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# Study the Dynamics of Commensalism Interaction with Michaels-Menten Type Prey Harvesting

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**Abstract**

This paper suggests and analyses a model consisting of two commensal populations with Michaelis-Menten type of harvesting for the first population. The first harvested commensal species draws strength from the second hosted species. The overall dynamics are provided to achieve the coexistence, stability and persistence of the equilibrium points for the proposed system. The local bifurcation near the positive equilibrium point is attained. Moreover, numerical simulation using MATLAB is investigated to the impact of the commensalism interaction on the behavior of the planned model. The analysis shows that the role of commensalism prevents the first population from extinction, which could be helpful for the survival of both species.

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## 1. Introduction

Mutualism is a relationship between two species when one of them get benefits from the activity of the other. During the past several decades, theoretical studies the stability and persistence of mutualism interaction. For example, in [1], a delay commensalism model has been proposed and investigated the periodic solution of the model with impulsive action. Moreover, several theoretical [2-5] also claimed that it might be more appropriate to assume that the connection between two species is non-linear. They recognized the commensalism interaction with functional response. Lei has developed a stage-structure commensalism system; he has concluded that the system might admit the unique positive fixed point that is globally stable [6].

On the other hand, harvesting species is necessary to obtain a human resource. Many researchers examined the impact of harvesting on population dynamics [7-9]. There are three kinds of harvesting: constant, linear, and non-linear harvesting. [10,11]. The latter is more realistic from the biological point of view [12]. In [13], a harvesting form known as the Michaelis-Menten type of harvesting has been suggested. In general, some harvesting might lead to the complex dynamic behaviors of the system; for instance, In [11], it has been shown that the Holling type II harvesting with the logistic model may admit zero, one or two positive equilibrium points. Further, in [14], it has been shown that the predator-prey model with Michaelis-Menten type harvesting might have a rich bifurcation phenomenon.

This paper purposes of studying the effect of Michaelis-Menten's type of harvesting in the commensal of two ecological populations. According to the Holling-type II functional response, the first population benefits from the second. The residual of this article is arranged as follows: Section two considers the equilibrium points for our model. In section three, the stability of the equilibrium points has been provided. Finally, some numerical analyses have been investigated to confirm our analytical result.

## 2. Assumptions of the Model

Suppose a two coexist species model that incorporates the non-linear term of harvesting for the first species. The first species benefit from the latter, while the latter neither benefit nor harm from the former. Based on assumptions. $u(t)$ is the density of the harvested first species, $v(t)$ is the density of the second.

Under the above assumptions, the model can be offered by the following system of differential equations:

$$\begin{aligned} \frac{du}{dt} &= ru\left(1-\frac{u}{K}\right)+\frac{\beta uv}{\alpha+u}-\frac{qEu}{cE+lu}=uf_1(u,v), \\ \frac{dv}{dt} &= sv\left(1-\frac{v}{m}\right)-dv=vf_2(u,v). \end{aligned} \qquad (1)$$

Here, model (1) has been analyzed with the initial conditions $u(0)\geq 0$ and $v(0)\geq 0$.

All system parameters (1) are assumed to be positive and described as: $k, m$ are the carrying capacities of the first and second species, respectively with intrinsic growth rate $r$, $s$; $\beta$ is a commensalism coefficient; $\alpha$ is half-saturation constant; $q, E$ are the effort, and the catchability rate applied on the first species, i.e.; $qE$ represents the harvesting rate of the first species; $c, l$ are suitable constants; $d$ represent the predator's natural death rate.

Figure 1 illustrates the schematic sketch of system (1) under examination.

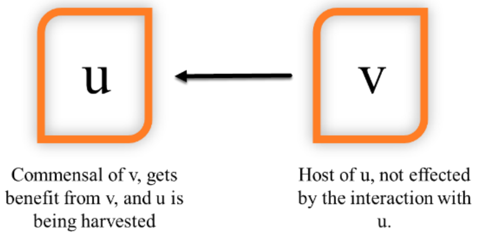


**Figure 1.** Schematic sketch of system (1).

The equations on the right-hand side of system (1) are continuously differentiable functions on $\mathbb{R}_+^2 = \{(u,v), u \geq 0, v \geq 0\}$. Therefore, there exists a unique solution for system (1).

The positive invariance of $\mathbb{R}_+^2$ for system (1) is studied first, and then boundedness is proven

## 3. Positivity and Boundedness of the Solution

**Lemma 1.** System (1) is positively invariant.

**Proof.** Let $U = (u,v)^T \in \mathbb{R}^2$ and, $F(U) = [f_1(U), f_2(U)]^T$, where, $F(U)\colon \mathbb{R}_+^2 \to \mathbb{R}^2$ and $f \in C_+^\infty(\mathbb{R}_+^2)$. Then the system (1) becomes:

$$\dot{U} = f(U), \tag{2}$$

with $U(0) = U_0$. It is clear for any $U(0) \in \mathbb{R}_+^2$, such that $U_i = 0$, then $[f_i(U)]_{n_i=0} \geq 0$ (for $i$ = 0; 1; 2). Now, any solution of the eq. (2) with $U_0 \in \mathbb{R}_+^2$, say $U(t) = U(t; U_0)$, is such that $U(t) \in \mathbb{R}_+^2$, for all $t > 0$. Thus, system (1) is positively invariant [15]. ■

**Theorem 1.** All solutions $u(t)$ and $v(t)$ of the system (1) with the initial conditions $(u, v)$ are uniformly bounded.

**Proof.** Let $(u(t), v(t))$ be any solution of system (1) with a non-negative initial condition. Then for $w(t) = u(t) + v(t)$, we have $\frac{dw}{dt} = \frac{du}{dt} + \frac{dv}{dt}$

$$\frac{dw}{dt} = ru\left(1 - \frac{u}{K}\right) + \frac{\beta uv}{\alpha + u} - \frac{qEu}{cE + lu} + sv\left(1 - \frac{v}{m}\right) - dv$$

Hence, $\frac{dw}{dt} + \eta\, w \;\leq\; ru + sv + \beta km = \mu$

where $\eta = min.\{eq, d\}$, Then $\frac{dw}{dt} + \eta w \leq \mu$, then:

$$0 \leq w(u(t), v(t)) \;\leq\; \frac{\mu}{\eta}\,(1 - e^{-\eta t}) + w(0)e^{-\eta t},$$

Hence:

$$0 \leq \limsup_{t\to\infty} \mathrm{w(t)} \leq \frac{\mu}{\eta},$$

Therefore, all the solutions of system (1) that are initiated in $\mathbb{R}_+^2$ are attracted to the region $\xi = \left\{(u,v) \in \mathbb{R}_+^2 : w = u + v \leq \frac{\mu}{\eta}\right\}$ under the given conditions. Thus, these solutions are uniformly bounded. ■

## 4. Existence of Equilibria and their Stability

In this section, the existence and stability analysis of the equilibrium points of system (1) is calculated. The computation shows that system (1) has the following equilibria

1. The vanishing equilibrium point: $F_1 = (0,0)$.
2. The second population equilibrium point: $F_2 = (0, \ddot{v})$, where $\ddot{v} = \frac{(s-d)m}{s}$, exists when:
$$s > d. \tag{3}$$
3. The first population equilibrium point $F_3 = (\tilde{u}, 0)$, where $\tilde{u}$ is the positive root of the following quadratic polynomial
$$\left(\frac{rl}{k}\right)u^2 - \left(rl - \frac{crE}{k}\right)u + qE - rcE = 0. \tag{4}$$
By discard rule of sign, eq. (4) has unique positive solution say $\tilde{u}$ if
$$q < rc. \tag{5}$$
Moreover, if $rl < \frac{crE}{k}$ and $q > rc$ are satisfied, then system (1) has no positive roots. Finally, if $rl > \frac{crE}{k}$ and $q > rc$, then system (1) has two positive roots.
4. The positive equilibrium point $F_4 = (u^*, v^*)$, where $v^* = \frac{(s-d)m}{s}$, exists when s $> d$ and $u$, is the positive solution of the following polynomial
$$Au^3 + Bu^2 + Cu + D = 0.$$
where $A = rsl > 0$,
$$B = cErs + rsl(\alpha - k) + sqEkl,$$
$$C = -[rks\alpha l + rkscE - rsacE + k\beta lm(s-d) - sqEk\alpha l - sqE^2kc],$$
$$D = -rks\alpha cE - k\beta mcE(s-d) + sqE^2k\alpha c$$
Therefore, by discard rule of sign, the above equation has a positive root, say $u^*$ either if the following conditions
$$\alpha > k$$
$$D < 0,$$
hold. Or when $B, C$ and $D$ are all negative.
Otherwise, system (1) could have no positive fixed point or has two fixed points depending on the sign of $B, C$ and $D$.

### 4.1 Stability of equilibria:

Now, local stability around the above equilibrium points is clarified. First, the variational matrix of the system (1) at each point is computed, and then, the eigenvalues of the resultant matrix are calculated.

1. The variational matrix of system (1) at the vanishing fixed point $F_1 = (0,0)$ can be written as:
$$J(F_1) = \begin{bmatrix} r - \frac{q}{c} & 0 \\ 0 & s - d \end{bmatrix}.$$
Then, the eigenvalues of $J(F_1)$ are given by $\lambda_{01} = r - \frac{q}{c}$ and $\lambda_{02} = s - d$. That means $F_1$ is a locally asymptotical stable point if and only if $r < \frac{q}{c}$ and $s < d$ hold. Otherwise, if one of the previous conditions are violated, then $F_1$ become a saddle point. Moreover, if both of the previous conditions are violated, then $F_1$ become an unstable node.
2. The Jacobian matrix of the system at $F_2$ can be written as:
$$J(F_2) = \begin{bmatrix} r - \frac{q}{c} + \frac{\beta \ddot{v}}{\alpha} & 0 \\ 0 & -(s-d) \end{bmatrix}.$$

Then, the eigenvalues of $J(F_2)$ are given by $\lambda_{11} = r - \frac{q}{c} + \frac{\beta \bar{v}}{\alpha}$ and $\lambda_{12} = -(s-d) < 0$. That means $F_2$ is a locally asymptotical stable point if the following is satisfied

$$r + \frac{\beta \bar{v}}{\alpha} < \frac{q}{c}, \qquad (6)$$

Otherwise, $F_2$ become a saddle point.

3. The Jacobian matrix of the system at $F_3$ can be written as:

$$J(F_3) = \begin{bmatrix} r - \frac{2r\tilde{u}}{k} - \frac{cqE^2}{(cE+l\tilde{u})^2} & \frac{\beta\tilde{u}}{\alpha+\tilde{u}} \\ 0 & s-d \end{bmatrix}.$$

Then, the eigenvalues of $J(F_3)$ are given by $\lambda_{21} = r - \frac{2r\tilde{u}}{k} - \frac{cqE^2}{(cE+l\tilde{u})^2}$ and $\lambda_{22} = s-d$. That means $F_3$ is a locally asymptotical stable point if the following conditions are satisfied

$$r < \frac{2r\tilde{u}}{k} + \frac{qE^2}{(cE+l\tilde{u})^2}$$
$$s < d.$$

On the other hand, if one of the previous conditions are violated, then $F_3$ become a saddle point. Moreover, if both of the previous conditions are violated, then $F_3$ become an unstable node.

4. The Jacobian matrix of the system at $F_4$ can be written as:

$$J(F_4) = \begin{bmatrix} r - \frac{2ru^*}{k} + \frac{\alpha\beta v^*}{(\alpha+u^*)^2} - \frac{cqE^2}{(cE+l\tilde{u})^2} & \frac{\beta u^*}{\alpha+u^*} \\ 0 & -(s-d) \end{bmatrix}.$$

Then, the eigenvalues of $J(F_4)$ are given by $\lambda_{31} = r - \frac{2ru^*}{k} + \frac{\alpha\beta v^*}{(\alpha+u^*)^2} - \frac{cqE^2}{(cE+l\tilde{u})^2}$ and $\lambda_{32} = -(s-d)$. That means $F_4$ is a locally asymptotical stable point in the $R_+^2$ if the following are satisfied

$$r + \frac{\alpha\beta v^*}{(\alpha+u^*)^2} < \frac{2ru^*}{k} + \frac{cqE^2}{(cE+l\tilde{u})^2},$$
$$s > d.$$

It is clear that if one of the previous conditions are violated, then $F_4$ become a saddle point. Moreover, if both of the previous conditions are violated, then $F_4$ become an unstable node.

In the following, it will be shown that system (1) has no periodic solution,

**Theorem 2.** System (1) has no periodic solution in $\mathbb{R}_+^2$, if the following condition is satisfied:

$$\frac{\beta}{(\alpha+u)^2} > \frac{qEl}{v(cE+lu)^2} \qquad (7)$$

**Proof.** For any initial value $(u,v)$ in $\mathbb{R}_+^2$, let $H(u,v) = \frac{1}{uv}$, $h_1(u,v) = ru\left(1 - \frac{u}{K}\right) + \frac{\beta uv}{\alpha+u} - \frac{qEu}{cE+lu}$ and $h_2(u,v) = sv\left(1 - \frac{v}{m}\right) - dv$.

Clearly, $H(u,v) > 0$, for all $(u,v) \in \mathbb{R}_+^2$ and its $C^1$ function in $\mathbb{R}_+^2$.

Now, since $Hh_1(u,v) = \frac{r}{v} - \frac{ru}{kv} + \frac{\beta}{\alpha+u} - \frac{qE}{v(cE+lu)}$ ; $Hh_2(u,v) = \frac{s}{u}(1 - \frac{v}{m}) - \frac{d}{u}$

Hence:

$$\Delta(u,v) = \frac{\partial Hh_1}{\partial u} + \frac{\partial Hh_2}{\partial v}.$$
$$= -\frac{r}{kv} - \frac{s}{lu} - \frac{\beta}{(\alpha+u)^2} + \frac{qEl}{v(cE+lu)^2} < 0$$

Note that $\Delta(u,v)$ does not change sign if condition (7) satisfies and is not identically zero in the $\mathbb{R}_+^2$. Then according to Bendixson-Dulic criteria, there is no periodic solution. ■

## 5. Persistence Analysis

In this section, the persistence of the system (1) is measured to indicate the existence of all system species for long time behavior. Recall that, from the mathematical point of view, the persistence of a system implies that there are no omega-limit sets on the boundary planes for the strictly positive trajectories that initiate in $\mathbb{R}_+^2$.

**Theorem 3.** Suppose that the boundary equilibrium points conditions hold, then system (1) is uniformly persistent if the following conditions are satisfied:

$$s > \frac{s\bar{v}}{m} + d, \qquad (8)$$
$$r > \frac{r\ddot{u}}{k} + \frac{qE}{cE+l\ddot{u}} \qquad (9)$$

**Proof.** Consider the following function $\varphi(u,v) = u^a v^b$, where $a$ and $b$ are positive constants. Obviously $\varphi(u,v) > 0$ for all $(u,v) \in \mathbb{R}_+^2$ and $\varphi(u,v) \to 0$ when $u \to 0$ or $v \to 0$. Consequently:

$$\omega(u,v) = \frac{\dot{\varphi}}{\varphi} = b\left[s\left(1 - \frac{v}{m}\right) - d\right] + a\left[r\left(1 - \frac{u}{K}\right) + \frac{\beta v}{\alpha+u} - \frac{qE}{cE+lu}\right].$$

Now, the only possible omega limit sets of the system (1) on the boundary of $uv$-plane are the equilibrium points $F_1, F_2$ and $F_3$. Thus according to the Gard method [16], the proof follows, and the system is uniformly persists provided that $\omega(u,v) > 0$ at the boundary fixed points.

Now, since:

$$\omega(F_1) = b(s-d) + a\left(r - \frac{q}{c}\right);$$
$$\omega(F_2) = b\left[s\left(1 - \frac{\bar{v}}{m}\right) - d\right] + a\left(r - \frac{q}{c}\right);$$
$$\omega(F_3) = b(s-d) + a\left(r[1 - \frac{\ddot{u}}{k}] - \frac{qE}{cE+l\ddot{u}}\right).$$

It follows that, $\omega(F_1) > 0, \omega(F_2) > 0$ and $\omega(F_3) > 0$ under conditions (8), (9) and the boundary equilibrium points conditions for all values of $a$ and $b$. Then, system (1) is uniformly persistent. ■

## 6. Bifurcation Analysis

This section determines the bifurcations condition for the positive equilibrium point of the system (1). For this purpose, system (1) can be rephrased in the following vector forms $\frac{dU}{dt} = F(U)$ with $U = \begin{bmatrix} u \\ v \end{bmatrix}$ and $F = \begin{bmatrix} uf_1(u,v) \\ vf_1(u,v) \end{bmatrix}$. For any non-zero vector $K = (k_1,k_2)^T$, the second derivate of $F$ to $U$ is given by:

$$D^2F(U)(K,K) = \begin{bmatrix} \left(-\frac{2r}{k} - \frac{2\alpha\beta v}{(\alpha+u)^3} + \frac{2cE^2ql}{(cE+lu)^3}\right)k_1^2 + \frac{2\alpha\beta k_2}{(\alpha+u)^2} \\ -\frac{2sk_2^2}{l} \end{bmatrix} \qquad (10)$$

**Theorem 4.** For the parameter value $d^* = s$, then system (1), at the equilibrium point $F_4$, has a saddle-node bifurcation.

**Proof.** According to the variational matrix $J(F_4)$, system (1), at the equilibrium point $F_4$, has a zero eigenvalue, say $\lambda_{32} = -(s-d) = 0$, at $d = d^*$, and the Jacobian matrix $J(F_4)$, becomes:

$$J^*(F_4) = \begin{bmatrix} r - \frac{2ru^*}{k} + \frac{\alpha\beta v^*}{(\alpha+u^*)^2} - \frac{cqE^2}{(cE+l\tilde{u})^2} & \frac{\beta u^*}{\alpha+u^*} \\ 0 & 0 \end{bmatrix}.$$

Now, suppose that $K^4 = (k_1^4, k_2^4)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_{32} = 0$. Thus, $(J^*(F_4) - \lambda_{32}I)K^4 = 0$, which implies:

$$k_2^4 = -\left(r - \frac{2ru^*}{k} + \frac{\alpha\beta v^*}{(\alpha+u^*)^2} - \frac{cqE^2}{(cE+l\tilde{u})^2}\right)\frac{(\alpha+u^*)k_1^4}{\beta u^*},$$

where $k_1^4$is any nonzero real number.

Let $X^4 = (x_1^4, x_2^4)^T$ be an eigenvector associated with the eigenvalue $\lambda_{32}$ of the matrix $(J^*(F_4))^T$. Then, $((J^*(F_4))^T - \lambda_{32}I)X^4 = 0$, Subsequently, by solving this equation for $X^4$, it is obtained that $x_1^4 = 0$ and $x_2^4$ is non zero real number.

Now, to confirm that the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial d} = F_d(U,d) = \left(\frac{\partial f_1}{\partial d}, \frac{\partial f_2}{\partial d}\right)^T = (0,-1)^T.$$

Therefore, $F_d(F_4, d^*) = (0,-1)^T$ and hence:

$$(X^4)^T F_d(F_4, d^*) = -x_2^4 \neq 0.$$

Thus, the first condition of saddle-node bifurcation is satisfied.

Now, by substituting $F_4, d^*$ and $K^4$ in (10), the following is obtained:

$$D^2F(F_4,d^*)(K^4,K^4) = \begin{bmatrix} \left(-\frac{2r}{k} - \frac{2\alpha\beta v^*}{(\alpha+u^*)^3} + \frac{2cE^2ql}{(cE+lu^*)^3}\right){k_1^4}^2 + \frac{2\alpha\beta k_2^4}{(\alpha+u^*)^2} \\ -\frac{2s{k_2^4}^2}{l} \end{bmatrix}.$$

Hence:

$$(X^4)^T D^2F(F_4,d^*)(K^4,K^4) = \frac{2s{k_2^4}^2}{l} \neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at $F_4$ with the parameter $d^*$. ■

**Corollary 1.** According to Sotomayor's theorem, the transcritical and pitchfork bifurcation cannot occur at $F_4$ with the parameter $d^*$ since $F_4(X^4)^T F_d(F_4, d^*) \neq 0$.

## 7. Numerical Analysis

This section discovers model (1) dynamics by performing numerical simulations using MATLAB R2018b. For this purpose, the following parameter set is measured for system (1).

$$r = 0.9, k = 5, \beta = 0.4, \alpha = 0.8, q = 0.02, E = 0.03\ c = l = 1, s = 0.8, m = 4, d = 1. \quad (11)$$

For the above set of parameters, it is clear that system (1) has the vanishing $F_1 = (0,0)$ and the first species $F_3 = (4.99,0)$ equilibriums point. Figure 2 and Figure 3 show the location and the stability behavior of these two equilibriums points.

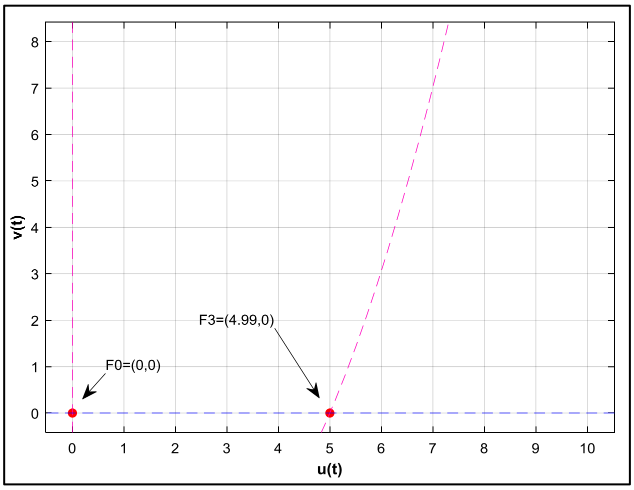


**Figure 2.** The two species nullcline which shows the number and location of the equilibrium point of system (1).

Moreover, the behavior for the $F_1 = (0,0)$ confirm the result, which states that $F_1$is a saddle point if one of the following conditions $r < \frac{q}{c}$ or $s < d$ is violated. Regarding the parameter given by (11), it is clear that the first condition is violated since $r = 0.9 > \frac{q}{c} = 0.02$. In this case, the eigenvalues of $J(F_1)$ have an opposite sign ($\lambda_{11} = -0.2, \lambda_{12} = 0.88$) (See Figure 3). On the other hand, Figure 3 shows that $F_3 = (4.99,0)$ is a stable fixed point since the eigenvalues of $J(F_3)$ are both negative ($\lambda_{21} = -0.2, \lambda_{22} = -0.88$) that confirms the stability conditions for $F_3$. Further, it is clear from Figure 3 the solutions that started from the different initial points settle down to the $F_3$ after some time.

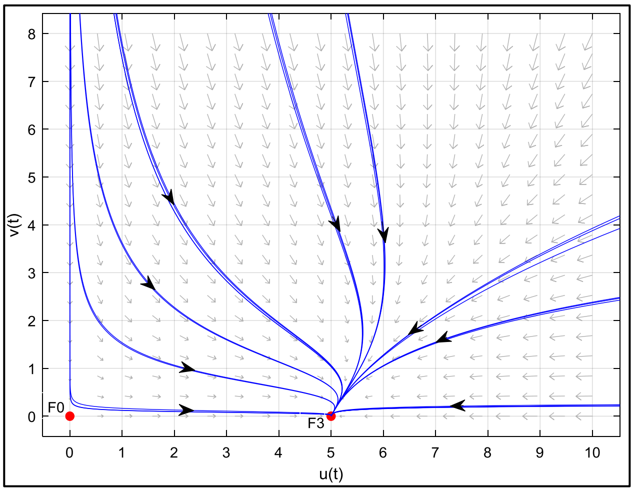


**Figure 3.** Phase plane examination with the data given by eq. (11).

Now, using the same parameter in eq. (11) with $d = 0.6$, system (1) has four different equilibrium points $F_1 =$

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$(0,0), F_2 = (0,1), F_3 = (4.99,0)$ and $F_4 = (5.36,1)$ (See Figure 4).

Moreover, Figure 5 describe the behavior around the above equilibrium points as:

1. The trivial fixed point $F_1 = (0,0)$ is a nodal source since the eigenvalues of $J(F_1)$ $\lambda_{11} = 0.2, \lambda_{12} = 0.55$ are both positive.
2. The second population equilibrium point $F_2 = (0,1)$ is a saddle point since the eigenvalues of $J(F_2)$ $\lambda_{21} = -0.2, \lambda_{22} = 1.83$ has the opposite of sign.
3. The first population equilibrium point $F_3 = (4.99,0)$ is a saddle point since the eigenvalues of $J(F_3)$ $\lambda_{31} = -0.8, \lambda_{32} = 0.2$ has the opposite of sign.
4. The positive equilibrium point $F_4 = (5.36,1)$ is a node sink saddle since the eigenvalues of $J(F_4)$ $\lambda_{41} = -0.2, \lambda_{42} = -1.02$ are both negative.

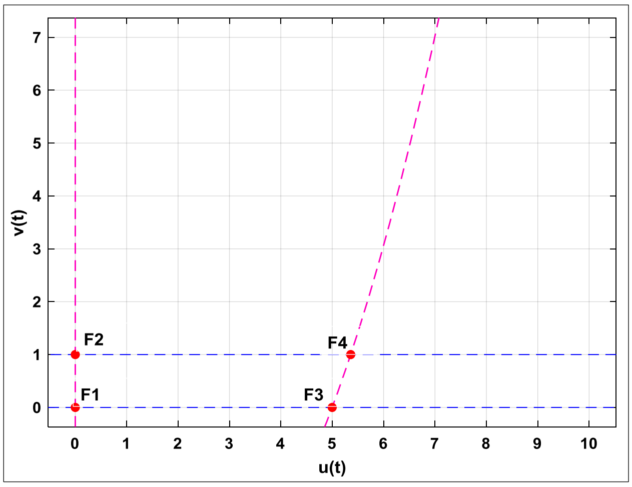


**Figure 4.** The two species nullcline which shows the number and location of the equilibrium point of system (1).

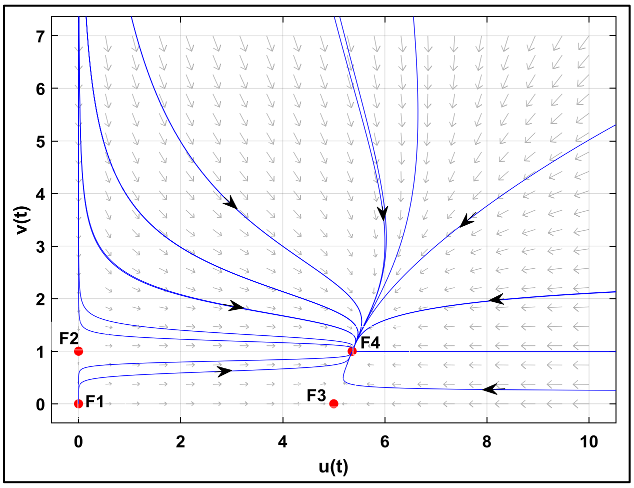


**Figure 5.** Phase plane examination with the data given by eq. (11) with $d = 0.6$.

Using the same parameter in eq. (11) with $\beta = 0.9$, system (1) has the same behaviour as above. The only difference is that the density of the first species rises from 5.36 to 5.76. This analysis shows that the increase in the commensalism coefficient leads to a rise in the first species (See Figure 6).

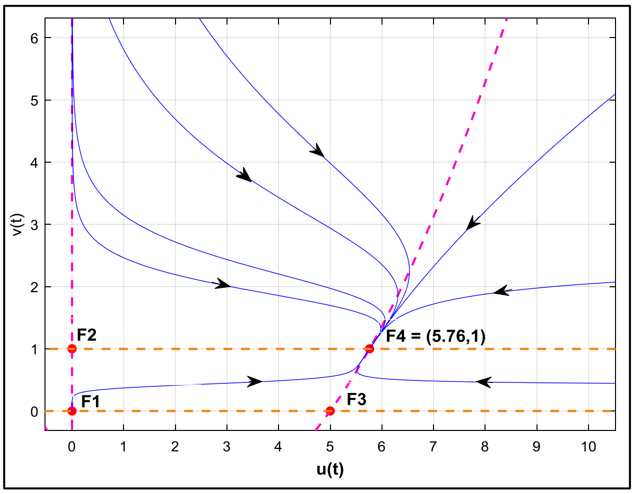


**Figure 6.** The two species nullcline and phase plane examination with the data given by eq. (11) with $\beta = 0.9$.

Furthermore, if we are substituting $d^* = s = 0.8$ in eq. (11), then system (1) has the same dynamic behavior as when $d = 1$ (See Figures 7 and 8). This result authorises the result that has been evidenced in Theorem 4. which says system (1) at the equilibrium point $F_4$ has a saddle-node bifurcation at $d^* = s$. As a result, system (1) with the parameter given in eq. (11) with $d \geq 0.8$, loses the persistent and the solution settle down to the first population equilibrium point $F_3$. And when $d < 0.8$, system (1) converge asymptotically to the positive equilibrium point $F_4$. Further, when the parameter $d$ passes $d^* = 0.8$, the number of equilibrium points decrease from four to only two equilibrium points. This result shows that $d^* = 0.8$ plays as a saddle-node bifurcation parameter at the equilibrium point $F_4$.

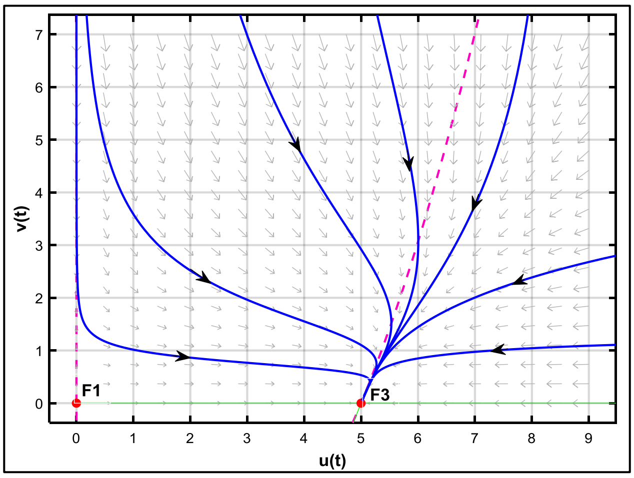


**Figure 7.** Phase plane examination with the data given by eq. (11) with $d = 0.8$.

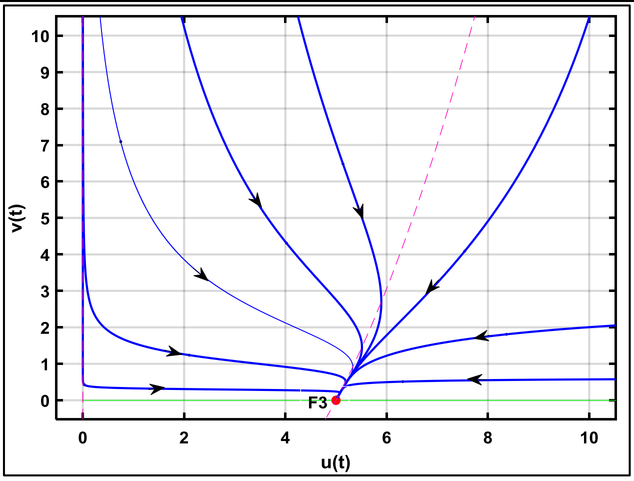

**Figure 8.** The two species nullcline and phase plane examination with the data given by eq. (11) with $d = 0.8$.

## 8. Conclusion

In the suggested model, it has been noticed that system (1) has at least four different equilibrium points. The existence of the equilibria has been calculated. The system's stability at $F_1, F_2, F_3$ and $F_4$ have described based on the detailed conditions. The persistence conditions of system (1) have been driven. The saddle-node bifurcation at the positive equilibrium point has been shown. The numerical simulations have been illustrated to confirm the logical results. The result has been exposed that the two commensalism populations can live together in abundance for a long time if the death rate is less than the intrinsic growth rate for the second population. Moreover, the commensalism coefficient positively impacts the density of the first species.

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