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Best One-Sided Multiplier Approximation of Unbounded Functions by Trigonometric Polynomials

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Abstract

The purpose of this paper is to find the best one – sided multiplier approximation of unbounded functions in $L\,p, \psi_n$ – space by trigonometric polynomials. Also, we will estimate the degree of the best one – sided multiplier approximation in term of averaged modulus.

1. Introduction

Babenko in 2012) [1] estimated the rate of best one-sided approximation of characteristic polynomials. In 2015, Soltani F. [2] studied some classes of Dunkl multiplier operators and gave an application of the theory reproducing kernels to Tikhonov of regularization which gives the best approximation of the operators on Hilbert space. In 2017, Saheb Kehaid Jassim and Abeer M. Salih [3] studied the multiplier approximation of unbounded functions by Bernstein-Durrmeyer operators.

In 2020, Saheb Al-Saidy and Ali H. Zaboon [4], obtained the multiplier approximation of periodical unbounded functions by Trigonometric operators. In 2020, A. M. Pasko [5], studied the point wise estimation of one-sided approximation of the class W_∞^r, $0 < r < 1$.

2. Basic Concepts

Definition 1, [6]. A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergent if there is a sequence $\{\psi_n\}_{n=0}^{\infty}$, such that $\sum_{n=0}^{\infty} a_n \psi_n < \infty$ and we will say that $\{\psi_n\}_{n=0}^{\infty}$ is a multiplier for the convergence.

Definition 2, [4]. For any real valued function f defined on $X = [0,\ 2\pi]$ if here is a sequence $\{\psi_n\}_{n=0}^{\infty}$, such that $\int_X f\,\psi_n(x)\,\mathrm{d}x < \infty$, $X = [0,2\pi]$, then we say that ψ_n is the multiplier integral.

Definition 3. Let $f \in Lp, \psi_n(X)$, $X = [\,0, 2\pi\,]$, $p \in [1,\infty)$ be the space of all real valued unbounded functions f, such that $\int_X f\ \psi_n(x)\,\mathrm{d}x < \infty$, then the norm:

$$\| f \|_{L\,p,\psi_n(X)} = \left(\int_X |f\psi_n(x)|^p\right)^{1/p},\ x \in X$$

is the norm on $Lp, \psi_n(X)$

For $f \in Lp, \psi_n(X)$, $X = [0,\ 2\pi]$, $0 < \delta$, we will define the following concepts:

$$\omega(f,\delta)p,\psi_n = \sup_{|h|<\delta} \|f(x+h) - f(x)\|_{L\,p,\psi_n(X)}$$

the multiplier modulus of continuity of function f

$$\tau_k(f,\delta)p,\psi_n = \|\omega_k(f,\delta)\|_{L\,p,\psi_n(X)},\ p \in [1,\infty),\ k \in \mathbb{N}$$

is the multiplier averaged modulus of smoothness of f of order k, where the k th modulus of smoothness.

The smoothness for f is defined by:

$$\omega(f,x,\delta)p,\psi_n = \sup_{|h|<\delta} \left\{ \left\{ \left\|\Delta_h^k f(f,t)\right\|_{L\,p,\psi_n(X)} : t,\ t+kh \in \left[\,x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\}\right.$$

where The k th symmetric difference of function f is:

$$\Delta_h^k\, f\,(x) = \sum_{i=0}^{k} \binom{k}{i}\ (-1)^{k-i}\, f\,(x - \frac{k\delta}{2} + ih),\ x \mp \frac{kh}{2} \in X$$

Definition 4, [7]. For $f \in Lp(X)$, $X = [\mathrm{a}, \mathrm{b}]$, $0 < \delta$, we will define the following concepts:

$$\omega(f,\delta)\,p = \sup_{|h|<\delta} \|f(x+h) - f(x)\|_{L\,p(X)}$$

the modulus of continuity of function f

$$\tau_k(f,\delta)\,p = \|\omega_k(f,\delta)\|_{L\,p\,(X)},\ p \in [1,\infty),\ k \in \mathbb{N}$$

is the averaged modulus of smoothness of f of order k. where the k th modulus of smoothness.

The smoothness for f is defined by:

$$\omega(f,x,\delta)p = \sup_{|h|<\delta} \left\{\Delta_h^k f(f,t)\|_{L\,p(X)} : t,\ t+kh \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}]\right\}$$

where the k th symmetric difference of function f is:

$$\Delta_h^k f(x) = \sum_{i=0}^{k} \binom{k}{i}\ (-1)^{k-i}\, f\,(x - \frac{k\delta}{2} + ih\,),\ x \mp \frac{kh}{2} \in X$$

Definition 5. For $f \in Lp, \psi_n(X)$, $X = [0, 2\pi]$, $0 < \delta$, we will define the following concepts:

$\omega(f, \delta)p, \psi_n = \sup_{|h| < \delta} \|f(x+h) - f(x)\|_{Lp, \psi_n(X)}$

the multiplier modulus of continuity of function f

$\tau_k(f, \delta)p, \psi_n = \|\omega_k(f, \delta)\|_{Lp, \psi_n(X)}, p \in [1, \infty), k \in \mathbb{N}$

is the multiplier averaged modulus of smoothness of f of order k, where the k^{th} modulus of smoothness.

The smoothness for f is defined by:

$\omega(f, x, \delta)p, \psi_n = \sup_{|h| < \delta} \{ \|\Delta_h^k f(f, t)\|_{Lp, \psi_n(X)} : t, t + kh \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}] \}$

Where The k^{th} symmetric difference of function f is

$\Delta_h^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - \frac{k\delta}{2} + ih), x \mp \frac{kh}{2} \in X$

Definition 6, [8]. The degree of best one – sided approximation of f is:

$\tilde{E}_n(f)p = \inf \{ \| p_n - q_n \|_{Lp(X)} : q_n(x) \leq f(x) \leq p_n(x) \}$

Also, the degree of best approximation of a function $f \in Lp(X)$ is define by:

$E_n(f)p = \inf \{ \| f - p_n \|_{Lp(X)} : p_n \in P_n \}$

Definition 7. The degree of best one-sided multiplier approximation of f is:

$\tilde{E}_n(f)p, \psi_n = \inf\{ \| p_n - q_n \|_{Lp, \psi_n(X)} : q_n(x) \leq f(x) \leq p_n(x) \}$

Also, the degree of best multiplier approximation of a function $f \in Lp, \psi_n(X)$ is define by:

$E_n(f)p, \psi_n = \inf\{ \| f - p_n \|_{Lp, \psi_n(X)} : p_n \in P_n \}$

3. Auxiliary Lemmas

Here we will study some propositions of a functions in the space $Lp, \psi_n(X)$, $X = [0, 2\pi]$.

Lemma 1. Let $f \in Lp, \psi_n(X)$, $X = [0, 2\pi]$, then:

$E_n(f)p, \psi_n \leq \tilde{E}_n(f)p, \psi_n$

Proof. Consider p_n^* to be the best polynomial approximation of f and let q_n, p_n be the best one-sided poly of f, where:

$q_n(x) \leq f(x) \leq p_n(x)$

$$
\begin{aligned}
E_n(f)p, \psi_n &= \inf\{ \| f - p_n \|_{Lp, \psi_n(X)} : p_n \in P_n \} \\
&= \| f - p_n^* \|_{Lp, \psi_n(X)} \\
&= \left(\int_x |(f - p_n^*)(x) \psi_n(x)|^p \, dx \right)^{1/p} \\
&\leq \left(\int_x |(p_n - q_n)(x) \psi_n(x)|^p \, dx \right)^{1/p} \\
&= \|p_n - q_n\|_{p, \psi_n} \\
&= \tilde{E}_n(f)p, \psi_n. \quad \blacksquare
\end{aligned}
$$

lemma 2. Let $f \in Lp, \psi_n(X)$, $X = [0, 2\pi]$, then:

$\tilde{E}_n(f)_{p,\psi_n} \leq cpE_n(f)_{p,\psi_n}$

Proof. Consider $p^* \in P$ is the best approximation of $f \in Lp, \psi_n(X)$ and let s_1, $s_2 \in P_n$ be the best one-sided approximation of f such that $s_2(x) \leq f(x) \leq s_1(x)$

$f(x) \leq s_1(x)$

$s_1(x) \leq 2f(x), s_m(x), s_n(x) \in P_n$

$$
\begin{aligned}
\tilde{E}_n(f)_{p,\psi_n} &= \inf\|s_n - s_m\|_{Lp, \psi_n(X)} \\
&= \|s_1 - s_2\|_{Lp, \psi_n(X)} \\
&= \left(\int_x |(s_1 - s_2)(x) \psi_n(x)|^p dx \right)^{1/p} \\
&= \left(\int_x |[s_1(x) - s_2(x)] \psi_n(x)|^p dx \right)^{1/p} \\
&\leq \left(\int_x |[2f(x) - f(x)] \psi_n(x)|^p dx \right)^{1/p} \\
&= \left(\int_x |[2f(x) - p^*(x) + p^*(x) - f(x)] \psi_n(x)|^p dx \right)^{1/p} \\
&= \left(\int_x |[2f(x) - p^*(x)] \psi_n(x)|^p dx \right)^{1/p} + \left(\int_x |[f(x) - p^*(x)] \psi_n(x)|^p dx \right)^{1/p} \\
&= \|2f - p^*\|_{Lp, \psi_n(X)} + \|f - p^*\|_{Lp, \psi_n(X)} \\
&\leq 2E_n(f)_{p,\psi_n}. \quad \blacksquare
\end{aligned}
$$

Lemma 3. Let f, g, $\varphi \in Lp, \psi_n(X)$, $X = [0, 2\pi]$ be 2π-periodic functions, if:

$|[f(x) - g(x)]\psi_n(x)| \leq \varphi(x)\psi_n(x)$

Then:

$\tilde{E}_n(f)p, \psi_n \leq Cp\left(\tilde{E}_n(g)p, \psi_n + 2\tilde{E}_n(\varphi)p, \psi_n + 2\|\varphi\|_{Lp, \psi_n(X)} \right)$

where C is constant depends on p.

Proof. Let p_n^* is the best approximation of f and g_n^* is the best approximation of φ

$$
\begin{aligned}
E_n(f)p, \psi_n &= \inf \|f - p_n\|_{Lp, \psi_n(X)}, p_n \in T_n \\
&= \|f - p_n^*\|_{Lp, \psi_n(X)}
\end{aligned}
$$

$$
\begin{aligned}
\|f - p_n^*\|_{Lp, \psi_n(X)} &= \left(\int_x |(f - p_n^*) \psi_n(x)|^p dx \right) \\
&= \left(\int_x |(f + g + \varphi + g_n - g - \varphi - g_n - p_n^x) \psi_n(x)|^p dx \right)^{1/p} \\
&\leq \left(\int_x |(g - p_n^*) \psi_n(x)|^p dx \right)^{1/p} + \left(\int_x |(f - g) \psi_n(x)|^p dx \right)^{1/p} + \left(\int_x |(\varphi - g_n^*) \psi_n(x)|^p dx \right)^{1/p} + \left(\int_x |(\varphi - g_n^*) \psi_n(x)|^p dx \right)^{1/p} \\
&\leq \|(g - p_n^*)\|_{Lp, \psi_n(X)} + \|(f - g)\|_{Lp, \psi_n(X)} + \|(\varphi - g_n^*)\|_{Lp, \psi_n(X)} + \|(\varphi + g_n^*)\|_{Lp, \psi_n(X)} \\
&\leq E_n(g)_{p,\psi_n} + E_n(f)p, \psi_n + \|(\varphi)\|p, \psi_n + E_n(\varphi)p, \psi_n + E_n(\varphi)p, \psi_n
\end{aligned}
$$

from Lemma 1 and Lemma 2, we get:

$E_n(f)p, \psi_n \leq \tilde{E}_n(g)p, \psi_n + 2\|\varphi\|_{Lp, \psi_n(X)} + 2\tilde{E}_n(\varphi)_{p,\psi_n}$

$\tilde{E}_n(f) \leq C(\tilde{E}_n(g)_{p,\psi_n} + 2\|(\varphi)\|p, \psi_n + 2\tilde{E}_n(\varphi)_{p,\psi_n})$ $\blacksquare$

Lemma 4. Let $f \in Lp, \psi_n(X)$, $X = [0, 2\pi]$, then:

$\omega(f, \delta)p, \psi_n \leq \delta \|f'\|p, \psi_n$

Proof.

$$
\begin{aligned}
\omega(f, \delta)p, \psi_n &= \sup\{ |[f(x+h) - f(x)]\psi_n(t)| : |h| \leq \delta, x, x+h \in X \} \\
&= \left| \int_x^{x+h} f'(t) \psi_n(t) \, dt \right|
\end{aligned}
$$

$\leq \sup \int_x^{x+h} |f'(t)\,\psi_n(x)|dt$
$\leq |h| \sup |f'(x)\,\psi_n(x)|$
$\leq |h|\, \|f'\|\, p,\psi_n$
$\leq \delta \|f'\|_{p,\psi_n}$. ■

Lemma 5. Let $f \in Lp,\psi_n(X)$, $X = [0,2\pi]$, $1 \leq p < \infty$, then:

$E_n(f)\,p,\psi_n \leq C(\kappa)\,\omega_k(f,\delta)\,p,\psi_n,\ \delta > 0$

Proof. Let p_n^* is the best approximation of f, where T_n is the set of all trigonometric polynomial.

$E_n(f)\,p,\psi_n = \inf \|f - p_n\|_{Lp,\psi_n(X)}$, $p_n \in T_n$
$= \|f - p_n^*\|_{Lp,\psi_n(X)}$
$= \left(\int_x |(f - p_n^*)(x)\,\psi_n(x)|^p dx\right)^{1/p}$
$\leq \sup\left(\int_x |[(f(x) - p_n^*(x)]\,\psi_n(x)|^p dx\right)^{1/p}$
$= \sup\left(\int_x |[(f(x) - f(x+h)]\,\psi_n(x)|^p dx\right)^{1/p}$
$= c_p \sup\left(\int_x |\Delta_h^k f(x)\,\psi_n(x)|^p dx\right)^{1/p}$
$= c_p \sup\|\Delta_h^k f(x)\|_{Lp,\psi_n(X)}$.
$= c_p\,\omega_k(f,\delta)\,p,\psi_n$

Then:

$E_n(f)p,\psi_n \leq c_p \omega_k(f,\delta)p,\psi_n$. ■

Lemma 6. Let $f \in Lp,\psi_n(X)$, then:

$\omega_k(f,\delta)p,\psi_n \leq \omega_{k-1}(f',\delta)p,\psi_n$

if f' exists.

Proof.

$\Delta_h^k f(x)\psi_n(x) = \Delta_h^{k-1}\Delta_h f(x)\,\psi_n(x)$
$= \Delta_h^{k-1}[(f(x+h) - f(x))\,\psi_n(x)]$
$= \Delta_h^{k-1}\int_0^h (f'(x+t)\,\psi_n(x)\,dt$

$|\Delta_h^{k-1} f(x)\,\psi_n(x)| = |(f'(x+t)\,\psi_n(x)\,dt|$
$\leq \int_{min(0,h)}^{max(0,h)} |\Delta_h^{k-1} f'(x+t)\psi_n(x)|dt$
$\leq \int_{min(o,h)}^{max(o,h)} \omega_{k-1}(f',\delta)\,p,\psi_n dt$
$\leq h\,\omega_{k-1}(f',\delta)\,p,\psi_n$
$\leq \delta\,\omega_{k-1}(f',\delta)\,p,\psi_n$

$\omega_k(f,\delta)\,p,\psi_n = \delta\,\omega_{k-1}(f',\delta)\,p,\psi_n$ ■

Lemma 7. Let $f \in Lp,\psi_n(X)$, then:

$\omega_k(f,\delta)p,\psi_n \leq \delta^k \|f^{(k)}\|p,\psi_n$

Proof. From Lemma 6 and Lemma 4

$\omega_k(f,\delta)\,p,\psi_n \leq \delta\,\omega_{k-1}(f',\delta)\,p,\psi_n$
$\leq \delta\,\delta\,\omega_{k-2}(f',\delta)\,p,\psi_n$
$\leq \delta^{k-1}\,\omega\left(f^{(k-1)},\delta\right)p,\psi_n$
$\leq \delta^{k-1}\,\delta\|f^{(k)}\|\,p,\psi_n$
$\leq \delta^k \|f^{(k)}\|\,p,\psi_n$ ■

Lemma 8. Let $f \in Lp,\psi_n(X), f^{(k)} \in Lp,\psi_n(X)$

Then:

$\tilde{E}_n(f)\,p,\psi_n \leq c_k n^{-k}\|f^{(k)}\|\,p,\psi_n,\ n > k$

Proof. From Lemmas 2, 5 and Lemma 7, we get:

$\tilde{E}_n(f)\,p,\psi_n \leq CpE_n(f)p,\psi_n$ …(1)
$E_n(f)p,\psi_n \leq C(k)\,\omega_k(f,\delta)p,\psi_n$ …(2)
$\omega_k(f,\delta)\,p,\psi_n \leq \delta^k\|f^{(k)}\|p,\psi_n$ …(3)

Then:

$\tilde{E}_n(f)p,\psi_n \leq C(k)\,\delta^k\|f^{(k)}\|p,\psi_n$ ■

Lemma 9. Let $f \in Lp,\psi_n(X)$, $X = [0,2\pi]$, with $\omega_k(f,x,h)p,\psi_n$ is a function of x, then:

$\tau_1(\omega_k(f,x,h),\delta)p,\psi_n \leq \tau_k(f,h+\frac{\delta}{k})p,\psi_n$

Proof. Set $g(x) = \omega_k(f,x,h)p,\psi_n$

$\omega_1(g,x,\delta)p,\psi_n = \sup\{|\Delta_\theta\, g(t)\,\psi_n(t)| : t, t+\theta \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]$
$= \sup\{|[g(t+\theta) - g(t)]_{\psi n}| : t, t+\theta \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$
$\leq \sup\{g(t): t \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$
$= \sup\{\sup\{|\Delta_m^k f(s)\psi_n(s)| : s, s+km \in [t - \frac{\delta}{2}, t + \frac{\delta}{2}]\} : t \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$
$\leq \sup\{|\Delta_m^k f(s)\psi_n(s)| : s, s+km \in [x - \frac{\delta}{2} - \frac{kh}{2}, x + \frac{\delta}{2} - \frac{kh}{2}]\}$
$= \omega_k(f,x,h+\frac{k}{2})\,p,\psi_n$

$\|\omega(g,x,s)\|p,\psi_n \leq \left\|\omega_k\left(f,x,h+\frac{\delta}{k}\right)\right\|p,\psi_n$

$\tau_1(g,x,h,\delta)p,\psi_n \leq \tau_k(f,x\ h+\frac{\delta}{k})p,\psi_n$

$\tau_1(\omega_k(f,x,h),\delta)p,\psi_n \leq \tau_k(f,x,h+\frac{\delta}{k})p,\psi_n$ ■

Lemma 10. Let $Lp,\psi_n(X)$, $X = [0,2\pi]$, $g_n(x) = \omega_k(f,x,n^{-1})p,\psi_n$, then:

$\tilde{E}_n(g_n)p,\psi_n \leq C\,\tau_k(f,2\,n^{-1})\,p,\psi_n$

where C is a constant.

Proof. From Lemma 9, letting $h = \delta = n^{-1}$, we have:

$\tau_1(g_n,n^{-1})p,\psi_n \leq \tau_k(f,n^{-1})p,\psi_n$

from Theorem 1

$\tilde{E_n}(g_n)p,\psi_n \leq C\tau_1(g_n,n^{-1})p,\psi_n$
$\leq C\tau_k(f,2\,n^{-1})p,\psi_n$ ■

Lemma 11. Let $f \in Lp,\psi_n(X)$, f' exists, then:

$\tau_k(f,\delta)p,\psi_n \leq \tau_{k-1}(f',\frac{\delta}{k-1}\delta)p,\psi_n$

Proof. Since:

$\Delta_h^k[f(t)\psi_n(t)] = \Delta_h^{k-1}\Delta_h[f(t)\psi_n(t)]$
$= \Delta_h^{k-1}([f(t+h) - f(t)]\,\psi_n(t))$
$= \Delta_h^{k-1}\left(\int_0^h [f'(u+t)\psi_n(u)]du\right), h > 0$

$|\Delta_h^k[f(t)\psi_n(t)]| \leq \int_0^h |\Delta_h^{k-1} f'(u+t)\psi_n(u)|\,du$

$\sup\{|\Delta_h^k[f(t)\psi_n(t)]| : t, t+k\,h \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}]$
$\leq \sup\{\int_0^h |\Delta_h^{k-1} f'(u+t)\psi_n(u)du|\} \cap [a, b]$

$\omega_k(f,x;\delta)p,\psi_n \leq h\omega_{k-1}(f',x;\delta)p,\psi_n$
$\leq \delta\omega_{k-1}(f',x;\delta)\,p,\psi_n$

$\|\omega_{k-1}(f,x;\delta)\|p,\psi_n \leq \delta\|\omega_{k-1}(f,x;\delta)\|\,p,\psi_n$

$\tau_k(f,\delta)\,p,\psi_n \leq \delta\,\tau_{k-1}(f',\frac{k}{k-1}\delta)\,p,\psi_n$ ■

Lemma 12. Let $f \in Lp,\psi_n(X)$, $X = [0, 2\pi]$

$\tau_1(f,\delta)\,p,\psi_n \leq \delta\|f'\|\,p,\psi_n,\ \delta \in [0, 2\pi]$

Proof.

$\omega(f, x, \delta)\, p, \psi_n = \sup\ \{|[f(t) - f(\acute{t})]\psi_n(t)| : \acute{t}, \acute{\acute{t}} \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$

$= \sup \{\left|\int_{t}^{\acute{t}} f(t)\psi_n(t)dt\right| : \acute{t}, \acute{\acute{t}} \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$

$\leq \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} |\acute{f}(t)\psi_n(t)|dt$

$= \leq \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |\acute{f}(x + t)\psi_n(t)|dt$

$\|\omega(f, x;\delta)\| p, \psi_n \leq \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \|\acute{f}(.+t)\|\, p, \psi_n\, dt$

$\tau_1(f, \delta)\, p, \psi_n \leq \delta\, \|\acute{f}\|\, p, \psi_n$ ■

Lemma 13. Let $f \in Lp, \psi_n(X)$, then:

$\tau_k(f, \delta)\, p, \psi_n \leq C(k)\, \delta^k \|f^{(k)}\|\, p, \psi_n$

Proof. From Lemma 11 and Lemma 12

$\tau_k(f, \delta)\, p, \psi_n \leq \delta\, \tau_{k-1}(\acute{f}, \frac{k}{k-1}\delta)\, p, \psi_n$

$\leq \delta\, \tau_{k-2}(\acute{f}, \frac{k-1}{k-2}\delta)\, p, \psi_n$

$\leq \delta\, \tau_{k-3}(\acute{f}, \frac{k-2}{k-3}\delta)\, p, \psi_n$

$\leq \delta^{k-1}\tau_1(f^{(k-1)}, 2\delta)\, p, \psi_n$

From Lemma 4, we get:

$\tau_k(f, \delta)\, p, \psi_n \leq 2\, \delta^k \|f^{(k)}\|\, p, \psi_n$

$\leq c(k)\, \delta^k \|f^{(k)}\|\, p, \psi_n$ ■

Lemma 14. For even natural number k and even $h > 0$ there exists a function $f_{k,h} \in L\, p, \psi_n(X)$, B = $[0,2\pi]$, then:

1- $|[f(x) - f_{k,\delta}(x)]\psi_n(x)| \leq c_1\omega_k(f, x, \delta)\, p, \psi_n$

2- $\|f - f_{k,\delta}\| \leq c_1\, \omega_k(f, \delta)\, p, \psi_n$

3- $\|f_{k,\delta}^r\|\, p, \psi_n \leq C_2 \frac{1}{\delta}\, \omega_r(f, \delta)\, p, \psi_n$

Proof (1). We can define the function [8]

$f_{k,\delta}(x) = (-\delta)^{-k} \int_0^{\delta} \ldots \int_0^{\delta} \{-f(x + t, \ldots + t_k)\, \psi_n(t) + (k_1) f\, [x + \frac{k-1}{k}(t, \ldots + t_k)]\, \psi_n(t) + \ldots \ldots + (-1)^k \binom{k}{k-1} f\, \{(x + \frac{t_1 + \cdots t_k}{k})\, \psi_n(t)\}\, dt_1 \ldots dt_2$

$|(f - f_{k,\delta})\, \psi_n(x)| \leq \frac{1}{-\delta} \int_0^{\delta} \ldots \int_0^{\delta} \left|\Delta^k_{\sum_{i=1}^k} + f(x)\right| \prod_{i-1}^k dt_i$

$\leq \omega_k(f, x;\delta)\, p, \psi_n$ 2) $f_{k,\delta}$

$= \frac{1}{(-\delta)^k} \int_0^{\delta} \ldots \int_0^{\delta} \{ f(x + \sum_{i=1}^k t_i)\, \psi_n(t) + \binom{k}{1} f[x + \frac{k-1}{k}(\sum_{i=1}^k t_i)]\, \psi_n(t) + \ldots.. + (-1)^k \binom{k}{k-1} f\, \{(x + \frac{1}{k}(\sum t_i)\, \psi_n(t)\} \prod_{i-1}^k dt$

$\|f - f_{k,\delta}\|\, p, \psi_n \leq \frac{1}{\delta^k} \int_0^{\delta} \ldots \int_0^{\delta} \left\|\Delta^k_{\frac{\sum_{i=1}^k t_i}{k}} f\right\|\, p, \psi_n \prod_{i-1}^k dt$

$\leq C_1\, \omega_k(f, \delta)\, p, \psi_n$ 3) $f_{k,\delta}^{(r)}(x)$

$= \frac{1}{(-\delta)^k} \int_0^{\delta} \ldots \int_0^{\delta} \{-\Delta_{\delta}^r f(x + \sum_{i=1}^{k-r} t_{k-r})\, \psi_n(t) + \binom{k}{1} \binom{k}{k-1}^r \Delta^r_{\frac{(k-1)\delta}{k}} f\, [x + \frac{(k-1)}{k}(\sum_{i=1}^{k-r} t_{k-r})\, \psi_n(t) + \ldots + (-1)^k \binom{k}{k-1}\, k_r\, \Delta^r_{\frac{\delta}{k}} f\, (x + \frac{1}{k}(\sum_{i=1}^{k-r} t_{k-r})\} \prod_{i-1}^k dt_i$

$\|f_{k,\delta}^{(r)}\|_{p,\psi_n} \leq \frac{1}{\delta^k} \int_0^{\delta} \ldots \int_0^{\delta} \{\|\Delta^r_{\delta} f\, (x + (\sum_{i=1}^{k-r} t_{k-r})\psi_n(t)\|_{p,\psi_n} + \binom{k}{1}\binom{k}{k-1}^r \left\|\Delta^r_{\frac{(k-1)\delta}{k}} f\, [x + \frac{k-1}{k}(\sum_{i=1}^{k-r} t_{k-r})]\psi_n(t)\right\|_{p,\psi_n} + \ldots. + \binom{k}{k-1} k^r \left\|\Delta^r_{\frac{\delta}{k}} f\, [x + \frac{1}{k}(\sum_{i=1}^{k-r} t_{k-r})]\psi_n(t)\right\|\} \prod_{i-1}^k dt_i$

$\|f_{k,\delta}^{(r)}\|_{p,\psi_n} \leq \frac{1}{\delta^k}\{\omega(f\,\delta)_{p,}\psi_n + \binom{k}{1}\binom{k}{k-1}^r \omega_r(f, \frac{(k-1)\delta}{k})p, \psi_n + \ldots + \binom{k}{k-1} k^r \omega_r(f, \frac{\delta}{k})_{p,\psi_n}\}$

$\leq C_2 \frac{1}{\delta^r}\, \omega_r(f, \delta)_{p,\psi_n}$ ■

4. Main Result

Theorem 1. Let $f \in Lp, \psi_n(X), X = [0, 2\pi]$, then:

$\tilde{E}_n(f)\, p, \psi_n \leq C\tau(f, n^{-1})\, p, \psi_n,\ 1 \leq i < \infty,$

where C is a constant.

Proof. Let $x_i = i\pi/n$, i = 0, 1, …, 2n

$y_i = (x_{i-1} + x_i)/2$, i = 1, 2, …, $2n$

Let us define the linear continuous 2π – periodic function s_n. and y_n for $x \in [x_{i-1}, y_i]$ and for $x \in [y_i, x_i]$, $i = 1, 2, \ldots, 2n$, $y_i = (x_{i-1} + x_i)/2$

$s_n(x) = \sup\{f(t)\psi_n(t) : t \in [x_{i-1}, x_i]\}$, $x = y_i$, $i = 1,2,\ldots,2n$

such that:

$s_n(0) = s_n(2\pi)$

$y_n(x) = \inf\{f(t)\psi_n(t) : t \in [x_{i-1}, x_i]\}$, for $x = y_i$, $i = 1,\ldots,2n$

such that $y_n(0) = y_n(2\pi)$.

Obviously:

$y_n(x) \leq f(x) \leq s_n(x),\ x \in [0, 2\pi]$

The derivatives $\acute{S_n}$ and $\acute{y_n}$ of the function $y_n(x)$ and $s_n(x)$ exist in $[0, 2\pi]$ except at the points $x_i, i = 1, \ldots 2n, y_i, i = 1, \ldots, 2n$. Using the definitions of the function s_n and y_n we obtain set $\delta = 4\pi n^{-1}$

Let $x \in (y_i, x_i)$ since s_n is linear in (y_i, x_i), we have:

$|\acute{s_n}(x)| \leq \frac{2n}{\pi} |s_n(y_{i-1}) - s_n(y_i)| \leq \frac{2n}{\pi}\, \omega(f, x, \delta)\, p, \psi_n$

Also $|\acute{y_n}(x)| \leq C_n = \omega_1(f, x, \delta)\, p, \psi_n$ again from the definition of the S_n and y_n, we obtain:

$0 \leq s_n(x) - y_n(x) \leq \omega(f, x, \delta)\, p, \psi_n$

$\|\acute{s_n}\| p, \psi_n \leq \|C_n\, \omega(f, x, \delta)\|\, p, \psi_n$

$\leq C_n\, \tau_1(f, \delta)\, p, \psi_n$ …(1)

$\|\acute{y_n}\| p, \psi_n \leq \|C_n\, \omega_1(f, x, \delta)\|\, p, \psi_n$

$\leq C_n\, \tau(f, \delta)\, p, \psi_n$ …(2)

Also:

$s_n(x) - y_n(x) \leq \omega(f, x, \delta)\, p, \psi_n$

$\|S_n(x) - y_n(x)\| p, \psi_n \leq \|\omega(f, x, \delta)\|\, p, \psi_n$

$\leq \tau_1(f, \delta)\, p, \psi_n$ …(3)

Applying Lemma 8 in (1), (2), we get:

$\tilde{E}_n(s_n) p, \psi_n \leq \frac{c_1}{n}\, C_n\, \tau_1(f, \delta)\, p, \psi_n$

$\leq C_2\, \tau_1(f, \delta)\, p, \psi_n$ …(4)

Also:

$\tilde{E}_n(y_n)p,\psi_n \le \frac{c_1}{n}C_n\tau_1(f,\delta)p,\psi_n$

$\le C_2\tau_1(f,\delta)p,\psi_n$ …(5)

Sine $\tilde{E}_n(f)p,\psi_n = \inf \|s_n - y_n\| p,\psi_n$

$\tilde{E}_n(f)p,\psi_n \le \|s_n - y_n\| p,\psi_n$

$\le \tilde{E}_n(s_n)p,\psi_n + \|s_n - y_n\| p,\psi_n + \tilde{E}_n(y_n)p,\psi_n$

From (3), (4) and (5), we get:

$\tilde{E}_n(f)p,\psi_n \le C_n\tau(f,\delta)p,\psi_n + \tau_1(f,\delta)p,\psi_n + C_n\tau_1(f,\delta)p,\psi_n$

$\le (2C_1+1)\tau_1(f,\delta)p,\psi_n$

$\le C\tau_1(f,\delta)p,\psi_n$

where $2C_2+1 = C$ is a constant. ■

Direct theorem

Theorem 2. Let $f \in Lp,\psi_n(X)$, for every natural number k there is a constant $C(k)$, then:

$\tilde{E}_n(f)p,\psi_n \le C(k)\tau_k(f,\frac{1}{n})p,\psi_n$

Proof. Applying Lemma 3 and Lemma 10 to the functions $f, f_{k,\delta}$ and

$\varphi(x) = \omega_k(f, 2n^{-1})p,\psi_n$

$\tilde{E}_n(f)p,\psi_n = C_p(\tilde{E}_n(f_{k,\delta})p,\psi_n + 2\tilde{E}_n(\varphi)p,\psi_n + 2\|\omega_k(f,x,\delta\| p,\psi_n$

$\le C_1(\tilde{E}_n(f_{k,\delta})p,\psi_n + 2Cp\,\tau_k(f,\delta)p,\psi_n, 2C_p\tau_k(f,\delta)p,\psi_n$ …(6)

$\tilde{E}_n(f_{k,\delta})p,\psi_n \le C(k)\|f_{k,\delta}^{(k)}\| p,\psi_n$

$\le C_1(k)\,\omega_k(f,\delta)p,\psi_n$

$\le C_2(k)\,\tau_k(f,\delta)p,\psi_n$ …(7)

From (6) and (7), then:

$\tilde{E}_n(f)p,\psi_n \le C_2(k)\,\tau_k(f,\delta)p,\psi_n + 2C_p\,\tau_k(f,\delta)p,\psi_n + 2C_p\,\tau_k(f,\delta)p,\psi_n$

$\le 4C_pC_2(k)\,\tau_k(f,\delta)p,\psi_n$

$\le C(p,k)\,\tau_k(f,\delta)p,\psi_n$ ■

Theorem 3. Let $f \in Lp,\psi_n(X)$, there exists $C(k)$

$\tau_k(f,n^{-1})p,\psi_n \le \frac{C(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}\tilde{E}_n(f)p,\psi_n$

Proof. Let $\alpha_n, \gamma_n, \in T_n$ are trigonometric polynomials

$\tilde{E}_n(f)p,\psi_n = \inf \|\alpha_n - \gamma_n\| p,\psi_n, \gamma_n(x) \le f(x) \le \alpha_n(x), x \in [0, 2\pi]$

If $\Delta_h^k f(t)\psi_n(t) \ge 0$, then:

$\Delta_h^k f(t)\psi_n(t) = \sum_{m=0}^{k}(-1)^m\binom{k}{m}f[t+(k-m)h]\psi_n(t)$

$\le \sum_{i=0}^{k/2}\binom{k}{2i}\alpha_n[t+(k-2i)h] - \sum_{i=0}^{k-1/2}\binom{k}{2i+1}\gamma_n[t+(k-2i-1)h]$

$= \Delta_h^k\alpha_n(t) + \sum_{i=0}^{k-1/2}\binom{k}{2i+1}\{\gamma_n[t+(k-2i-1)h] - \alpha_n[t+(k-2i-1)h]\}$

$= \Delta_h^k\alpha_n(t) + \sum_{i=0}^{k-1/2}\binom{k}{2i+1}\{\alpha_n[t+(k-2i-1)h - \alpha_n[t+(k-2i-1)h] - [\alpha_n(x) - \gamma_n(x)]\} + \sum_{i=0}^{k-1/2}\binom{k}{2i+1}[\alpha_n(x) - \gamma_n(x)]$

$\le \Delta_h^k\alpha_n(t) + 2^k\{\omega_1(\alpha_n - \gamma_n, x; k\delta)p,\psi_n + [\alpha_n(x) - \gamma_n(x)]\}$

$|\Delta_h^k f(t)\gamma_n(t)| \le |\Delta_h^k\alpha_n(t)| + 2^k[\omega_1(\alpha_n - \gamma_n, x; k\delta)p,\psi_n + |\alpha_n(x) - \gamma_n(x)|]$ …(8)

Now if $\Delta_h^k f(t)\psi_n(t) \le 0$, then in the same way, we obtain:

$|\Delta_h^k f(t)\psi_n(t)| \le |\Delta_h^k\gamma_n(t)| + 2^k[\omega_1(\alpha_n - \gamma_n, x; k\delta)p,\psi_n + [\alpha_n(x) - \gamma_n(x)]$ …(9)

From equations (8) and (9), we get

$\omega_k(f,x,\delta)_{p,\psi_n} \le \omega_k(\alpha_n, x\,\delta)p,\psi_n + \omega_k(\gamma_n,x,\delta)p,\psi_n + 2^k[\omega_1(\alpha_n - \gamma_n,x,\delta)p,\psi_n + |\alpha_n(x) - \gamma_n(x)|]$

$\tau_k(f,\delta)_{p,\psi_n} \le \tau_k(\alpha_n,\delta)p,\psi_n + \tau_k(\gamma_n,\delta)p,\psi_n + 2^k[\tau_1(\alpha_n - \gamma_n, k\delta)p,\psi_n + \tilde{E}_n(f)p,\psi_n]$

From Lemma 4, we get:

$\tau_1(\alpha_n - \gamma_n, k\delta)p,\psi_n \le k\delta\|\acute{\alpha_n} - \acute{\gamma_n}\| p,\psi_n$

Using the Bernstein inequality:

$\|\acute{T}\|_{p,\psi_n} \le n\|T\|_{p,\psi_n}$

Thus:

$\tau_1(\alpha_n - \gamma_n, k\delta)p,\psi_n \le nk\,\delta\|\alpha_n - \gamma_n\|_{p,\psi_n}$

$= k\delta\,n\,\tilde{E}_n(f)p,\psi_n$ …(10)

Equation (10) implies

$\tau_k(f,\delta)p,\psi_n \le \tau_k(\alpha_n,\delta)p,\psi_n + \tau_k(\gamma_n,\delta)p,\psi_n + 2^k(k\delta n+1)\tilde{E_n}(f)p,\psi_n$

Let us set $n = 2^{s_0}$, then:

$\tau_k(f,\delta)p,\psi_n \le \sum_{i=1}^{s_0}[\tau_k(\alpha_{2^i} - \alpha_{2^i},\delta)p,\psi_n + \tau_k(\gamma_{2^i} - \gamma_{2^{i-1}},\delta]p,\psi_n + \tau_k(\alpha_1 - \alpha_0,\delta)p,\psi_n\tau_k(\gamma_1 - \gamma_0,\delta)p,\psi_n + 2^k(k\,\delta n+1)\tilde{E_n}(f)p,\psi_n$ …(11)

Now

$\tau_k(\alpha_{2^i} - \alpha_{2^{i-1}},\delta)p,\psi_n \le k\delta^k\left\|(\alpha_{2^i} - \alpha_{2^{i-1}})^k\right\|_{p,\psi_n}$

$\le k\delta^k 2^{ik}\|\alpha_{2^i} - \alpha_{2^{i-1}}\|_{p,\psi_n}$

$\le k\delta^k 2^{ik}[\|\alpha_{2^i} - f\| p,\psi_n + \|\alpha_{2^{i-1}} - f\| p,\psi_n]$

$\le k\delta^k 2^{ik}[\|\alpha_{2^i} - \gamma_{2^i}\|p,\psi_n + \|\alpha_{2^{i-1}} - \gamma_{2^{i-1}}\|p,\psi_n]$

$\le 2k\delta^k 2^{ik}\tilde{E}_{2^{i-1}}(f)p,\psi_n$ …(12)

$\tau_k(\gamma_{2^i} - \gamma_{2^{i-1}})p,\psi_n \le k\delta^k\left\|(\gamma_{2^i} - \gamma_{2^{i-1}})^k\right\|_{p,\psi_n}$

$\le k\delta^k 2^{ik}\|\gamma_{2^i} - \gamma_{2^{i-1}}\|_{p,\psi_n}$

$\le k\delta^k 2^{ik}[\|\gamma_{2^i} - f\| p,\psi_n + \|\gamma_{2^{i-1}} - f\| p,\psi_n]$

$\le k\delta^k 2^{ik}[\|\alpha_{2^i} - \gamma_{2^i}\|p,\psi_n + \|\alpha_{2^{i-1}} - \gamma_{2^{i-1}}\|p,\psi_n]$

$\le 2k\delta^k 2^{ik}\tilde{E}_{2^{i-1}}(f)p,\psi_n$ …(13)

From inequalities (11), (12) and (13), we get:

$\tau_k(f,\delta)p,\psi_n \le 4k\delta^k\sum_{i=1}^{s_0}2^{ik}\tilde{E}_{2^{i-1}}(f)p,\psi_n + 2k\delta^k\tilde{E_0}(f)p,\psi_n + 2^k(k\,\delta_{n+1})\tilde{E}_n(f)p,\psi_n$

Let $\delta = n^{-1}$, then:

$\tau_k(f,\delta)p,\psi_n \le 4^{k+1}kn^{-k}\sum_{s=0}^{n}(s+1)^{k-1}\tilde{E}_s(f)p,\psi_n + 2^k(k+1)\tilde{E}_n(f)p,\psi_n$

$\le 2^{3k+1}n^{-k}\sum_{s=0}^{n}(s+1)^{k-1}\tilde{E}_s(f)p,\psi_n$

$\tau_k(f,\delta)p,\psi_n\frac{c(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}\tilde{E}_s(f)p,\psi_n$ ■

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