-Cl(V). Hence $m_{\mathcal{X}}$-Cl(V) ⊆ $m_{\mathcal{X}}$-ω-Cl(V) and therefore $m_{\mathcal{X}}$-Cl(V) = $m_{\mathcal{X}}$-ω-Cl(V). ■

Corollary 2.1. If each non-empty $m_{\mathcal{X}}$-O set of an m-space $\mathcal{X}$ is an infinite and $\mathcal{X}$ have the Property (I), then $m_{\mathcal{X}}$-ω-int(V) = $m_{\mathcal{X}}$-Int(V) for each $m_{\mathcal{X}}$-closed set V of $\mathcal{X}$.

Proof. Clearly $m_{\mathcal{X}}$-Int(V) ⊆ $m_{\mathcal{X}}$-ω-Int(V) by Proposition 2.3. Now, suppose $x \in m_{\mathcal{X}}$-ω-int(V). Then $x \notin \mathcal{X} \setminus m_{\mathcal{X}}$-ω-Int($V$) and so $x \notin m_{\mathcal{X}}$-ω-Cl(V^c). By Theorem 2.3, we conclude that $x \notin m_{\mathcal{X}}$-Cl(V^c) and so $x \in m_{\mathcal{X}}$-Int(V). Hence $m_{\mathcal{X}}$-ω-Int(V) ⊆ $m_{\mathcal{X}}$-Int(V) and therefore $m_{\mathcal{X}}$-ω-Int(V) = $m_{\mathcal{X}}$-Int(V). ■

Definition 2.15. Let $\mathcal{X}$ be a minimal space, $V \subseteq \mathcal{X}$, V is called $m_{\mathcal{X}}$-ω-dense in $\mathcal{X}$ if $m_{\mathcal{X}}$-ω-cl(V) = $\mathcal{X}$.

Definition 2.16. An m-space ($\mathcal{X}$, $m_{\mathcal{X}}$) is called minimal compact if every $m_{\mathcal{X}}$-O cover of $\mathcal{X}$ has a finite subcover.

Definition 2.17. A minimal space ($\mathcal{X}$,$m_{\mathcal{X}}$) is called m-b-compact if for every $m_{\mathcal{X}}$-b-O cover has a finite subcover.

Definition 2.18. A minimal space ($\mathcal{X}$,$m_{\mathcal{X}}$) is called m-ω-compact if for every $m_{\mathcal{X}}$-ω-O cover has a finite subcover.

Definition 2.19. A minimal space ($\mathcal{X}$,$m_{\mathcal{X}}$) is called m-ω_b-compact if for every $m_{\mathcal{X}}$-ω_b-O cover has a finite subcover.

Proposition 2.5. Let $(\mathcal{X}, m_{\mathcal{X}})$ be minimal space, then:
(i) Every m-b-compact space is m-compact.
(ii) Every m-ω-compact space is m-compact.
(iii) Every m-ω_b-compact space is m-compact, m-b-compact and m-ω-compact.

Proof. (i) Let $(\mathcal{X}, m_{\mathcal{X}})$ be m-b-compact space and $c = \{U_\alpha : \alpha \in \Lambda\}$ be $m_{\mathcal{X}}$-open cover for $\mathcal{X}$, but every $m_{\mathcal{X}}$-open set in $m_{\mathcal{X}}$-b-open set, so c is $m_{\mathcal{X}}$-b-open cover for m-b-compact space $\mathcal{X}$, so $\mathcal{X} \subseteq \bigcup_{i=1}^{n}\{U_{\alpha_i}\}$, then $\mathcal{X}$ is m-compact space

The proof of other parts is similar to the proof of (i). ■

Proposition 2.6. Every $m_{\mathcal{X}}$-ω_b-closed subset of m-ω_b-compact space is also m-ω_b-compact.

Proof. Let F be $m_{\mathcal{X}}$-ω_b-closed subset of m-ω_b-compact space $\mathcal{X}$ and $c = \{U_\alpha : \alpha \in \Lambda\}$ be $m_{\mathcal{X}}$-ω_b-open cover to F, that is $F \subseteq \bigcup_{\alpha\in\Lambda}\{U_\alpha\}$, so:

$$\mathcal{X} \subseteq (\bigcup_{\alpha\in\Lambda}\{U_\alpha\}) \cup (\mathcal{X} - F)$$

but $\mathcal{X}$ is m-ω_b-compact, then $\mathcal{X} \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \cup (\mathcal{X} - F)$ that is $F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, therefore F is m-ω_b-compact. ■

Definition 2.20. A function $f: (\mathcal{X}, m_{\mathcal{X}}) \to (Y, m_Y)$ is said to be m-ω_b-continuous if $f^{-1}(U)$ is $m_{\mathcal{X}}$-ω_b-open in $\mathcal{X}$ for every $m_{\mathcal{X}}$-open set in Y.

Proposition 2.7. Let $f: (\mathcal{X}, m_{\mathcal{X}}) \to (Y, m_Y)$ be surjective m-ω_b-continuous function, if $\mathcal{X}$ is m-ω_b-compact space, then Y is m-compact.

Proof. Let $c = \{U_\alpha : \alpha \in \Lambda\}$ be $m_{\mathcal{X}}$-open cover for Y, that is $Y \subseteq \bigcup_{\alpha\in\Lambda}\{U_\alpha\}$, then:

$$\mathcal{X} = f^{-1}(Y) \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda}\{U_\alpha\}) = \bigcup_{\alpha\in\Lambda} f^{-1}(\{U_\alpha\})$$

but $\mathcal{X}$ is m-$\omega_b$compact, then:

$$\mathcal{X} \subseteq \bigcup_{i=1}^{n} f^{-1}(\{U_{\alpha_i}\})$$

and so:

$$Y = f(\mathcal{X}) \subseteq \bigcup_{i=1}^{n} f f^{-1}(\{U_{\alpha_i}\}) = \bigcup_{i=1}^{n}(\{U_{\alpha_i}\})$$

Therefore, Y is m-compact space. ■

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Conflicts of Interest
The authors declare that there is no conflict of interest.

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