Al-Nahrain Journal of Science
ANJS, Vol.25 (2), June, 2022, pp. 25-28

Solving Schrodinger Equations Using Kashuri and Fundo Transform Decomposition Method

Huda J. Kadhum* and Ali H. Al-Fayadh

Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad-Iraq

Article's Information

Received:
04.10.2021
Accepted:
28.02.2022
Published:
30.06.2022

Keywords:
Kashuri and Fundo transform
Adomian polynomials
Schrödinger equation
Integral transform

Abstract

This paper proposes a hybrid method that combines the Kashuri and Fundo integral transform and the Adomian decomposition method to solve two types of Schrödinger equations. The exact solutions are successfully found, and the results are compared with those obtained by the existing methods. The obtained results show the accuracy and efficiency of the method.

DOI: 10.22401/ANJS.25.2.05
*Corresponding author: hudajawad91@yahoo.com

1. Introduction

Differential equations (DEs) play a major role in describing a wide range of natural phenomena undergoing change. They relate an unknown function to its derivatives. Partial differential equations (PDEs), as well as, ordinary differential equations (ODEs) usually arise in many diverse applications, such as fluid flow, mechanical systems, physics, and relativity [1]. There are a variety of numerical and analytical techniques for solving the PDEs.

The analytical solution for a given PDE is always preferable because of the restriction of the numerical solutions which may not offer us much about the behavior of systems. However, in many instances, systems described by PDEs are frequently so intricate or so large where a purely analytical solution to the equations is not tractable, or many of the modeled problems are resulted in nonlinear PDEs in which the exact solutions are difficult to obtain by using analytical methods. In this case, the numerical methods are needed to be developed for solving DEs despite these methods have some errors.

Many numerical methods were developed over a period of time [2-7], in order to find more accurate approximation solutions to these nonlinear equations. On the other hand, predict the future behavior of a dynamic system play a major role in various fields of applied sciences such as plasma physics, hydrodynamic, nonlinear optics, and quantum mechanics. The exact solutions of these DEs are very important.

Schrödinger equations (SchEs) play a crucial role in numerous spaces of practical sciences [8] and the references therein. In classical mechanics, the Schrödinger linear partial differential equation (SchLPDE) is a wave equation that plays an important role of Newton's second law where the future behavior of a dynamic system can be predicted. The wave function predicts the path of a given dynamical system or the distribution of results over time. The SchLPDE is given by:

$$u_t(x,t) = i u_{xx},\ i = \sqrt{-1},\ t \geq 0 \qquad \ldots(1)$$

with:

$$u(x,0) = h(x) \qquad \ldots(2)$$

where $h(x)$ is a continuous and square integrable function. On the other hand, the non-linear Schrödinger equation (NLSchE) is given by:

$$i u_t + u_{xx} + \rho |u|^2 u = 0 \qquad \ldots(3)$$

with:

$$u(x,0) = h(x) \qquad \ldots(4)$$

such that ρ is a constant term and u is complex.

Several different numerical schemes have been used for solving SchLPDEs and NLSchEs [9-14], and so on.

The main objective of this topic is to investigate the use of Kashuri-Fundo Decomposition Method (KF-DM) for solving SchLPDE and NLSchE. Using this method, we found the exact solutions of some applications successfully. The results are compared with the those obtained by other methods in literatures. The findings demonstrate the method's dependability, precision, and adaptability.

2. Preliminaries

In this part, we will look at the Kashuri and Fundo transforms and their definitions and characteristics. by [15,16]:

Definition 2.1. The Kashuri and Fundo transformation of the function $k(t) \in H$ where t is a positive real number, is defined by:

$$Ķ[k(t)](v) = A(v) = \frac{1}{V}\int_0^{\infty} e^{\frac{-1}{v2}} f(t)\, dt, -r_1 < v < r_2$$

where $Ķ[k(t)]$ denotes the Kashuri-Fundo transformation of the function $k(t)$, and $H = \{k(t)$: there exists positive real numbers M, r_1, and r_2 such that $|k(t)| < Me^{\frac{|t|}{r_i^2}}$, if $t \in (-1)^j \times [0, \infty)\}$

We give some useful properties and basic theorems of Kashuri-Fundo transform.

Theorem 2.2 (Duality relation). Let $k(t) \in H$ with Laplace transform $F(s)$. The Kashuri-Fundo $A(v)$ of $k(t)$ is given by:

$$A(v) = \frac{1}{v}\, F\left(\frac{1}{v^2}\right) \qquad \ldots(5)$$

Theorem 2.3 (Linearity of Kashuri-Fundo Transform). Let $k(t)$, $k_1(t)$ and $k_2(t)$ be functions whose a new integral transforms exists for $|v| < r$ and let c be a constant. Then for $|v| < k$,

1. $Ķ[(k_1 + k_2)(t)] = Ķ[k_1(t)] + Ķ[k_2(t)]$.
2. $Ķ[(ck)(t)] = cK[k(t)]$.

Theorem 2.4 (Some fundamental properties of Kashuri-Fundo transform). Let $A(v)$ be a Kashuri-Fundo transform of $k(t)$. Then:

1. $Ķ[k`(t)] = \frac{A(v)}{v^2} - \frac{k(0)}{v}$.
2. $Ķ[k``(t)] = \frac{A(v)}{v^4} - \frac{k(0)}{v^3} - \frac{k`(0)}{v}$.
3. $Ķ\,[k^n(t)] = \frac{A(v)}{v^{2n}} - \sum_{j=0}^{n-1} \frac{k^j(0)}{v^{2(n-j)-1}}$.

Table 1. List of some special Kashuri-Fundo transformation of $k(t)$, [17].

$k(t)$	$Ķ[k(t)]$
1	v
$t^n, n \geq 0$	$n!\, v^{2n+1}$
$e^{-\mu t}$	$\frac{v}{1+\mu v^2}$
$\sin(\mu t)$	$\frac{a v^2}{1+\mu^2\, v^4}$
$\cos(\mu t)$	$\frac{v^2}{1+\mu^2 v^4}$

3. Kashuri-Fundo Decomposition Method (KF-DM)

In this section, we provide the KF-DM to standard NLSchE (3)-(4). We start by insertion the Kashuri-Fundo transform on each side of Eq. (3):

$$\frac{1}{v^2}Ķ[ɥ(x,t)] - \frac{1}{v}ɥ(x,0) - iĶ\left[ɥ_{xx}\right] - iĶ\,[\rho|ɥ|^2ɥ] = 0 \qquad \ldots(6)$$

and substituting Eq. (4) into Eq. (6), we get:

$$Ķ[ɥ(x,t)] = v\, k(x) + iv^2Ķ\left[ɥ_{xx} + [\rho|ɥ|^2ɥ]\right] \qquad \ldots(7)$$

Using the inverse of Kashuri-Fundo transformation to Eq. (7):

$$ɥ(x,t) = k(x) + Ķ^1\left[iv^2Ķ\left[ɥ_{xx} + \rho|ɥ|^2ɥ\right]\right] \qquad \ldots(8)$$

where ɥ(x,,t) is the unknown function and k(x) is the source terms.

We will now suppose an infinite series solution for the function ɥ(x,,t) of the form:

$$ɥ(x,t) = \sum_{n=0}^{\infty} ɥ_n(x,t) \qquad \ldots(9)$$

Note that $|v|^2 v = v^2\bar{v}$ and it can be written as follows:

$$|ɥ|^2\bar{ɥ} = \sum_{n=0}^{\infty} A_n \qquad \ldots(10)$$

where A_n are Adomian polynomials, which can be computed using the following formula:

$$A_n = \frac{1}{n!}\frac{d^n}{dx^n}\left[F\left(\sum_{r=0}^{n}\omega^r\, ɥ_r\right)\right]_r \qquad \ldots(11)$$

where n is a nonnegative integer.

Some of A_n's are:

$A_0 = F(ɥ_0)$
$\quad = ɥ_0^2\overline{ɥ_0}$
$A_1 = ɥ_1\, F\,'(ɥ_0)$
$\quad = 2ɥ_0ɥ_1\overline{ɥ_0} + ɥ_0^2\overline{ɥ_1}$
$A_2 = ɥ_2\, F\,'(ɥ_0) + \frac{1}{2!}ɥ_1^2 F\,''(ɥ_0)$
$\quad = 2ɥ_0ɥ_2\overline{ɥ_0} + ɥ_1^2\overline{ɥ_0} + 2ɥ_0\, ɥ_1\overline{ɥ_1} + ɥ_0^2\overline{ɥ_2}$

Putting Eq. (9) and Eq. (10) into Eq. (8), yields:

$$\sum_{n=0}^{\infty} ɥ_n(x,t) = k(x) + iĶ^{-1}\left[v^2Ķ\left[\sum_{n=0}^{\infty} ɥ_{nxx} + \sum_{n=0}^{\infty} A_n\right]\right] \qquad \ldots(12)$$

The following recursive relation can be generated through comparing each sides of Eq. (12):

$ɥ_0(x,t) = k(x)$
$ɥ_1(x,t) = iĶ^{-1}[v^2\, Ķ[\, ɥ_{0xx} + A_0]$
$ɥ_2(x,t) = iĶ^{-1}[v^2\, Ķ[\, ɥ_{1xx} + A_1]$
$ɥ_3(x,t) = iĶ^{-1}[v^2\, Ķ[\, ɥ_{2xx} + A_2]$

We can write the general recursive relation of the unknown function ɥ(x, t) as:

$$ɥ_{n+1}(x,t) = iĶ^{-1}[v^2\, Ķ[\, ɥ_{nxx} + A_n] \qquad \ldots(13)$$

where n is a nonnegative positive integer number.

Solutions of ɥ(x,t) may be given exactly or approximately as:

$$ɥ(x,t) = \sum_{n=0}^{\infty} ɥ_n(x,t) \qquad \ldots(14)$$

4. Applications

In this section, the KF-DM will be applied to solve some SchLPDEs and NLSchEs.

Example 4.1. Consider the SchLPDE**:**

$$iɥ_t + ɥ_{xx} = 0 \qquad \ldots(15)$$

with the initial condition given by:

$$ɥ(x,0) = \cosh(3x) \qquad \ldots(16)$$

Applying the KF-DM to Eq. (15) and using the initial condition given in Eq. (16), we get the recurrence relation given by:

$ɥ_0(x,t) = cosh(3x)$
$ɥ_{n+1}(x,t) = iĶ^{-1}[v^2Ķ\,[ɥ_{nxx}]],\ n \geq 0 \qquad \ldots(17)$

We express few components as:

$ɥ_0(x,t) = cosh(3x) \qquad \ldots(18)$
$ɥ_1(x,t) = iĶ^{-1}[v^2Ķ\,[ɥ_{0xx}]] \qquad \ldots(19)$
$\qquad = i\,Ķ^{-1}[9\, v^3 cosh(3x)]$

$= 9\ it\ cosh(3x)$

$ч_2(x,t) = i\text{К}^{-1}[v^2\text{К}\,[ч_{1xx}]]$ …(20)

$= i\,\text{К}^{-1}[81iv^5\ cosh(3\,x)]$

$= \frac{(i9t)^2}{2!}\ Cosh(3\,x)$

$ч_3(x,t) = i\text{К}^{-1}[v^2\text{К}\,[ч_{2xx}]]$ …(21)

$= i\text{К}^{-1}[\frac{277}{2!}\ i^2\ v^7\ cosh(3x)]$

$= \frac{(i9t)^3}{3!}\ cosh(3x)$

and so on. Thus, on summing the above iterations, we obtain:

$ч\,(x,t) = cosh(3x))\left[9it + \frac{(9it)^2}{2!} + \frac{(9it)^3}{3!} + \cdots\right]$ …(22)

which is leading to the exact solution:

$ч(x,t) = cosh(3x)\ e^{9ix}$ …(23)

This agrees with the result found by (ADM) [18].

Example 4.2. Consider the following NLSchE of the form:

$iч_t + ч_{xx} + 6|ч|^2ч = 0$ …(24)

with the initial condition given by:

$ч(x,t) = e^{3ix}$ …(25)

Appling the Kashuri-Fundo transform coupled with ADM:

$ч_0(x,t) = ч(x,0)$ …(26)

$ч_{n+1}(x,t = i\text{К}^{-1}\left[v^2\text{К}\,(ч_{nxx} + 6\,A_n)\right]$ …(27)

where:

$A_0 = \overline{ч_0}\ ч_0{}^2 = e^{3ix}$

$A_1 = 2ч_0\ ч_1\ \overline{ч_0} + ч_0{}^2\ \overline{ч_1} = -3it\ e^{3ix}$

$A_3 = 2\ ч_0\ ч_2\ \overline{ч_0} + ч_1{}^2\ \overline{ч_0} + 2ч_0\ ч_1\ \overline{ч_1} + ч_0{}^2\ \overline{ч_2}$

$= \frac{9}{2}\ i^2\ t^2 e^{3ix}$

⋮

and so on.

$ч_0(x,t) = e^{3ix}$ …(28)

$ч_1(x,t) = i\text{К}^{-1}\left[v^2\text{К}\,(ч_{0xx} + 6\,A_0)\right]$ …(29)

$= i\,\text{К}^{-1}\,[v^2\text{К}(-3e^{3ix})]$

$= i\,\text{К}^{-1}[-3v^3e^{3ix}]$

$= -3it\ e^{3ix}$

$ч_2(x,t) = i\text{К}^{-1}\,[v^2\text{К}\,(ч_{1xx} + 6\,A_1)]$ …(30)

$= i\,\text{К}^{-1}[v^2\ \text{К}(9ite^{3ix})]$

$= i\,\text{К}^{-1}[\,9v^5\ ie^{3ix}\}]$

$= \frac{9}{2!}\ i^2t^2\ e^{3ix}$

$= \frac{(3it)^2}{2!}\ e^{3ix}$

$ч_3(x,t) = i\text{К}^{-1}\,[v^2\text{К}\,(ч_{2xx} + 6\,A_2)]$ …(31)

$= i\,\text{К}^{-1}\left[v^2\text{К}\left\{\frac{-27}{2}\ i^2\ t^2e^{3ix}\right\}\right]$

$= i\,\text{К}^{-1}[-27\,i^2\ v^7e^{3ix}]$

$= \frac{-27}{3!}\ i^3\ t^3e^{3ix}$

$= -\frac{(3it)^3}{3!}e^{3ix}$

⋮

Then:

$ч(x,t) = \sum_{n=0}^{\infty} ч_n(x,t)$ …(32)

$= e^{3ix}(1 - 3it + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!}\ \ldots.)$

$= e^{3i(x-t)}$ …(33)

Then the exact solution is:

$ч(x,t) = e^{3i(x-t)}$ …(34)

The same exact agrees with the result found by (ADM) [9] and (VIM) [10,11].

Example 4.3. Consider the following NLSchE of the form:

$iч_t + ч_{xx} + 2|ч|^2ч = 0$ …(35)

with the initial condition given by:

$ч(x,0) = e^{ix}$ …(36)

Appling the Kashuri-Fundo transform coupled with ADM:

$ч_0(x,t) = ч(x,0)$ …(37)

$ч_{n+1}(x,t) = i\text{К}^{-1}[\,v^2\text{К}(ч_{nxx} + 2A_n)]$ …(28)

where:

$A_0 = ч_0^2\ \overline{ч_0} = e^{ix}$,

$A_1 = 2ч_0ч_1\overline{ч_0} + ч_0^2\overline{ч_1} = ie^{ix}$,

$A_2 = 2ч_0ч_2\overline{ч_0} + ч_1^2\overline{ч_0} + 2ч_0ч_1\overline{ч_1} + ч_0^2\overline{ч_2} = i^2e^{ix}$,

⋮

and so on.

$ч_0\,(x,t) = e^{ix}$ …(39)

$ч_1(x,t) = i\text{К}^{-1}[\,v^2\text{К}(ч_{0xx} + 2A_0)]$ …(40)

$= i\text{К}^{-1}[v^3e^{ix}]$

$= ite^{ix}$

$ч_2(x,t) = i\text{К}^{-1}[\,v^2\text{К}\{ч_{1xx} + 2A_1\}]$ …(41)

$= i\text{К}^{-1}[iv^5e^{ix}]$

$= \frac{1}{2!}\ t^2i^2e^{ix}$

$ч_3(x,t) = i\text{К}^{-1}[\,v^2\text{К}\{ч_{2xx} + 2A_2\}]$ …(42)

$= i\text{К}^{-1}[v^7\ e^{ix}]$

$= \frac{1}{3!}\ i^3t^3e^{ix}$

Then:

$ч_0(x,t) = \sum_{n=0}^{\infty} ч_n(x,t)$ …(43)

$= e^{ix}\left(1 + it + \frac{i^2}{2!}\ t + \frac{i^3}{3!}\ t + \cdots\right)$

which is leading to the exact solution:

$ч(x,t) = e^{i(x+t)}$ …(44)

The same exact which agrees with the result found by (ADM) [17].

5. Conclusion

Our goal of this work is to provide a combined computational method consisting of an integral transform and a semi-analytical method for solving SchLPDEs and NLSchEs. It is an application of the Kashuri-Fundo transform with the Adomain method. The exact solutions of some SchLPDEs and NLSchEs were successfully found and the obtained results are compared to ADM and VIM. This approach shows its efficiency, accuracy, and simplicity. This approach can be used to solve various types of differential equations in physics, engineering, and applied mathematics.

Acknowledgments

We would like to thanks Al-Nahrain University their support.

Conflicts of Interest

The authors declare that there is no conflict of interest.

References

[1] Brezis H. and Browder F.; "Partial differential equations in the 20th century", Advances in mathematics, 135 (Apr 15): 76-144, 1998.
[2] Smith G. D.; "Numerical Solution of Partial differential equations: Finite difference methods", Oxford: Oxford University Press, 1978.
[3] Strang G. and Fix G.; "An analysis of the finite element method", Englewood Cliffs, Prentice, Hall, 1973.
[4] LeVeque R.; "Finite volume methods for hyperbolic problems", Cambridge: Cambridge University Press, 2002.
[5] Banerjee P. K.; "Boundary element methods in engineering", London: McGraw-Hill, 1994.
[6] Liu G. R.; "Mesh free methods", Singapore: CRC Press, 2002.
[7] Gottlieb D. and Orzag S.; "Numerical analysis of spectral methods: Theory and applications", Philadephia: SIAM, 1977.
[8] Sadighi A. and Ganji D.D.; "Analytic treatment of linear and nonlinear Schrödinger equations: A study with homotopy perturbation and Adomian's decomposition method", Phys. Lett. A, 372 (4): 465-469, 2008.
[9] Wazwaz A.M., "Partial differential equations and solitary waves theory", Springer-Verlag, Heidelberg 2009.
[10] Sweilam N. H.; "Variational iteration method for solving cubic nonlinear schrödinger equation", Journal of Computational and Applied Mathematics, 207: 155-163, 2007
[11] Nuruddeen R. I.; "Elzaki decomposition method and its applications in solving linear and nonlinear Schrödinger equations", Sohag Journal of Mathematics, 4 (2): 1-5, 2017.
[12] Biazar J. andGhazvini H.; "Exact solution for nonlinear Schrödinger equation by He's Homotopy Pertubation Method", Phys. Lett. A, 366 (1-2): 79-84, 2007.
[13] Ablowitz M. and Segur H.; "Solitons and the Inverse Scattering Transform", SIAM, 1981.
[14] Kumar A. and Pankaj R. D.; "Solitary wave solutions of schrödinger equation by Laplace-Adomian decomposition method", Physical Review and Reseach International, 3 (4): 702-712, 2013.
[15] Kashuri A.; Fundo A. and Kreku M.; "Mixture of a new integral transform and homotopy perturbation method for solving nonlinear partial differential equations", Advances in Pure Mathematics, 3: 317-323, 2013.
[16] A. Kashuri A. and Fundo A.; "A new integral transform", Advances in Theoretical and Applied Mathematics, 8 (1): 27-43, 2013.
[17] Nuruddeen R. I.; Muhammad L.; Nass A. M. and Sulaiman T. A.; "A review of the integral transforms-based decomposition methods and their applications in solving nonlinear PDEs", Palestine Journal of Mathematics, 1 (7): 262-280, 2018.
[18] Nuruddeen R. I.; "Elzaki decomposition method and its applications in solving linear and nonlinear Schrödinger equations", Sohag Journal of Mathematics, 4 (2): 1-5, 2017.