Global Stability of Harmful Phytoplankton and Herbivorous Zooplankton with Holling Type IV Functional Response

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Abstract

In this paper harmful phytoplankton and herbivorous zooplankton model with Hollimg type IV functional response is proposed and analyzed. The local stability analysis of the system is carried out. The global dynamics of the system is investigated with the help of the Lyapunov function. Finally, the analytical obtained results are supported with numerical simulation.

Keywords: Holling type IV functional response, harmful phytoplankton, herbivorous zooplankton, global stability.

Introduction

Plankton is the basis of the entire aquatic food chain. Phytoplankton, in particular, occupies the first trophic level. Plankton performs services for the Earth: it serves as food for marine life, gives off oxygen and also absorbs half of the carbon dioxide from the Earth's atmosphere. The dynamics of a rapid (or massive) increase or decrease of plankton populations is an important subject in marine plankton ecology and generally termed as a 'bloom'. Harmful algal blooms (HABs) have adverse effects on human health, fishery, tourism, and the environment. In recent years, considerable scientific attention has been given to HABs, see for example [5,7,12,21,23,25]. On the other hand, ecology relates to the study of living beings in relation to their living styles. Research in the area of the theoretical ecology was initiated by Lotka (1925) and by Volterra (1926). Since then many mathematicians and ecologists contributed to the growth of this area of knowledge. Consequently, several mathematical models deal with the dynamics of prey predator models involving different types of functional responses have been proposed and studied, see for example [1,2,9,11,13,14] and the references therein.

Keeping the above in view, in this chapter a harmful phytoplankton interacting herbivorous zooplankton with Holling type IV functional response have been proposed and studied.

Mathematical model formulation

Consider the simple phytoplankton-zooplankton system with Holling typeIV functional response which can be written as:

$$\begin{aligned}\frac{dx}{dt} &= (a-bx)x - \frac{\alpha\gamma xy}{x^2+\gamma x+\gamma\beta}\\ \frac{dy}{dt} &= \frac{e\alpha\gamma xy}{x^2+\gamma x+\gamma\beta} - hy - \theta Dxy\end{aligned} \qquad \text{.......................} (1)$$

Here $x(t)$ and $y(t)$ represent the densities of phytoplankton and zooplankton at time t respectively. While the parameters $a>0$ is the intrinsic growth rate of the phytoplankton population; $b>0$ is the strength of intra-specific competition among the phytoplankton species; the parameter $\beta>0$ can be interpreted as the half-saturation constant in the absence of any inhibitory effect; the parameters $\gamma>0$ is a direct measure of the predator immunity from the phytoplankton; the zooplankton consumer consume their food according to Holling type IV of functional response, where $\alpha>0$ is the predation rate on the zooplankton; $e>0$ is the conversion rate of predation into higher level species; here $\theta>0$ represents the liberation rate of toxic substance by the harmful phytoplankton x; $D>0$ represent the maximum zooplankton ingestion rates for the toxic substance produced by phytoplankton x. Finally $h>0$ represent the natural death rate for the zooplankton. The initial condition for system (1) may be taken as any point in the region $R_+^2=\{(x,y): x\geq 0, y\geq 0\}$. Obviously, the interaction functions in the right hand side of system (1) are continuously differentiable functions on R_+^2, hence they are Lipschitizian. Therefore the solution of system (1) exists and is unique. Further, all the solutions of system (1) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem (1):
All the solutions of system (1) which initiate in R_+^2 are uniformly bounded.

***Proof*:**
Let $((x(t), y(t))$ be any solution of the system (1) with non-negative initial condition (x_0, y_0). According to the first equation of system (1) we have

$$\frac{dx}{dt} \le (a - bx)x$$

Then by solving this differential inequality we obtain that

$$x(t) \le \frac{ax_0}{ae^{-at} + (1 - e^{-at})bx_0}$$

Thus $\lim_{t\to\infty} Sup\, x(t) \le M$ where $M = \max\left\{\frac{a}{b}, x_0\right\}$. Define the function: $W(x,y) = x + \frac{1}{e} y$

So the time derivative of $W(t)$ along the solution of the system (1)

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{e}\frac{dy}{dt}$$
$$\frac{dW}{dt} \le (a+h)x - h(x + \frac{1}{e} y)$$
$$\frac{dW}{dt} + hW \le (a+h)M$$

Again by solving the above linear differential inequality we get

$$\frac{dW}{dt} \le \frac{(a+h)M}{h} + W(0)e^{-ht} - \frac{(a+h)}{h} Me^{-ht}$$

Consequently, for $t \to \infty$ we have

$$0 \le W(t) \le \frac{(a+h)M}{h}$$

Hence all solution of system (1) enter the region

$$\Omega = \{((x(t), y(t) \in R_+^2 : x(t) + \frac{1}{e} y(t) \le \frac{(a+h)M}{h} + \varepsilon \;\; for\, any\, \varepsilon > 0\}$$ ■

Existence of equilibrium points
The system (1) have at most three non-negative equilibrium points, two of them namely $F_0 = (0,0)$, $F_x = (\frac{a}{b}, 0)$ always exist. While the existence of other equilibrium points is shown in the following:
The positive equilibrium point $F_{xy} = (x^*, y^*)$ exists in the interior of the first quadrant if and only if there is a positive solution to the following set of algebraic nonlinear equations:

$$a - bx - \frac{\alpha\gamma y}{x^2 + \gamma x + \gamma\beta} = 0 \text{ ....(2a)}$$
$$\frac{e\alpha\gamma x}{x^2 + \gamma x + \gamma\beta} - h - \theta Dx = 0 \text{ .... (2b)}$$

From (2a) we have

$$y^* = \left(a - bx^*\right)\frac{{x^*}^2 + \gamma x^* + \gamma\beta}{\alpha\gamma}$$

Clearly, $y^* > 0$ if the following condition holds
$a > bx^*$
while x^*, represents the positive root to the following equation

$$f(x) = A_3 x^3 + A_2 x^2 + A_1 x + A_0 \text{ .... (3)}$$

Where
$A_3 = -D\theta$,
$A_2 = -(D\theta\gamma + h)$, $A_1 = \gamma(e\alpha - \theta D\beta - h)$,
$A_0 = -h\gamma\beta$

So by using Descartes rule of signs, Eq. (3) has either no positive root and hence there is no equilibrium point or two positive roots depending on the following condition holds:

$$e\alpha > \theta D\beta + h$$

The stability analysis
In this section the stability (locally as well as globally) analysis of the above mentioned equilibrium points of system (1) are investigated analytically.

The Jacobian matrix of system (1) at the equilibrium point $F_0 = (0,0)$ can be written as

$$J_0 = J(F_0) = \begin{bmatrix} a & 0 \\ 0 & -h \end{bmatrix}$$

$\lambda_{01} = a > 0$, $\lambda_{02} = -h < 0$

Therefore, the equilibrium point F_0 is a saddle point.

The Jacobian matrix of system (1) at the equilibrium point $F_x = (\frac{a}{b}, 0)$ can be written as

$$J_x = J(F_x) = \begin{bmatrix} -a & \frac{-\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} \\ 0 & \frac{e\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} - h - \frac{a\theta D}{b} \end{bmatrix}$$

Hence, the eigenvalues of J_x are:

$$\hat{\lambda}_1 = -a < 0,\ \hat{\lambda}_2 = \frac{e\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} - h - \frac{a\theta D}{b}$$

Therefore, F_x is locally asymptotically stable if and only if

$$\frac{e\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} < h + \frac{a\theta D}{b} \quad \text{.................................(4a)}$$

While F_x is saddle point provided that

$$\frac{e\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} > h + \frac{a\theta D}{b} \quad \text{................................. (4b)}$$

Finally, the Jacobian matrix of system (1) at the positive equilibrium point $F_{xy} = (x^*, y^*)$ in the $Int.R_+^2$ can be written as:

$$J_{xy} = J(F_{xy}) = \begin{bmatrix} \left(-b + \frac{\alpha\gamma y^*(2x^*+\gamma)}{R^{*2}}\right)x^* & \frac{-\alpha\gamma x^*}{R} \\ \left(\frac{e\gamma\alpha(\gamma\beta - x^{*2})}{R^{*2}} - \theta D\right)y^* & 0 \end{bmatrix}$$

Note that according to the stability theorem for the two dimensional dynamical system, $F_{xy} = (x^*, y^*)$ is locally asymptotically stable provided that

$Trace(J_{xy}) = T = a_{11} < 0$,

$|J_{xy}| = D = -a_{12}a_{21} > 0$

Now since

$$T = \left(-b + \frac{\alpha\gamma y^*(2x^*+\gamma)}{R^{*2}}\right)x^* \quad \text{.................................(5a)}$$

$$D = \frac{\alpha\gamma x^* y^*}{R^{*3}}\left(e\gamma\alpha(\gamma\beta - x^{*2}) - \theta D\ R^{*2}\right) \quad \text{................. (5b)}$$

Therefore the positive equilibrium point $F_{xy} = (x^*, y^*)$ of system (1) is locally asymptotically stable in $Int.R_+^2$ under the following necessary and sufficient conditions

$$b > \frac{\alpha\gamma y^*(2x^*+\gamma)}{R^{*2}} \quad \text{...(6a)}$$

$$e\gamma\alpha(\gamma\beta - x^{*2}) > \theta D\ R^{*2} \quad \text{.................................. (6b)}$$

In the following the persistence of system (1) is studied. It is well known that the system is said to be persists if and only if each species is persist. Mathematically, this is means that, system (1) is persists if the solution of system with positive initial condition does not have omega limit sets on the boundary planes of its domain. However, biologically means that, all the species are survivor. In the following theorem the persistence condition of the system (1) is established using the Gard and Hallam technique [10].

<u>Theorem (2):</u>

System (1) is uniformly persist provided that condition (4b) holds.

Proof:

Consider the following function, $\sigma(x,y) = x^{p_1} y^{p_2}$ where $p_i, i = 1,2$ are undetermined positive constants. Obviously, $\sigma(x,y)$ is C^1 positive function defined on R_+^2 and $\sigma(x,y) \to 0$, if $x \to 0$ or $y \to 0$. Now since

$$\Psi(x,y) = \frac{\sigma'(x,y)}{\sigma(x,y)} = p_1 \frac{x'}{x} + p_2 \frac{y'}{y}$$

Therefore

$$\Psi(x,y) = p_1\left[a - bx - \frac{\alpha\gamma y}{x^2+\gamma x+\gamma\beta}\right] + p_2\left[\frac{e\alpha\gamma x}{x^2+\gamma x+\gamma\beta} - h - \theta D x\right]$$

Note that, since $F_0 = (0,0)$ and $F_x = (\frac{a}{b}, 0)$ are the only possible omega limit sets of the solution of system (1) on the boundary of $Int.R_+^2$, in addition

$\Psi(E_0) = ap_1 - dp_2$

$$\Psi(E_x) = \left(\frac{e\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} - h - \frac{a\theta D}{b} \right) p_2 > 0$$

Clearly $\Psi(E_0) > 0$ for all sufficiently large positive value of p_1 with respect to p_2, while $\Psi(E_x) > 0$, for all values of p_2 under condition (4b).Hence σ represents persistence function and system (1) is uniformly persistent. ■

Since system (1) may have either two equilibrium points or no equilibrium points in the $Int.R_+^2$ of the F_{xy}. The global stability of the equilibrium point F_x in R_+^2 is investigated as shown in the following.

Global stability of the system

In this section the global stability of the equilibrium points F_x in R_+^2 is investigated as shown in the following theorem.

Theorem (3):

Assume that the equilibrium point F_x is locally asymptotically stable in the R_+^2, and let the following conditions:

$$h \geq \frac{e\alpha\gamma ab}{a^2+\gamma b(a+\beta b)} \quad \text{.. (7)}$$

Hold, then F_x is globally asymptotically stable in R_+^2.

***Proof*:**

Consider the following positive definite function:

$$U_1(x,y) = c_1\left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + c_2 y$$

Where $\bar{x} = \frac{a}{b}$. Clearly $U_1 : R_+^2 \to R,$ and is a C^1 positive definite function, where $c_i, (i = 1,2)$ are positive constants to be determined. Now, since the derivative of U_1 along the trajectory of system (1) can be written as:

$$\frac{dU_1}{dt} = -c_1 b(x - \bar{x})^2 - (c_1 - c_2 e)\frac{\alpha\gamma xy}{x^2+\gamma x+\gamma\beta} - \left(c_2 h - \frac{c_1\alpha\gamma\bar{x}}{x^2+\gamma x+\gamma\beta}\right)y - c_2 D\theta xy$$

And since, we have $x \leq \frac{a}{b}$ then by choosing the positive constants as $c_1 = 1$ and $c_2 = \frac{1}{e}$ we get:

$$\frac{dU_1}{dt} \leq -b\left(x - \frac{a}{b}\right)^2 - \left(\frac{h}{e} - \frac{\alpha\gamma ab}{a^2+\gamma b(a+\beta b)}\right)y$$

Therefore, $\frac{dU_1}{dt} < 0$ under condition (7), hence U_1 is strictly Lyapunov function. Therefore, F_x is globally asymptotically stable in the R_+^2. ■

Numerical analysis

In this section the global dynamics of system (1) is studied numerically. System (1) is solved numerically for different sets of parameters and for different sets of initial conditions, and then the attracting sets and their time series are drown as shown below. Now, for the following set of hypothetical parameters

$$\begin{gathered} a = 0.25, b = 0.2, \alpha = 1, \gamma = 0.75, \beta = 2, \\ h = 0.01, e = 0.25, \theta = 0.02, D = 0.01. \end{gathered} \quad \text{.... (8)}$$

The attracting sets along with their time series of system (1) are drown in Fig. (1). Note that from now onward, in the time series figures, we will use the following representation: blue color represents the trajectory of phytoplankton, green color represents the trajectory of zooplankton.

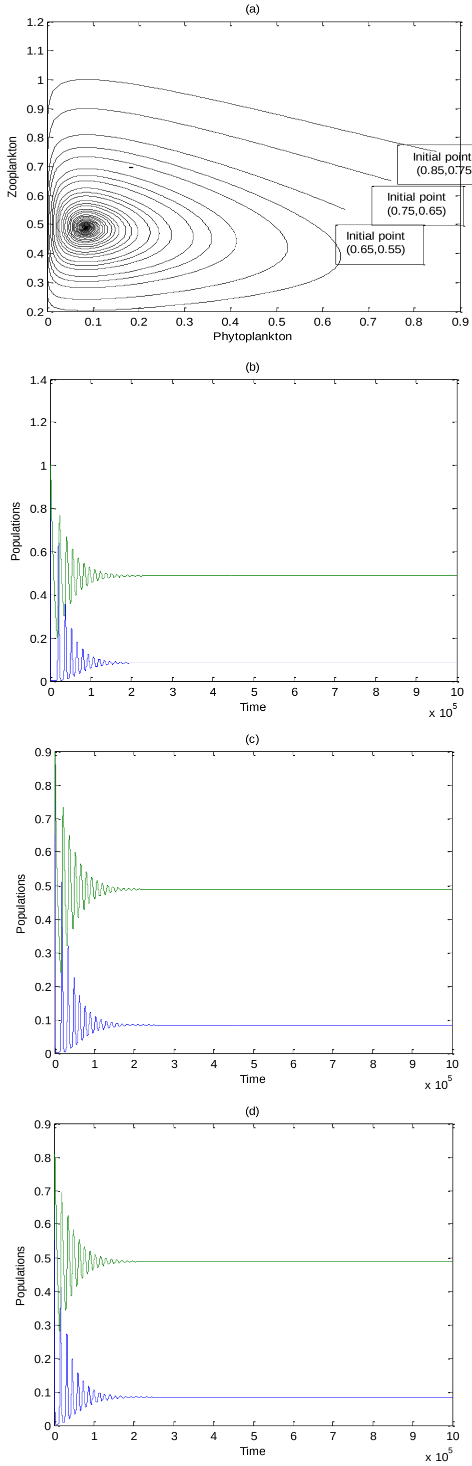

Fig. (1): (a) The solution of system (1) approaches asymptotically to the positive equilibrium point starting from different initial values for the data given by Eq. (8). (b) Time series of the attractor in (a) starting at (0.85, 0.75). (c) Time series of the attractor in (a) starting at (0.75, 0.65) (d) Time series of the attractor in (a) starting at (0.65, 0.55).

Clearly, as shown in Fig. (1), system (1) has a globally stable positive equilibrium point $F_{xy} = (0.08, 0.48)$ in the $Int.R_{+}^{2}$, hence all the species coexists and the system persists. However, for the parameters values given by Eq. (8) with the intrinsic growth rate $a = 0.5$, system (1) approaches to the periodic dynamics in $Int.R_{+}^{2}$, see the following figure.

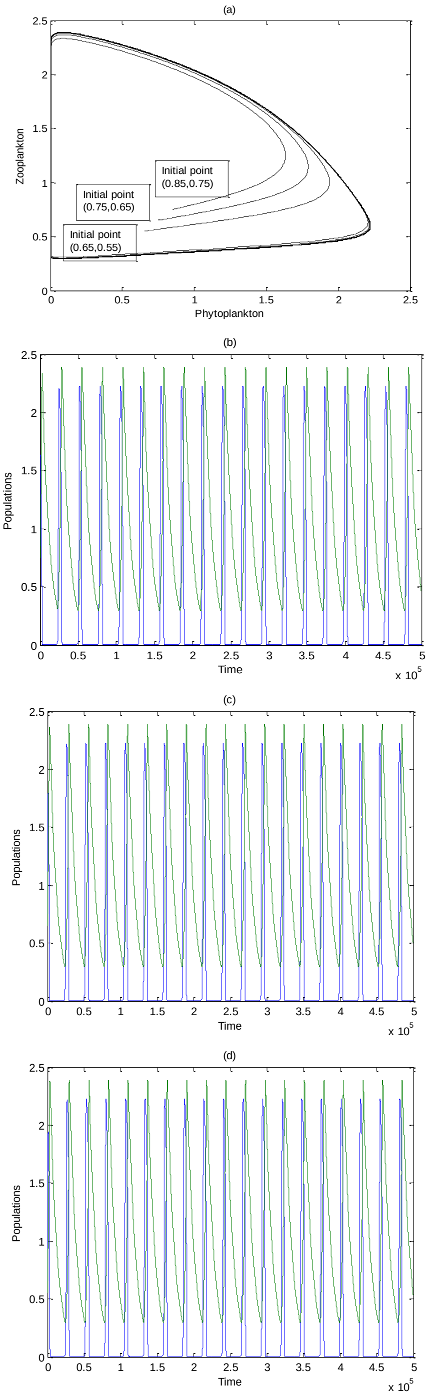

Fig. (2): (a) Globally asymptotically stable limit cycle of system (1) starting from different initial values for the data given by Eq. (8) with $a = 0.5$. (b) Time series of the attractor in (a) starting at (0.85, 0.75). (c) Time series of the attractor in (a) starting at (0.75, 0.65) (d) Time series of the attractor in (a) starting at (0.65, 0.55).

Discussion and Conclusion

In this paper, a mathematical model consisting of a Holling type IV phytoplankton-zooplankton model with intra specific competition has been studied analytically as well as numerically. The condition for the system (1) to be uniformly bounded and persistence have been derived. The local as well as global stability of the proposed system has been studied. The effect of intrinsic growth rate of the phytoplankton species on the dynamical behavior of system (1) is studied numerically and the trajectories of the system are drowned. According to these formats the following conclusions are obtained:

1. For the set of hypothetical parameters values given in Eq.(8), system (1) approaches asymptotically to a globally asymptotically stable point $F_{xy} = (x^*, y^*)$.
2. As the intrinsic growth rate of the phytoplankton decreasing then the system (1) approaches to an asymptotically stable positive equilibrium point, otherwise the system has periodic dynamics. So this parameter has a stabilizing effect on the system.

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الخلاصة

في هذا البحث تم اقتراح و تحليل نظام العوالق النباتية والعوالق الحيوانية مع دالة الاستجابة لهولنك ذات النوع الرابع. السلوك الديناميكي المحلي للنظام تمت دراسته. كما تم مناقشة السلوك الديناميكي الشامل بمساعدة دالة ليابانوف, واخيرا تم تدعيم النتائج التحليلية الناتجة باستخدام المحاكاة العددية وقد لوحظ بان النظام يمتلك اما نقطة استقرار او ديناميكية دورية.