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Modular Irreducible Representations of the F_pW_4-Submodules $N_{F_p}(\lambda,\mu)$ of the Modules $M_{F_p}(\lambda,\mu)$ as Linear Codes, where W_4 is the Weyl Group of Type B_4

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Abstract

The modular representations of the F_pW_n-Specht modules $S_K(\lambda,\mu)$ as linear codes is given in our paper [6], and the modular irreducible representations of the F_pW_4-submodules $N_{F_p}(\lambda,\mu)$ of the Specht modules $S_{F_p}(\lambda,\mu)$ as linear codes where W_4 is the Weyl group of type B_4 is given in our paper [5]. In this paper we are concerning of finding the linear codes of the representations of the irreducible F_pW_4-submodules $N_{F_p}(\lambda,\mu)$ of the F_pW_4-modules $M_{F_p}(\lambda,\mu)$ for each pair of partitions (λ,μ) of a positive integer $n = 4$, where $F_p = \mathrm{GF}(p)$ is the Galois field (finite field) of order p, and p is a prime number greater than or equal to 3. We will find in this paper a generator matrix of a subspace $U_{(p)}^{((2,1),(1))}$ representing the irreducible F_pW_4-submodules $N_{F_p}((2,1),(1))$ of the F_pW_4-modules $M_{F_p}((2,1),(1))$ and give the linear code of $U_{(p)}^{((2,1),(1))}$ for each prime number p greater than or equal to 3. Then we will give the linear codes of all the subspaces $U_{(p)}^{(\lambda,\mu)}$ for all pair of partitions (λ,μ) of a positive integer $n = 4$, and for each prime number p greater than or equal to 3.

We mention that some of the ideas of this work in this paper have been influenced by that of Adalbert Kerber and Axel Kohnert [13], even though that their paper is about the symmetric group and this paper is about the Weyl groups of type B_n.

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Remarks: Throughout this paper, let:

i- F_p be the Galois field (finite field) of order p ([8], p.429), that is $F_p = \mathrm{GF}(p)$.

ii- K be a field which is infinite (of characteristic 0) or finite of order a prime number $p \geq 3$, and $x_1, x_2, \ldots, x_n$ be independent indeterminates over K.

iii- W_n be the Weyl group of type B_n, which is the group of all permutations w of $\{x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n\}$, such that $w(-x_i) = -w(x_i)$, for each $i = 1, 2, \ldots, n$.

iv- KW_n be the group ring of W_n with coefficients in K. KW_n is also a group algebra of W_n over K.

1. Introduction

There are many types of Weyl groups which are; (1) The infinite family of Weyl groups of type A_n, namely symmetric groups, (2) The infinite family of Weyl groups of type B_n, namely hyperoctahedral groups, (3) The infinite family of Weyl groups of type C_n, (4) The infinite family of Weyl groups of type D_n, (5) The Weyl groups of type G_2, (6) The Weyl groups of type F_4, (7) The Weyl groups of types E_6, E_7, and E_8 (see [9], p.40; [12], p.134; and [15], p.36). In this paper we are concern with the Weyl groups of type B_n, and the connection of the representations of the Weyl groups W_n of type B_n with the linear codes and more precisely we are concern with the modular irreducible representations of the F_pW_n-submodules $N_{F_p}(\lambda,\mu)$ of the F_pW_n-modules $M_{F_p}(\lambda,\mu)$ as linear codes when $n = 4$, and for each prime number p greater than or equal to 3.

2. Preliminaries

Definition 2.1. Let $\{y_1,\dots,y_r\}\subseteq\{\pm x_1,\dots,\pm x_n\}$, such that $y_i\neq\pm y_j$ for each $i,j=1,\dots,r$ and $i\neq j$, then we define:

$$\Delta_1(y_1,\dots,y_r)=\begin{cases}\prod_{1\le i<j\le r}(y_j^2-y_i^2)\prod_{\ell=1}^{r}y_\ell & \text{if } r>1\\ y_1 & \text{if } r=1\end{cases}$$

$$\Delta_2(y_1,\dots,y_r)=\begin{cases}\prod_{1\le i<j\le r}(y_j^2-y_i^2) & \text{if } r>1\\ 1 & \text{if } r=1\end{cases}$$

([2], p.8 and [4], p.15).

Example 2.2.

$$\begin{aligned}\Delta_2(x_4,x_9,x_3)&=(x_3^2-x_4^2)(x_3^2-x_9^2)(x_9^2-x_4^2)\\&=x_3^4x_9^2-x_3^4x_4^2+x_3^2x_4^4-x_3^2x_9^4+\\&\quad x_4^2x_9^4-x_4^4x_9^2\end{aligned}$$

and

$$\begin{aligned}\Delta_1(x_4,x_9,x_3)&=x_3x_4x_9\,(\Delta_2(x_4,x_9,x_3))\\&=x_3^5x_4x_9^3-x_3^5x_4^3x_9+x_3^3x_4^5x_9\\&\quad -x_3^3x_4x_9^5+x_3x_4^3x_9^5-x_3x_4^5x_9^3.\end{aligned}$$

Definition 2.3. Let $(\lambda,\mu)=((\lambda_1,\lambda_2,\dots,\lambda_s),(\mu_1,\mu_2,\dots,\mu_t))$ be a pair of partitions of a positive integer *n*, and let $Z^{(\lambda,\mu)}$ be any (λ,μ)-tableau, then:

$$f(Z^{(\lambda,\mu)})=\begin{cases}f_1(Z^\lambda) & \text{if } |\mu|=0\\ f_2(Z^\mu) & \text{if } |\lambda|=0\\ f_1(Z^\lambda)f_2(Z^\mu) & \text{otherwise}\end{cases}$$

such that:

$$f_1(Z^\lambda)=\prod_{j=1}^{\lambda_1}\Delta_1(Z^{(\lambda,\mu)}(1,j,1),\dots,Z^{(\lambda,\mu)}(\lambda'_j,j,1))$$

where λ'_j is the number of the indeterminates in the j^{th} column of the first tableau Z^λ, and

$$f_2(Z^\mu)=\prod_{j=1}^{\mu_1}\Delta_2(Z^{(\lambda,\mu)}(1,j,2),\dots,Z^{(\lambda,\mu)}(\mu'_j,j,2))$$

where μ'_j is the number of the indeterminates in the j^{th} column of the second tableau Z^μ, $f(Z^{(\lambda,\mu)})$ is called the Specht polynomial of (λ,μ)-tableau $Z^{(\lambda,\mu)}$ ([2], p.9 and [4], p.15).

Example 2.4. Let $Z^{((2,1),(1))}$ be the following $((2,1),(1))$-tableau:

$$\begin{matrix}x_3 & x_4\\ x_1 & \end{matrix}\quad ;\quad -x_2$$

$$\begin{aligned}f(Z^{((2,1),(1))})&=\prod_{j=1}^{\lambda_1}\Delta_1(Z^{((2,1),(1))}(1,j,1),\dots,\\&\qquad Z^{((2,1),(1))}(\lambda'_j,j,1))\prod_{j=1}^{\mu_1}\Delta_2(Z^{((2,1),(1))}(1,j,2)\\&\qquad ,\dots,Z^{((2,1),(1))}(\mu'_j,j,2))\\&=\Delta_1(Z^{((2,1),(1))}(1,1,1),Z^{((2,1),(1))}(2,1,1))\\&\qquad\Delta_1(Z^{((2,1),(1))}(1,2,1))\cdot\Delta_2(Z^{((2,1),(1))}(1,1,2))\\&=\Delta_1(x_3,x_1)\cdot\Delta_1(x_4)\cdot\Delta_2(-x_2)\\&=(x_1^2-x_3^2)x_3x_1\cdot x_4\cdot 1\\&=x_1^3x_3x_4-x_1x_3^3x_4\end{aligned}$$

Definition 2.5. Let $(\lambda,\mu)=((\lambda_1,\lambda_2,\dots,\lambda_s),(\mu_1,\mu_2,\dots,\mu_t))$ be a pair of partitions of a positive integer n, and let $Z^{(\lambda,\mu)}$ be any (λ,μ)-tableau. Then the cyclic KW_n-module $S_K(\lambda,\mu)$ generated over KW_n by $f(Z^{(\lambda,\mu)})$ (i.e., $S_K(\lambda,\mu)=KW_nf(Z^{(\lambda,\mu)})$) is called the Specht module over *K* corresponding to the pair of partitions (λ,μ) of *n* ([2], p.10 and [4], p.16).

Theorem 2.6. Let $(\lambda,\mu)=((\lambda_1,\lambda_2,\dots,\lambda_s),(\mu_1,\mu_2,\dots,\mu_t))$ be a pair of partitions of a positive integer n. Then there are exactly $\frac{n!}{H_{\lambda,\mu}}$ distinct (λ,μ)-standard tableaux, where $H_{\lambda,\mu}=H_\lambda\cdot H_\mu$, such that $H_\lambda=\prod_{i=1}^{s}\prod_{j=1}^{\lambda_i}h_{ij}$, where $h_{ij}=\lambda_i+\lambda'_j-i-j+1$, and $H_\mu=\prod_{i=1}^{t}\prod_{j=1}^{\mu_i}e_{ij}$, where $e_{ij}=\mu_i+\mu'_j-i-j+1$ ([2], p.20 & p.21 and [4], p.13).

Theorem 2.7. Let $(\lambda,\mu)=((\lambda_1,\lambda_2,\dots,\lambda_s),(\mu_1,\mu_2,\dots,\mu_t))$ be a pair of partitions of a positive integer *n*. Then the Specht module $S_K(\lambda,\mu)$ has a *K*-basis $B(\lambda,\mu)=\{f(Z^{(\lambda,\mu)})\mid Z^{(\lambda,\mu)}$ is a standard (λ,μ)-tableau$\}$, and $\dim_K S_K(\lambda,\mu)=\frac{n!}{H_{\lambda,\mu}}$ ([2], p.21, [3], p.305 and [4], p.17).

Theorem 2.8 ([1], p.68 & p.87). Let $((m-2,2),(n-m))$ be a pair of partitions of a positive integer *n*, where $4\le m\le n$.

1. If *p* divides neither $(m-1)$ nor $(m-2)$. Then $S_K((m-2,2),(n-m))$ is irreducible KW_n-module.

2. If p divides either $(m-1)$ or $(m-2)$. Then the KW_n- module $S_K((m-2,2),(n-m))$ has the following composition series:
   $0 \subset N_K((m-2,2),(n-m)) \subset S_K((m-2,2),(n-m))$.

Theorem 2.9 ([2], p.64). Let K be a field of characteristic p not equal to 2.

1. If p does not divide m, then:
   $S_K((m-r+1,1^{r-1}),(n-m))$,
   $S_K((m-r+1,1^{r-1}),(1^{n-m}))$,
   $S_K((n-m),(m-r+1,1^{r-1}))$, and
   $S_K((1^{n-m}),(m-r+1,1^{r-1}))$ are irreducible KW_n-modules where $0<r\le m\le n$.
2. If p divides m, then we have the following composition series:
   $0 \subset N_K((m-r+1,1^{r-1}),(n-m)) \subset S_K((m-r+1,1^{r-1}),(n-m))$,
   $0 \subset N_K((m-r+1,1^{r-1}),(1^{n-m})) \subset S_K((m-r+1,1^{r-1}),(1^{n-m}))$,
   $0 \subset N_K((n-m),(m-r+1,1^{r-1})) \subset S_K((n-m),(m-r+1,1^{r-1}))$,
   $0 \subset N_K((1^{n-m}),(m-r+1,1^{r-1})) \subset S_K((1^{n-m}),(m-r+1,1^{r-1}))$.
   (i.e., $S_K((m-r+1,1^{r-1}),(n-m))$,
   $S_K((m-r+1,1^{r-1}),(1^{n-m}))$,
   $S_K((n-m),(m-r+1,1^{r-1}))$, and
   $S_K((1^{n-m}),(m-r+1,1^{r-1}))$ are reducible KW_n-modules and each one of them has only one proper irreducible KW_n-module), where
   $1<r\le m-1\le n$, and $S_K((m),(n-m))$,
   $S_K((m),(1^{n-m}))$, $S_K((1^m),(n-m))$, and
   $S_K((1^m),(1^{n-m}))$ are irreducible KW_n- modules.

3. The Specht Polynomials of the Standard ((2,1),(1))- Tableaux

We know that $\dim_K S_K((2,1),(1)) = \dfrac{4!}{3\cdot1\cdot1\cdot1} = 8$, and thus we have eight standard ((2,1),(1))-tableaux, which are:

$$Z_1^{((2,1,(1))} = \begin{matrix} x_1 & x_3 \\ x_2 & \end{matrix} \; ; \; x_4 \, ,$$
$$Z_2^{((2,1,(1))} = \begin{matrix} x_1 & x_2 \\ x_3 & \end{matrix} \; ; \; x_4 \, ,$$
$$Z_3^{((2,1,(1))} = \begin{matrix} x_1 & x_4 \\ x_2 & \end{matrix} \; ; \; x_3 \, ,$$
$$Z_4^{((2,1,(1))} = \begin{matrix} x_1 & x_2 \\ x_4 & \end{matrix} \; ; \; x_3 \, ,$$
$$Z_5^{((2,1,(1))} = \begin{matrix} x_1 & x_4 \\ x_3 & \end{matrix} \; ; \; x_2 \, ,$$
$$Z_6^{((2,1,(1))} = \begin{matrix} x_1 & x_3 \\ x_4 & \end{matrix} \; ; \; x_2 \, ,$$
$$Z_7^{((2,1,(1))} = \begin{matrix} x_2 & x_4 \\ x_3 & \end{matrix} \; ; \; x_1 \, ,$$
$$Z_8^{((2,1,(1))} = \begin{matrix} x_2 & x_3 \\ x_4 & \end{matrix} \; ; \; x_1 \, .$$

The corresponding Specht polynomials are:

$$f(Z_1^{((2,1),(1))}) = (x_2^2 - x_1^2)\, x_1x_2x_3 = x_1x_2^3x_3 - x_1^3x_2x_3,$$
$$f(Z_2^{((2,1),(1))}) = (x_3^2 - x_1^2)\, x_1x_2x_3 = x_1x_2x_3^3 - x_1^3x_2x_3,$$
$$f(Z_3^{((2,1),(1))}) = (x_2^2 - x_1^2)\, x_1x_2x_4 = x_1x_2^3x_4 - x_1^3x_2x_4,$$
$$f(Z_4^{((2,1),(1))}) = (x_4^2 - x_1^2)\, x_1x_2x_4 = x_1x_2x_4^3 - x_1^3x_2x_4,$$
$$f(Z_5^{((2,1),(1))}) = (x_3^2 - x_1^2)\, x_1x_3x_4 = x_1x_3^3x_4 - x_1^3x_3x_4,$$
$$f(Z_6^{((2,1),(1))}) = (x_4^2 - x_1^2)\, x_1x_3x_4 = x_1x_3x_4^3 - x_1^3x_3x_4,$$
$$f(Z_7^{((2,1),(1))}) = (x_3^2 - x_2^2)\, x_2x_3x_4 = x_2x_3^3x_4 - x_2^3x_3x_4,$$
$$f(Z_8^{((2,1),(1))}) = (x_4^2 - x_2^2)\, x_2x_3x_4 = x_2x_3x_4^3 - x_2^3x_3x_4.$$

The above polynomials $f(Z_1^{((2,1),(1))})$, $f(Z_2^{((2,1),(1))})$, …, $f(Z_8^{((2,1),(1))})$ mod 3 will be:

$$f_{(3)}(Z_1^{((2,1),(1))}) = x_1x_2^3x_3 + 2x_1^3x_2x_3,$$
$$f_{(3)}(Z_2^{((2,1),(1))}) = x_1x_2x_3^3 + 2x_1^3x_2x_3,$$
$$f_{(3)}(Z_3^{((2,1),(1))}) = x_1x_2^3x_4 + 2x_1^3x_2x_4,$$
$$f_{(3)}(Z_4^{((2,1),(1))}) = x_1x_2x_4^3 + 2x_1^3x_2x_4,$$
$$f_{(3)}(Z_5^{((2,1),(1))}) = x_1x_3^3x_4 + 2x_1^3x_3x_4,$$
$$f_{(3)}(Z_6^{((2,1),(1))}) = x_1x_3x_4^3 + 2x_1^3x_3x_4,$$
$$f_{(3)}(Z_7^{((2,1),(1))}) = x_2x_3^3x_4 + 2x_2^3x_3x_4,$$
$$f_{(3)}(Z_8^{((2,1),(1))}) = x_2x_3x_4^3 + 2x_2^3x_3x_4.$$

The above polynomials $f(Z_1^{((2,1),(1))})$, $f(Z_2^{((2,1),(1))})$, …, $f(Z_8^{((2,1),(1))})$ mod 5 will be:

$$f_{(5)}(Z_1^{((2,1),(1))}) = x_1x_2^3x_3 + 4x_1^3x_2x_3,$$
$$f_{(5)}(Z_2^{((2,1),(1))}) = x_1x_2x_3^3 + 4x_1^3x_2x_3,$$
$$f_{(5)}(Z_3^{((2,1),(1))}) = x_1x_2^3x_4 + 4x_1^3x_2x_4,$$

$f_{(5)}(Z_4^{((2,1),(1))}) = x_1x_2x_4^3 + 4x_1^3x_2x_4$,
$f_{(5)}(Z_5^{((2,1),(1))}) = x_1x_3^3x_4 + 4x_1^3x_3x_4$,
$f_{(5)}(Z_6^{((2,1),(1))}) = x_1x_3x_4^3 + 4x_1^3x_3x_4$,
$f_{(5)}(Z_7^{((2,1),(1))}) = x_2x_3^3x_4 + 4x_2^3x_3x_4$,
$f_{(5)}(Z_8^{((2,1),(1))}) = x_2x_3x_4^3 + 4x_2^3x_3x_4$.

4. The Symmetrized Specht Polynomials of the Standard ((2,1),(1))- Tableaux

Definition 4.1. Let $Z_\ell^{(\lambda,\mu)}$ be any (λ,μ)-tableau. Then $\underline{R}(Z_\ell^{(\lambda,\mu)})$ will be defined as the set of all permutations *w* belong to the Weyl group W_n of type B_n, which permute the variables in each row of Z_ℓ^{λ} and in each row of Z_ℓ^{μ} without changing the sign of any variable in $Z_\ell^{(\lambda,\mu)}$, i.e., $\underline{R}(Z_\ell^{(\lambda,\mu)}) = \{w \in W_n \mid w\, Z_\ell^{(\lambda,\mu)}(i, j_1, 1) = Z_\ell^{(\lambda,\mu)}(i, j_2, 1)$, $i = 1, \ldots, s$ and $1 \le j_1, j_2 \le \lambda_i$; and $w\, Z_\ell^{(\lambda,\mu)}(i, j_1, 2) = Z_\ell^{(\lambda,\mu)}(i, j_2, 2)$, $i = 1, \ldots, t$ and $1 \le j_1, j_2 \le \mu_i\}$ ([5], p.211).

Definition 4.2. Let $Z_\ell^{(\lambda,\mu)}$ be any (λ,μ)-tableau, then the symmetrized Specht polynomial $f\left[Z_\ell^{(\lambda,\mu)}\right]$ will be defined by $f\left[Z_\ell^{(\lambda,\mu)}\right] = \sum_{w \in \underline{R}(Z_\ell^{(\lambda,\mu)})} f\left(w\, Z_\ell^{(\lambda,\mu)}\right)$.

If we take the coefficients of the polynomial $f\left[Z_\ell^{(\lambda,\mu)}\right]$ modulo a prime number *p*, then $f\left[Z_\ell^{(\lambda,\mu)}\right]$ will be denoted by $f_{(p)}\left[Z_\ell^{(\lambda,\mu)}\right]$, which will be called the *p*-reduced symmetrized Specht polynomial of the (λ,μ)-tableau $Z_\ell^{(\lambda,\mu)}$ ([5], p.211).

Remark 4.3. The F_pW_n-module generated by any *p*-reduced symmetrized Specht polynomial $f_{(p)}\left[Z_\ell^{(\lambda,\mu)}\right]$ will be denoted by $N_{F_p}(\lambda,\mu)$, i.e., $N_{F_p}(\lambda,\mu) = F_pW_nf_{(p)}\left[Z_\ell^{(\lambda,\mu)}\right]$. $N_{F_p}(\lambda,\mu)$ will be irreducible F_pW_n-submodule of the Specht module $S_{F_p}(\lambda,\mu)$, where F_p is a field of order *p* ([5], p.211).

The pair of partitions ((2,1),(1)) of 4 have the following symmetrized Specht polynomials of the standard ((2,1),(1))-tableaux:

$f\left[Z_1^{((2,1),(1))}\right] = f\left(Z_1^{((2,1),(1))}\right) + f\left((x_1\ x_3)Z_1^{((2,1),(1))}\right)$
$= \left(x_1x_2^3x_3 - x_1^3x_2x_3\right) + (x_1\ x_3)\left(x_1x_2^3x_3 - x_1^3x_2x_3\right)$

(since $f\left(w\, Z_\ell^{(\lambda,\mu)}\right) = w\, f\left(Z_\ell^{(\lambda,\mu)}\right)\ \forall\, Z_\ell^{(\lambda,\mu)} \in \mathrm{T}^{(\lambda,\mu)}$ and $\forall\, w \in W_n$ by [4], Remark 1.6.7, p.16)
$= x_1x_2^3x_3 - x_1^3x_2x_3 + x_1x_2^3x_3 - x_1x_2x_3^3$
$= 2x_1x_2^3x_3 - x_1^3x_2x_3 - x_1x_2x_3^3$,

$f\left[Z_2^{((2,1),(1))}\right] = f\left(Z_2^{((2,1),(1))}\right) + f\left((x_1\ x_2)Z_2^{((2,1),(1))}\right)$
$= \left(x_1x_2x_3^3 - x_1^3x_2x_3\right) + (x_1\ x_2)\left(x_1x_2x_3^3 - x_1^3x_2x_3\right)$
$= x_1x_2x_3^3 - x_1^3x_2x_3 + x_1x_2x_3^3 - x_1x_2^3x_3$
$= 2x_1x_2x_3^3 - x_1^3x_2x_3 - x_1x_2^3x_3$,

$f\left[Z_3^{((2,1),(1))}\right] = f\left(Z_3^{((2,1),(1))}\right) + f\left((x_1\ x_4)Z_3^{((2,1),(1))}\right)$
$= \left(x_1x_2^3x_4 - x_1^3x_2x_4\right) + (x_1\ x_4)\left(x_1x_2^3x_4 - x_1^3x_2x_4\right)$
$= x_1x_2^3x_4 - x_1^3x_2x_4 + x_1x_2^3x_4 - x_1x_2x_4^3$
$= 2x_1x_2^3x_4 - x_1^3x_2x_4 - x_1x_2x_4^3$,

$f\left[Z_4^{((2,1),(1))}\right] = f\left(Z_4^{((2,1),(1))}\right) + f\left((x_1\ x_2)Z_4^{((2,1),(1))}\right)$
$\left(x_1x_2x_4^3 - x_1^3x_2x_4\right) + (x_1\ x_2)\left(x_1x_2x_4^3 - x_1^3x_2x_4\right)$
$= x_1x_2x_4^3 - x_1^3x_2x_4 + x_1x_2x_4^3 - x_1x_2^3x_4$
$= 2x_1x_2x_4^3 - x_1^3x_2x_4 - x_1x_2^3x_4$,

$f\left[Z_5^{((2,1),(1))}\right] = f\left(Z_5^{((2,1),(1))}\right) + f\left((x_1\ x_4)Z_5^{((2,1),(1))}\right)$
$= \left(x_1x_3^3x_4 - x_1^3x_3x_4\right) + (x_1\ x_4)\left(x_1x_3^3x_4 - x_1^3x_3x_4\right)$
$= x_1x_3^3x_4 - x_1^3x_3x_4 + x_1x_3^3x_4 - x_1x_3x_4^3$
$= 2x_1x_3^3x_4 - x_1^3x_3x_4 - x_1x_3x_4^3$,

$f\left[Z_6^{((2,1),(1))}\right] = f\left(Z_6^{((2,1),(1))}\right) + f\left((x_1\ x_3)Z_6^{((2,1),(1))}\right)$
$= \left(x_1x_3x_4^3 - x_1^3x_3x_4\right) + (x_1\ x_3)\left(x_1x_3x_4^3 - x_1^3x_3x_4\right)$
$= x_1x_3x_4^3 - x_1^3x_3x_4 + x_1x_3x_4^3 - x_1x_3^3x_4$
$= 2x_1x_3x_4^3 - x_1^3x_3x_4 - x_1x_3^3x_4$,

$f\left[Z_7^{((2,1),(1))}\right] = f\left(Z_7^{((2,1),(1))}\right) + f\left((x_2\ x_4)Z_7^{((2,1),(1))}\right)$
$= \left(x_2x_3^3x_4 - x_2^3x_3x_4\right) + (x_2\ x_4)\left(x_2x_3^3x_4 - \right.$
$= x_2x_3^3x_4 - x_2^3x_3x_4 + x_2x_3^3x_4 - x_2x_3x_4^3$
$= 2x_2x_3^3x_4 - x_2^3x_3x_4 - x_2x_3x_4^3$,

$f\left[Z_8^{((2,1),(1))}\right] = f\left(Z_8^{((2,1),(1))}\right) + f\left((x_2\ x_3)Z_8^{((2,1),(1))}\right)$
$= \left(x_2x_3x_4^3 - x_2^3x_3x_4\right) + (x_2\ x_3)\left(x_2x_3x_4^3 - x_2^3x_3x_4\right)$
$= x_2x_3x_4^3 - x_2^3x_3x_4 + x_2x_3x_4^3 - x_2x_3^3x_4$
$= 2x_2x_3x_4^3 - x_2^3x_3x_4 - x_2x_3^3x_4$.

Then $f\left[Z_1^{((2,1),(1))}\right]$, $f\left[Z_2^{((2,1),(1))}\right]$, ..., $f\left[Z_8^{((2,1),(1))}\right]$ (mod 3) will be:

$f_{(3)}\left[Z_1^{((2,1),(1))}\right] = 2x_1^3x_2x_3 + 2x_1x_2^3x_3 + 2x_1x_2x_3^3$,
$f_{(3)}\left[Z_2^{((2,1),(1))}\right] = 2x_1^3x_2x_3 + 2x_1x_2^3x_3 + 2x_1x_2x_3^3$,
$f_{(3)}\left[Z_3^{((2,1),(1))}\right] = 2x_1^3x_2x_4 + 2x_1x_2^3x_4 + 2x_1x_2x_4^3$,
$f_{(3)}\left[Z_4^{((2,1),(1))}\right] = 2x_1^3x_2x_4 + 2x_1x_2^3x_4 + 2x_1x_2x_4^3$,

$f_{(3)}\left[Z_5^{((2,1),(1))}\right]=2x_1^3x_3x_4+2x_1x_3^3x_4+2x_1x_3x_4^3,$
$f_{(3)}\left[Z_6^{((2,1),(1))}\right]=2x_1^3x_3x_4+2x_1x_3^3x_4+2x_1x_3x_4^3,$
$f_{(3)}\left[Z_7^{((2,1),(1))}\right]=2x_2^3x_3x_4+2x_2x_3^3x_4+2x_2x_3x_4^3,$
$f_{(3)}\left[Z_8^{((2,1),(1))}\right]=2x_2^3x_3x_4+2x_2x_3^3x_4+2x_2x_3x_4^3.$

Let $b_1=f_{(3)}\left[Z_1^{((2,1),(1))}\right]=f_{(3)}\left[Z_2^{((2,1),(1))}\right]$,
$b_2=f_{(3)}\left[Z_3^{((2,1),(1))}\right]=f_{(3)}\left[Z_4^{((2,1),(1))}\right]$,
$b_3=f_{(3)}\left[Z_5^{((2,1),(1))}\right]=f_{(3)}\left[Z_6^{((2,1),(1))}\right]$,
$b_4=f_{(3)}\left[Z_7^{((2,1),(1))}\right]=f_{(3)}\left[Z_8^{((2,1),(1))}\right]$.

Then $B_{(3)}^{((2,1),(1))}=\{b_1,b_2,b_3,b_4\}$ is a basis of the submodule $N_{F_3}((2,1),(1))=F_3W_4\,f_{(3)}\left[Z_1^{((2,1),(1))}\right]$ of the Specht module $S_{F_3}((2,1),(1))=F_3W_4f_{(3)}\left(z_1^{((2,1),(1))}\right)$.

$f\left[Z_1^{((2,1),(1))}\right], \ldots, f\left[Z_8^{((2,1),(1))}\right]$ (mod 5) will be:

$f_{(5)}\left[Z_1^{((2,1),(1))}\right]=4x_1^3x_2x_3+2x_1x_2^3x_3+4x_1x_2x_3^3,$
$f_{(5)}\left[Z_2^{((2,1),(1))}\right]=4x_1^3x_2x_3+4x_1x_2^3x_3+2x_1x_2x_3^3,$
$f_{(5)}\left[Z_3^{((2,1),(1))}\right]=4x_1^3x_2x_4+2x_1x_2^3x_4+4x_1x_2x_4^3,$
$f_{(5)}\left[Z_4^{((2,1),(1))}\right]=4x_1^3x_2x_4+4x_1x_2^3x_4+2x_1x_2x_4^3,$
$f_{(5)}\left[Z_5^{((2,1),(1))}\right]=4x_1^3x_3x_4+2x_1x_3^3x_4+4x_1x_3x_4^3,$
$f_{(5)}\left[Z_6^{((2,1),(1))}\right]=4x_1^3x_3x_4+4x_1x_3^3x_4+2x_1x_3x_4^3,$
$f_{(5)}\left[Z_7^{((2,1),(1))}\right]=4x_2^3x_3x_4+2x_2x_3^3x_4+4x_2x_3x_4^3,$
$f_{(5)}\left[Z_8^{((2,1),(1))}\right]=4x_2^3x_3x_4+4x_2x_3^3x_4+2x_2x_3x_4^3.$

Let $b_i=f_{(5)}\left[Z_i^{((2,1),(1))}\right], i=1,\ldots,8,$ then $B_{(5)}^{((2,1),(1))}=\{b_1,b_2,\ldots,b_8\}$ is a basis of the submodule $N_{F_5}((2,1),(1))=F_5W_4f_{(5)}\left[Z_1^{((2,1),(1))}\right]$ of the Specht module $S_{F_5}((2,1),(1))=F_5W_4\,f_{(5)}\left(z_1^{((2,1),(1))}\right)$.

5. The Subspace $U_{(3)}^{((2,1),(1))}$ as a Linear Code

The 3-reduced symmetrized Specht polynomials $f_{(3)}\left[Z_1^{((2,1),(1))}\right],\ldots,f_{(3)}\left[Z_8^{((2,1),(1))}\right]$ give the following matrix:

$$\begin{bmatrix} 2&2&2&0&0&0&0&0&0&0&0&0\\ 2&2&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&2&2&2\\ 0&0&0&0&0&0&0&0&0&2&2&2 \end{bmatrix} \xrightarrow[\substack{R_2+2R_1\to R_5\\ R_4+2R_3\to R_6\\ R_6+2R_5\to R_7\\ R_8+2R_7\to R_8}]{\substack{R_1\to R_1\\ R_3\to R_2\\ R_5\to R_3\\ R_7\to R_4}}$$

$$\begin{bmatrix} 2&2&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&2&2&2\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The above matrix will give the following generator matrix $\chi_{(3)}^{((2,1),(1))}$ ([10], p.2 & [11], p.49) of the subspace $U_{(3)}^{((2,1),(1))}$ (which represents the submodule $N_{F_3}((2,1),(1))$) of the vector space F_3^{12} (which represents the module $M_{F_3}((2,1),(1))$):

$$\chi_{(3)}^{((2,1),(1))}=\begin{bmatrix} 2&2&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&2&2&2 \end{bmatrix}.$$

The rows of the above generator matrix $\chi_{(3)}^{((2,1),(1))}$ are representing the elements of the basis $B_{(3)}^{((2,1),(1))}=\{b_1,b_2,b_3,b_4\}$ of the submodule $N_{F_3}((2,1),(1))=F_3W_4\,f_{(3)}\left[Z_1^{((2,1),(1))}\right]$ of the module $M_{F_3}((2,1),(1))$, where:

$b_1=2x_1^3x_2x_3+2x_1x_2^3x_3+2x_1x_2x_3^3,$
$b_2=2x_1^3x_2x_4+2x_1x_2^3x_4+2x_1x_2x_4^3,$
$b_3=2x_1^3x_3x_4+2x_1x_3^3x_4+2x_1x_3x_4^3,$
$b_4=2x_2^3x_3x_4+2x_2x_3^3x_4+2x_2x_3x_4^3.$

Hence $k_3=\dim_{F_3}N_{F_3}((2,1),(1))=4,$ and the minimum distance $d_3=3.$

Therefore the four-dimensional subspace $U_{(3)}^{((2,1),(1))}$ (which represents the submodule $N_{F_3}((2,1),(1))$) of the vector space F_3^{12} (which represents the module $M_{F_3}((2,1),(1))$) can be considered as a linear (12, 4, 3, 3)-code ([7], p.16), where 12 means that each vector of this subspace has 12 coordinates, and 4 means that the dimension k_3 of this subspace $U_{(3)}^{((2,1),(1))}$ is 4, and 3 means that the minimum number of nonzero coordinates of any nonzero element of the subspace $U_{(3)}^{((2,1),(1))}$ is 3 (the minimum distance of this code $U_{(3)}^{((2,1),(1))}$ is 3 ([14], p.195)), and 3 means that this subspace $U_{(3)}^{((2,1),(1))}$ is over a field of order 3.

6. The Subspace $U_{(5)}^{((2,1),(1))}$ as a Linear Code

The 5-reduced symmetrized Specht polynomials $f_{(5)}\left[Z_1^{((2,1),(1))}\right],\ldots,f_{(5)}\left[Z_8^{((2,1),(1))}\right]$ give the following matrix:

$$\begin{bmatrix} 4&2&4&0&0&0&0&0&0&0&0&0\\ 4&4&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&4&2&4&0&0&0&0&0&0\\ 0&0&0&4&4&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&4&2&4&0&0&0\\ 0&0&0&0&0&0&4&4&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&4&2&4\\ 0&0&0&0&0&0&0&0&0&4&4&2 \end{bmatrix} \xrightarrow[\substack{R_2+4R_1\to R_2\\ R_4+4R_3\to R_4\\ R_6+4R_5\to R_6\\ R_8+4R_7\to R_8}]{\substack{R_1\to R_1\\ R_3\to R_3\\ R_5\to R_5\\ R_7\to R_7}}$$

$$\begin{bmatrix} 4&2&4&0&0&0&0&0&0&0&0&0\\ 0&2&3&0&0&0&0&0&0&0&0&0\\ 0&0&0&4&2&4&0&0&0&0&0&0\\ 0&0&0&0&2&3&0&0&0&0&0&0\\ 0&0&0&0&0&0&4&2&4&0&0&0\\ 0&0&0&0&0&0&0&2&3&0&0&0\\ 0&0&0&0&0&0&0&0&0&4&2&4\\ 0&0&0&0&0&0&0&0&0&0&2&3 \end{bmatrix} = \chi_{(5)}^{((2,1),(1))}$$

which is the generator matrix of the subspace $U_{(5)}^{((2,1),(1))}$ of the vector space F_5^{12}. The eight rows of the above generator matrix $\chi_{(5)}^{((2,1),(1))}$ are representing the elements of the basis $B_{(5)}^{((2,1),(1))}=\{b_1,b_2,\ldots,b_8\}$ of the submodule $N_{F_5}((2,1),(1))=F_5W_4f_{(5)}\left[Z_1^{((2,1),(1))}\right]$ of the module $M_{F_5}((2,1),(1))$, where:

$b_1=4x_1^3x_2x_3+2x_1x_2^3x_3+4x_1x_2x_3^3,$
$b_2=4x_1^3x_2x_3+4x_1x_2^3x_3+2x_1x_2x_3^3,$
$b_3=4x_1^3x_2x_4+2x_1x_2^3x_4+4x_1x_2x_4^3,$
$b_4=4x_1^3x_2x_4+4x_1x_2^3x_4+2x_1x_2x_4^3,$
$b_5=4x_1^3x_3x_4+2x_1x_3^3x_4+4x_1x_3x_4^3,$
$b_6=4x_1^3x_3x_4+4x_1x_3^3x_4+2x_1x_3x_4^3,$
$b_7=4x_2^3x_3x_4+2x_2x_3^3x_4+4x_2x_3x_4^3,$
$b_8=4x_2^3x_3x_4+4x_2x_3^3x_4+2x_2x_3x_4^3.$

Hence $k_5=\dim_{F_5}N_{F_5}((2,1),(1))=8$, and the minimum distance $d_5=2$.

Therefore the eight-dimensional subspace $U_{(5)}^{((2,1),(1))}$ (which represents the submodule $N_{F_5}((2,1),(1))$) of the vector space F_5^{12} (which represents the module $M_{F_5}((2,1),(1))$) can be considered as a linear (12,8,2,5)-code ([7], p.16), where 12 means that each vector of this subspace has 12 coordinates, and 8 means that the dimension k_5 of this subspace $U_{(5)}^{((2,1),(1))}$ is 8, and 2 means that the minimum number of nonzero coordinates of any nonzero element of the subspace $U_{(5)}^{((2,1),(1))}$ is 2 (the minimum distance of this code $U_{(5)}^{((2,1),(1))}$ is 2 ([14], p.195)), and 5 means that this subspace $U_{(5)}^{((2,1),(1))}$ is over a field of order 5.

7. The p-Modular Irreducible Representations for the Submodules $N_{F_p}(\lambda,\mu)$ of the Modules $M_{F_p}(\lambda,\mu)$ Corresponding to all Pairs of Partitions (λ,μ) of 4 as Linear Codes when p is a Prime Number and $p\geq 3$

The linear codes of the representations of the submodules $N_{F_p}(\lambda,\mu)$ of the modules $M_{F_p}(\lambda,\mu)$ corresponding to all pairs of partitions (λ,μ) of 4, when $p\geq 3$, are as follows:

1) For the pair of partitions ((4),( )), we have that $m(M)=\dim_{F_p}M_{F_p}((4),(\ ))=\frac{4!}{4!}=1$, and we have $\dim_{F_p}S_{F_p}((4),(\ ))=\frac{4!}{4\cdot 3\cdot 2\cdot 1}=1$, and thus we have only one standard ((4),( ))-tableau $Z_1^{((4),(\ ))}$ whose Specht polynomial is $f_{(p)}\left(Z_1^{((4),(\ ))}\right)=x_1x_2x_3x_4$.
   (i) If $p=3$, then $f_{(3)}\left[Z_1^{((4),(\ ))}\right]=4!\ (\mathrm{mod}\ 3)$ $f_{(3)}\left(Z_1^{((4),(\ ))}\right)=24\ (\mathrm{mod}\ 3)\cdot x_1x_2x_3x_4=0$ (since 24 (mod 3) = 0). Thus:
   $k_3=\dim_{F_3}N_{F_3}((4),(\ ))$
   $=\dim_{F_3}F_3W_4f_{(3)}\left[Z_1^{((4),(\ ))}\right]=0$,
   hence the minimum distance d_3 does not exist. Therefore, the subspace $U_{(3)}^{((4),(\ ))}$ (which represents the submodule $N_{F_3}((4),(\ ))$) of the vector space F_3 (which represents the module $M_{F_3}((4),(\ ))$) is a linear (1, 0, – , 3)-code.
   (ii) If $p\geq 5$, then $f_{(p)}\left[Z_1^{((4),(\ ))}\right]=4!\ (\mathrm{mod}\ p)$ $f_{(p)}\left(Z_1^{((4),(\ ))}\right)=24(\mathrm{mod}\,p)\cdot x_1x_2x_3x_4\neq 0$ (since $24\ (\mathrm{mod}\,p)\neq 0$). Thus:
   $k_p=\dim_{F_p}N_{F_p}((4),(\ ))$
   $=\dim_{F_p}F_pW_4f_{(p)}\left[Z_1^{((4),(\ ))}\right]=1,$
   since $N_{F_p}((4),(\ ))$ is a nontrivial submodule of the module $M_{F_p}((4),(\ ))$, hence the minimum distance $d_p=1$. Therefore, the subspace $U_{(p)}^{((4),(\ ))}$

(which represents the submodule $N_{F_p}((4),(\ ))$) of the vector space F_p (which represents the module $M_{F_p}((4),(\ ))$) is a linear $(1,1,1,p)$-code.

2) For the pair of partitions ((3,1), ( )), we have that $m(M)=\dim_{F_p} M_{F_p}((3,1),(\ ))=\frac{4!}{3!\cdot 1!}=4,$ and we have $\dim_{F_p} S_{F_p}((3,1),(\ ))=\frac{4!}{4\cdot 2\cdot 1\cdot 1}=3,$ and thus we have 3 standard $((3,1),(\ ))$-tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((3,1),(\ ))}\right)=x_1x_2x_3x_4^3+(p-1)\,x_1^3x_2x_3x_4,$

$f_{(p)}\left(Z_2^{((3,1),(\ ))}\right)=x_1x_2x_3^3x_4+(p-1)\,x_1^3x_2x_3x_4,$

$f_{(p)}\left(Z_3^{((3,1),(\ ))}\right)=x_1x_2^3x_3x_4+(p-1)\,x_1^3x_2x_3x_4.$ If $p\geq 3,$ then by theorem 2.9 (1), we have that $S_{F_p}((3,1),(\ ))$ is irreducible $F_p\,W_4$-module. Hence:

$$N_{F_p}((3,1),(\ ))=F_pW_4f_{(p)}\left[Z_1^{((3,1),(\ ))}\right]$$
$$=S_{F_p}((3,1),(\ )),$$

Since:

$$f_{(p)}\left[Z_1^{((3,1),(\ ))}\right]=6\,(\mathrm{mod}\,p)x_1x_2x_3x_4^3+(p-2)x_1x_2x_3^3x_4+(p-2)x_1x_2x_3^3x_4+(p-2)\cdot x_1x_2^3x_3x_4+(p-2)\,x_1^3x_2x_3x_4\neq 0.$$

Thus:

$k_p=\dim_{F_p} N_{F_p}((3,1),(\ ))=\dim_{F_p} S_{F_p}((3,1),(\ ))=3,$ and the minimum distance $d_p=2,$ since each Specht polynomial (which we give above) consists of 2 monomials.

Therefore for each $p\geq 3,$ the subspace $U_{(p)}^{((3,1),(\ ))}$ (which represents the submodule $N_{F_p}((3,1),(\ ))$) of the vector space F_p^4 (which represents the module $M_{F_p}((3,1),(\ ))$) is a linear $(4,3,2,p)$-code.

3) For the pair of partitions $((2,2),(\ ))$, we have that $m(M)=\dim_{F_p} M_{F_p}((2,2),(\ ))=\frac{4!}{2!\cdot 2!}=6,$ and we have $\dim_{F_p} S_{F_p}((2,2),(\ ))=\frac{4!}{3\cdot 2\cdot 2\cdot 1}=2,$ and thus we have 2 standard $((2,2),(\ ))$-tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((2,2),(\ ))}\right)=x_1x_2x_3^3x_4^3+(p-1)\cdot x_1x_2^3x_3^3x_4+(p-1)\,x_1^3x_2x_3x_4^3+x_1^3x_2^3x_3x_4,$$

$$f_{(p)}\left(Z_2^{((2,2),(\ ))}\right)=x_1x_2^3x_3x_4^3+(p-1)\cdot x_1x_2^3x_3^3x_4+(p-1)\,x_1^3x_2x_3x_4^3+x_1^3x_2x_3^3x_4.$$

(i) If $p=3,$ then:

$$f_{(3)}\left[Z_1^{((2,2),(\ ))}\right]=x_1^3x_2^3x_3x_4+x_1^3x_2x_3^3x_4+x_1^3x_2x_3x_4^3+x_1x_2^3x_3^3x_4+x_1x_2^3x_3x_4^3+x_1x_2x_3^3x_4^3,$$

$$f_{(3)}\left[Z_2^{((2,2),(\ ))}\right]=x_1^3x_2^3x_3x_4+x_1^3x_2x_3^3x_4+x_1^3x_2x_3x_4^3+x_1x_2^3x_3^3x_4+x_1x_2^3x_3x_4^3+x_1x_2x_3^3x_4^3.$$

The above polynomials modulo 3 give the following matrix:

$$\begin{bmatrix}1&1&1&1&1&1\\1&1&1&1&1&1\end{bmatrix}\xrightarrow[R_2+2R_1\to R_2]{R_1\to R_1}\begin{bmatrix}1&1&1&1&1&1\\0&0&0&0&0&0\end{bmatrix}.$$

The first row of the matrix above form a basis of the subspace $U_{(3)}^{((2,2),(\ ))}$. Hence $k_3=\dim_{F_3} N_{F_3}((2,2),(\ ))=1,$ and the minimum distance $d_3=6.$

Therefore, the subspace $U_{(3)}^{((2,2),(\ ))}$ (which represents the submodule $N_{F_3}((2,2),(\ ))$) of the vector space F_3^6 (which represents the module $M_{F_3}((2,2),(\ ))$) is a linear $(6,1,6,3)$-code.

(ii) If $p\geq 5,$ then by theorem 2.8 (1), we have that $S_{F_p}((2,2),(\ ))$ is irreducible F_pW_4-module.

Hence $N_{F_p}((2,2),(\ ))=F_p\,W_4\,f_{(p)}\left[Z_1^{((2,2),(\ ))}\right]=S_{F_p}((2,2),(\ )),$ since:

$$f_{(p)}\left[Z_1^{((2,2),(\ ))}\right]=4x_1^3x_2^3x_3x_4+(p-2)x_1^3x_2x_3^3x_4+(p-2)x_1^3x_2x_3x_4^3+(p-2)x_1x_2^3x_3^3x_4+(p-2)\cdot x_1x_2^3x_3x_4^3+4x_1x_2x_3^3x_4^3\neq 0.$$

Thus:

$$k_p=\dim_{F_p} N_{F_p}((2,2),(\ ))$$
$$=\dim_{F_p} S_{F_p}((2,2),(\ ))=2,$$

and $d_p=4,$ since each Specht polynomial (which we give above) consists of 4 monomials.

Therefore for each $p\geq 5,$ the subspace $U_{(p)}^{((2,2),(\ ))}$ (which represents the submodule $N_{F_p}((2,2),(\ ))$) of the vector space F_p^6 (which represents the module $M_{F_p}((2,2),(\ ))$) is a linear $(6,2,4,p)$-code.

4) For the pair of partitions $((2,1,1),(\ )),$ we have that $m(M)=\dim_{F_p} M_{F_p}((2,1,1),(\ ))=\frac{4!}{2!\cdot 1!\cdot 1!}=12,$ and we have $\dim_{F_p} S_{F_p}((2,1,1),(\ ))=\frac{4!}{4\cdot 1\cdot 2\cdot 1}=3,$ and thus we have 3 standard $((2,1,1),(\ ))$-tableaux whose Specht polynomials are:

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$$f_{(p)}\left(Z_1^{((2,1,1),(\ ))}\right)=x_1x_2x_3^3x_4^5+(p-1)\cdot x_1^3x_2x_3x_4^5+(p-1)x_1^5x_2x_3^3x_4+(p-1)\cdot x_1x_2x_3^5x_4^3+x_1^5x_2x_3x_4^3+x_1^3x_2x_3^5x_4,$$

$$f_{(p)}\left(Z_2^{((2,1,1),(\ ))}\right)=x_1x_2^3x_3x_4^5+(p-1)\cdot x_1^3x_2x_3x_4^5+(p-1)x_1^5x_2^3x_3x_4+(p-1)\cdot x_1x_2^5x_3x_4^3+x_1^5x_2x_3x_4^3+x_1^3x_2^5x_3x_4,$$

$$f_{(p)}\left(Z_3^{((2,1,1),(\ ))}\right)=x_1x_2^3x_3^5x_4+(p-1)\cdot x_1^3x_2x_3^5x_4+(p-1)x_1^5x_2^3x_3x_4+(p-1)\cdot x_1x_2^5x_3^3x_4+x_1^5x_2x_3^3x_4+x_1^3x_2^5x_3x_4.$$

If $p\geq 3$, then by Theorem 2.9 (1), we have that $S_{F_p}((2,1,1),(\ ))$ is irreducible F_pW_4-module. Hence:

$$N_{F_p}((2,1,1),(\ ))=F_pW_4f_{(p)}\left[Z_1^{((2,1,1),(\ ))}\right]=S_{F_p}((2,1,1),(\ )),$$

Since:

$$f_{(p)}\left[Z_1^{((2,1,1),(\ ))}\right]=(p-1)x_1^5x_2x_3^3x_4+(p-1)x_1x_2^5x_3^3x_4+x_1^3x_2x_3^5x_4+x_1x_2^3x_3^5x_4+x_1^5x_2x_3x_4^3+x_1x_2^5x_3x_4^3+(p-2)x_1x_2x_3^5x_4^3+(p-1)x_1^3x_2x_3x_4^5+(p-1)x_1x_2^3x_3x_4^5+2x_1x_2x_3^3x_4^5\neq 0.$$

Thus:

$$k_p=\dim_{F_p}N_{F_p}((2,1,1),(\ ))=\dim_{F_p}S_{F_p}((2,1,1),(\ ))=3,$$

and the minimum distance $d_p=6$, since each Specht polynomial (which we give above) consists of 6 monomials.

Therefore for each $p\geq 3$, the subspace $U_{(p)}^{((2,1,1),(\ ))}$ (which represents the submodule $N_{F_p}((2,1,1),(\ ))$) of the vector space F_p^{12} (which represents the module $M_{F_p}((2,1,1),(\ ))$) is a linear $(12, 3, 6, p)$-code.

5) For the pair of partitions $((1,1,1,1),(\ ))$, we have that $m(M)=\dim_{F_p}M_{F_p}((1,1,1,1),(\ ))=\dfrac{4!}{1!\cdot 1!\cdot 1!\cdot 1!}=24,$ and we have $\dim_{F_p}S_{F_p}((1,1,1,1),(\ ))=\dfrac{4!}{4\cdot 3\cdot 2\cdot 1}=1.$

Thus we have only one standard $((1,1,1,1),(\ ))$-tableau whose Specht polynomial is:

$$f_{(p)}\left(Z_1^{((1,1,1,1),(\ ))}\right)=x_1x_2^3x_3^5x_4^7+(p-1)\cdot x_1^3x_2x_3^5x_4^7+(p-1)x_1^5x_2^3x_3x_4^7+(p-1)\cdot x_1^7x_2^3x_3^5x_4+(p-1)x_1x_2^5x_3^3x_4^7+(p-1)\cdot x_1x_2^7x_3^5x_4^3+(p-1)x_1x_2^3x_3^7x_4^5+x_1^5x_2x_3^3x_4^7+x_1^3x_2^5x_3x_4^7+x_1^7x_2x_3^5x_4^3+x_1^3x_2^7x_3^5x_4+x_1^7x_2^3x_3x_4^5+x_1^5x_2^3x_3^7x_4+x_1x_2^7x_3^3x_4^5+x_1x_2^5x_3^7x_4^3+x_1^3x_2x_3^7x_4^5+x_1^5x_2^7x_3x_4^3+x_1^7x_2^5x_3^3x_4+(p-1)x_1^7x_2x_3^3x_4^5+(p-1)\cdot x_1^5x_2x_3^7x_4^3+(p-1)x_1^7x_2^5x_3x_4^3+(p-1)\cdot x_1^3x_2^7x_3x_4^5+(p-1)x_1^5x_2^7x_3^3x_4+(p-1)x_1^3x_2^5x_3^7x_4,$$

and $f_{(p)}\left[Z_1^{((1,1,1,1),(\ ))}\right]=f_{(p)}\left(i\ Z_1^{((1,1,1,1),(\ ))}\right)=f_{(p)}\left(Z_1^{((1,1,1,1),(\ ))}\right)$, for each $p\geq 3$. Hence:

$N_{F_p}((1,1,1,1),(\ ))=S_{F_p}((1,1,1,1),(\ ))$.

Thus:

$$k_p=\dim_{F_p}N_{F_p}((1,1,1,1),(\ ))=\dim_{F_p}S_{F_p}((1,1,1,1),(\ ))=1,$$

and $d_p=24$, since the Specht polynomial $f_{(p)}\left(Z_1^{((1,1,1,1),(\ ))}\right)$ has 24 monomials.

Therefore for each $p\geq 3$, the subspace $U_{(p)}^{((1,1,1,1),(\ ))}$ (which represents the submodule $N_{F_p}((1,1,1,1),(\ ))$) of the vector space F_p^{24} (which represents the module $M_{F_p}((1,1,1,1),(\ ))$) is a linear $(24, 1, 24, p)$- code.

6) For the pair of partitions $((3),(1))$, we have that $m(M)=\dim_{F_p}M_{F_p}((3),(1))=\dfrac{4!}{3!\cdot 1!}=4,$ and we have $\dim_{F_p}S_{F_p}((3),(1))=\dfrac{4!}{3\cdot 2\cdot 1\cdot 1}=4$.

(i) If $p=3$, then $f_{(3)}\left[Z_1^{((3),(1))}\right]=3!\ (\text{mod } 3)\ f_{(3)}\left(Z_1^{((3),(1))}\right)=6\ (\text{mod } 3)\cdot x_1x_2x_3=0$ (since $6\ (\text{mod } 3)=0$), where $Z_1^{((3),(1))}$ is a standard $((3),(1))$-tableau. Thus:

$$k_3=\dim_{F_3}N_{F_3}((3),(1))=\dim_{F_3}F_3W_4\ f_{(3)}\left[Z_1^{((3),(1))}\right]=0$$

Hence the minimum distance d_3 does not exist.

Therefore, the subspace $U_{(3)}^{((3),(1))}$ (which represents the submodule $N_{F_3}((3),(1))$) of the vector space F_3^4 (which represents the module $M_{F_3}((3),(1))$) is a linear (4, 0, – , 3)-code.

(ii) If $p\geq 5$, then by theorem 2.9 (1), we have that $S_{F_p}((3),(1))$ is irreducible F_pW_4-module. Hence $N_{F_p}((3),(1))=F_p\,W_4\ f_{(p)}\left[Z_1^{((3),(1))}\right]=S_{F_p}((3),(1))$, since $f_{(p)}\left[Z_1^{((3),(1))}\right]=3!\ (\text{mod}\,p)\cdot f_{(p)}\left(Z_1^{((3),(1))}\right)=6\ (\text{mod}\,p)\,x_1x_2x_3\neq 0$ (since $6\ (\text{mod}\,p)\neq 0$). Thus:

$k_p = \dim_{F_p} N_{F_p}((3),(1))$
$= \dim_{F_p} S_{F_p}((3),(1)) = 4,$
and the minimum distance $d_p = 1,$ since the Specht polynomial $f_{(p)}\left(Z_1^{((3),(1))}\right) = x_1x_2x_3$ (which has only one monomial).
Therefore for $p \geq 5,$ the subspace $U_{(p)}^{((3),(1))}$ (which represents the submodule $N_{F_p}((3),(1))$) of the vector space F_p^4 (which represents the module $M_{F_p}((3),(1))$) is a linear $(4, 4, 1, p)$- code.

7) For the pair of partitions $((2,1),(1))$, we have that $m(M) = \dim_{F_p} M_{F_p}((2,1),(1)) = \frac{4!}{2!\cdot 1!\cdot 1!} = 12,$ and we have $\dim_{F_p} S_{F_p}((2,1),(1)) = \frac{4!}{3\cdot 1\cdot 1\cdot 1} = 8.$
   (i) If $p = 3,$ then the subspace $U_{(3)}^{((2,1),(1))}$, which represents the submodule $N_{F_3}((2,1),(1))$) of the vector space F_3^{12} (which represents the module $M_{F_3}((2,1),(1))$) is a linear $(12, 4, 3, 3)$- code (see (5) of this paper for the full details).
   (ii) If $p \geq 5,$ then by theorem 2.9 (1), we have that $S_{F_p}((2,1),(1))$ is irreducible F_pW_4-module. Hence $N_{F_p}((2,1),(1)) = F_pW_4 f_{(p)}\left[Z_1^{((2,1),(1))}\right] = S_{F_p}((2,1),(1)),$ since $f_{(p)}\left[Z_1^{((2,1),(1))}\right] = (p-1)x_1^3x_2x_3 + (p-1)x_1x_2^3x_3 + 2x_1x_2x_3^3 \neq 0.$
   Thus:
   $k_p = \dim_{F_p} N_{F_p}((2,1),(1))$
   $= \dim_{F_p} S_{F_p}((2,1),(1)) = 8,$
   and the minimum distance $d_p = 2,$ since the Specht polynomial:
   $f_{(p)}\left(Z_1^{((2,1),(1))}\right) = x_1x_2x_3^3 + (p-1)x_1^3x_2x_3$
   (which consists of 2 monomials).
   Therefore for each $p \geq 5,$ the subspace $U_{(p)}^{((2,1),(1))}$ (which represents the submodule $N_{F_p}((2,1),(1))$) of the vector space F_p^{12} (which represents the module $M_{F_p}((2,1),(1))$) is a linear $(12, 8, 2, p)$-code.

For the pair of partitions $((1,1,1),(1))$, we have that:
$m(M) = \dim_{F_p} M_{F_p}((1,1,1),(1))$
$= \frac{4!}{1!\cdot 1!\cdot 1!\cdot 1!} = 24,$
and we have $\dim_{F_p} S_{F_p}((1,1,1),(1)) = \frac{4!}{3\cdot 2\cdot 1\cdot 1} = 4.$
Thus we have 4 standard $((1,1,1),(1))$- tableaux whose Specht polynomials are:
$f_{(p)}\left(Z_1^{((1,1,1),(1))}\right) = x_1x_2^3x_3^5 + (p-1)x_1^3x_2x_3^5 + (p-1)x_1^5x_2^3x_3 + (p-1)x_1x_2^5x_3^3 + x_1^5x_2x_3^3 + x_1^3x_2^5x_3,$
$f_{(p)}\left(Z_2^{((1,1,1),(1))}\right) = x_1x_2^3x_4^5 + (p-1)x_1^3x_2x_4^5 + (p-1)x_1^5x_2^3x_4 + (p-1)x_1x_2^5x_4^3 + x_1^5x_2x_4^3 + x_1^3x_2^5x_4,$
$f_{(p)}\left(Z_3^{((1,1,1),(1))}\right) = x_1x_3^3x_4^5 + (p-1)x_1^3x_3x_4^5 + (p-1)x_1^5x_3^3x_4 + (p-1)x_1x_3^5x_4^3 + x_1^5x_3x_4^3 + x_1^3x_3^5x_4,$
$f_{(p)}\left(Z_4^{((1,1,1),(1))}\right) = x_2x_3^3x_4^5 + (p-1)x_2^3x_3x_4^5 + (p-1)x_2^5x_3^3x_4 + (p-1)x_2x_3^5x_4^3 + x_2^5x_3x_4^3 + x_2^3x_3^5x_4$
and $f_{(p)}\left[Z_1^{((1,1,1),(1))}\right] = f_{(p)}\left(i\,Z_1^{((1,1,1),(1))}\right) = f_{(p)}\left(Z_1^{((1,1,1),(1))}\right)$, for each $p \geq 3.$
Hence $N_{F_p}((1,1,1),(1)) = F_pW_4 f_{(p)}\left[Z_1^{((1,1,1),(1))}\right] = F_pW_4 f_{(p)}\left(Z_1^{((1,1,1),(1))}\right) = S_{F_p}((1,1,1),(1)),$ for each $p \geq 3.$
Thus:
$k_p = \dim_{F_p} N_{F_p}((1,1,1),(1))$
$= \dim_{F_p} S_{F_p}((1,1,1),(1)) = 4,$
and the minimum distance $d_p = 6,$ since each Specht polynomial (which we give above) consists of 6 monomials.
Therefore for each $p \geq 3,$ the subspace $U_{(p)}^{((1,1,1),(1))}$ (which represents the submodule $N_{F_p}((1,1,1),(1))$) of the vector space F_p^{24} (which represents the module $M_{F_p}((1,1,1),(1))$) is a linear $(24, 4, 6, p)$- code.

8) For the pair of partitions $((2),(2))$, we have that $m(M) = \dim_{F_p} M_{F_p}((2),(2)) = \frac{4!}{2!\cdot 2!} = 6,$ and we have $\dim_{F_p} S_{F_p}((2),(2)) = \frac{4!}{2\cdot 1\cdot 2\cdot 1} = 6.$ Thus we have 6 standard $((2),(2))$-tableaux whose Specht polynomials are:
$f_{(p)}\left(Z_1^{((2),(2))}\right) = x_1x_2,\ f_{(p)}\left(Z_2^{((2),(2))}\right) = x_1x_3,$
$f_{(p)}\left(Z_3^{((2),(2))}\right) = x_1x_4,\ f_{(p)}\left(Z_4^{((2),(2))}\right) = x_2x_3,$
$f_{(p)}\left(Z_5^{((2),(2))}\right) = x_2x_4,\ f_{(p)}\left(Z_6^{((2),(2))}\right) = x_3x_4,$ and
$f_{(p)}\left[Z_1^{((2),(2))}\right] = 4\ (\text{mod}\, p)\ x_1x_2 = 4\ (\text{mod}\, p) f_{(p)}\left(Z_1^{((2),(2))}\right),$ for each $p \geq 3.$

Hence:

$N_{F_p}((2),(2)) = F_p W_4 f_{(p)}\left[Z_1^{((2),(2))}\right]$
$= F_p W_4 f_{(p)}\left(Z_1^{((2),(2))}\right)$
$= S_{F_p}((2),(2))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((2),(2)) = \dim_{F_p} S_{F_p}((2),(2)) = 6$, and the minimum distance $d_p = 1$, since each Specht polynomial (which we give above) consists of only one monomial.

Therefore for each $p \geq 3$, the subspace $U_{(p)}^{((2),(2))}$ (which represents the submodule $N_{F_p}((2),(2))$) of the vector space F_p^6 (which represents the module $M_{F_p}((2),(2))$) is a linear $(6, 6, 1, p)$- code.

9) For the pair of partitions $((1,1),(2))$, we have that $m(M) = \dim_{F_p} M_{F_p}((1,1),(2)) = \frac{4!}{1!\cdot 1!\cdot 2!} = 12$, and we have $\dim_{F_p} S_{F_p}((1,1),(2)) = \frac{4!}{2\cdot 1\cdot 2\cdot 1} = 6$. Thus we have 6 standard $((1,1),(2))$- tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((1,1),(2))}\right) = x_1 x_2^3 + (p-1)\, x_1^3 x_2$,
$f_{(p)}\left(Z_2^{((1,1),(2))}\right) = x_1 x_3^3 + (p-1)\, x_1^3 x_3$,
$f_{(p)}\left(Z_3^{((1,1),(2))}\right) = x_1 x_4^3 + (p-1)\, x_1^3 x_4$,
$f_{(p)}\left(Z_4^{((1,1),(2))}\right) = x_2 x_3^3 + (p-1)\, x_2^3 x_3$,
$f_{(p)}\left(Z_5^{((1,1),(2))}\right) = x_2 x_4^3 + (p-1)\, x_2^3 x_4$,
$f_{(p)}\left(Z_6^{((1,1),(2))}\right) = x_3 x_4^3 + (p-1)\, x_3^3 x_4$,

and

$f_{(p)}\left[Z_1^{((1,1),(2))}\right] = 2\, x_1 x_2^3 + (p-2)\, x_1^3 x_2$
$= 2\, f_{(p)}\left(Z_1^{((1,1),(2))}\right)$, for each $p \geq 3$.

Hence:

$N_{F_p}((1,1),(2)) = F_p W_4 f_{(p)}\left[Z_1^{((1,1),(2))}\right]$
$= F_p W_4 f_{(p)}\left(Z_1^{((1,1),(2))}\right)$
$= S_{F_p}((1,1),(2))$, for each $p \geq 3$.

Thus:

$k_p = \dim_{F_p} N_{F_p}((1,1),(2)) = \dim_{F_p} S_{F_p}((1,1),(2)) = 6$,

and the minimum distance $d_p = 2$, since each Specht polynomial (which we give above) consists of 2 monomials.

Therefore for each $p \geq 3$, the subspace $U_{(p)}^{((1,1),(2))}$ (which represents the submodule $N_{F_p}((1,1),(2))$) of the vector space F_p^{12} (which represents the module $M_{F_p}((1,1),(2))$) is a linear $(12, 6, 2, p)$- code.

10) For the pair of partitions $((2),(1,1))$, we have that $m(M) = \dim_{F_p} M_{F_p}((2),(1,1)) = \frac{4!}{2!\cdot 1!\cdot 1!} = 12$, and we have $\dim_{F_p} S_{F_p}((2),(1,1)) = \frac{4!}{2\cdot 1\cdot 2\cdot 1} = 6$. Thus we have 6 standard $((2),(1,1))$- tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((2),(1,1))}\right) = x_1 x_2 x_4^2 + (p-1)\, x_1 x_2 x_3^2$,
$f_{(p)}\left(Z_2^{((2),(1,1))}\right) = x_1 x_3 x_4^2 + (p-1)\, x_1 x_2^2 x_3$,
$f_{(p)}\left(Z_3^{((2),(1,1))}\right) = x_1 x_3^2 x_4 + (p-1)\, x_1 x_2^2 x_4$,
$f_{(p)}\left(Z_4^{((2),(1,1))}\right) = x_2 x_3 x_4^2 + (p-1)\, x_1^2 x_2 x_3$,
$f_{(p)}\left(Z_5^{((2),(1,1))}\right) = x_2 x_3^2 x_4 + (p-1)\, x_1^2 x_2 x_4$,
$f_{(p)}\left(Z_6^{((2),(1,1))}\right) = x_2^2 x_3 x_4 + (p-1)\, x_1^2 x_3 x_4$,

and

$f_{(p)}\left[Z_1^{((2),(1,1))}\right] = 2\, x_1 x_2 x_4^2 + (p-2)\, x_1 x_2 x_3^2$
$= 2\, f_{(p)}\left(Z_1^{((2),(1,1))}\right)$, for each $p \geq 3$.

Hence $N_{F_p}((2),(1,1)) = F_p W_4 f_{(p)}\left[Z_1^{((2),(1,1))}\right] = F_p W_4 f_{(p)}\left(Z_1^{((2),(1,1))}\right) = S_{F_p}((2),(1,1))$, for each $p \geq 3$.

Thus:

$k_p = \dim_{F_p} N_{F_p}((2),(1,1))$
$= \dim_{F_p} S_{F_p}((2),(1,1)) = 6$,

and the minimum distance $d_p = 2$, since each Specht polynomial (which we give above) consists of 2 monomials.

Therefore for each $p \geq 3$, the subspace $U_{(p)}^{((2),(1,1))}$ (which represents the submodule $N_{F_p}((2),(1,1))$) of the vector space F_p^{12} (which represents the module $M_{F_p}((2),(1,1))$) is a linear $(12, 6, 2, p)$- code.

11) For the pair of partitions $((1,1),(1,1))$, we have that $m(M) = \dim_{F_p} M_{F_p}((1,1),(1,1)) = \frac{4!}{1!\cdot 1!\cdot 1!\cdot 1!} = 24$, and we have $\dim_{F_p} S_{F_p}((1,1),(1,1)) = \frac{4!}{2\cdot 1\cdot 2\cdot 1} = 6$. Thus we have 6 standard $((1,1),(1,1))$- tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((1,1),(1,1))}\right) = x_1 x_2^3 x_4^2 + (p-1) x_1 x_2^3 x_3^2 +$
$(p-1)\, x_1^3 x_2 x_4^2 + x_1^3 x_2 x_3^2$,
$f_{(p)}\left(Z_2^{((1,1),(1,1))}\right) = x_1 x_3^3 x_4^2 + (p-1) x_1 x_2^2 x_3^3 +$
$(p-1)\, x_1^3 x_3 x_4^2 + x_1^3 x_2^2 x_3$,
$f_{(p)}\left(Z_3^{((1,1),(1,1))}\right) = x_1 x_3^2 x_4^3 + (p-1) x_1 x_2^2 x_4^3 +$
$(p-1)\, x_1^3 x_3^2 x_4 + x_1^3 x_2^2 x_4$,

$f_{(p)}\left(Z_4^{((1,1),(1,1))}\right)=x_2x_3^3x_4^2+(p-1)x_1^2x_2x_3^3+(p-1)x_2^3x_3x_4^2+x_1^2x_2^3x_3,$

$f_{(p)}\left(Z_5^{((1,1),(1,1))}\right)=x_2x_3^2x_4^3+(p-1)x_1^2x_2x_4^3+(p-1)x_2^3x_3^2x_4+x_1^2x_2^3x_4,$

$f_{(p)}\left(Z_6^{((1,1),(1,1))}\right)=x_2^2x_3x_4^3+(p-1)x_1^2x_3x_4^3+(p-1)x_2^2x_3^3x_4+x_1^2x_3^3x_4,$

and

$f_{(p)}\left[Z_1^{((1,1),(1,1))}\right]=f_{(p)}\left(i\,Z_1^{((1,1),(1,1))}\right)=f_{(p)}\left(Z_1^{((1,1),(1,1))}\right),$

for each $p\ge 3.$ Hence:

$$N_{F_p}((1,1),(1,1))=F_pW_4\,f_{(p)}\left[Z_1^{((1,1),(1,1))}\right]=F_pW_4\,f_{(p)}\left(Z_1^{((1,1),(1,1))}\right)=S_{F_p}((1,1),(1,1)),\text{ for each } p\ge 3.$$

Thus

$$k_p=\dim_{F_p}N_{F_p}((1,1),(1,1))=\dim_{F_p}S_{F_p}((1,1),(1,1))=6,$$

and the minimum distance $d_p=4$, since each Specht polynomial (which we give above) consists of 4 monomials.

Therefore for each $p\ge 3,$ the subspace $U_{(p)}^{((1,1),(1,1))}$ (which represents the submodule $N_{F_p}((1,1),(1,1))$) of the vector space F_p^{24} (which represents the module $M_{F_p}((1,1),(1,1))$) is a linear $(24,6,4,p)$- code.

12) For the pair of partitions $((1),(3)),$ we have that $m(M)=\dim_{F_p}M_{F_p}((1),(3))=\frac{4!}{1!\cdot 3!}=4$, and we have $\dim_{F_p}S_{F_p}((1),(3))=\frac{4!}{1\cdot 3\cdot 2\cdot 1}=4.$

(i) If $p=3,$ then:

$$f_{(3)}\left[Z_1^{((1),(3))}\right]=3!\,(\mathrm{mod}\ 3)f_{(3)}\left(Z_1^{((1),(3))}\right)=6\,(\mathrm{mod}\ 3)\cdot x_1=0$$

(since $6\,(\mathrm{mod}\ 3)=0$), where $Z_1^{((1),(3))}$ is a standard $((1),(3))$-tableau.

Thus:

$$k_3=\dim_{F_3}N_{F_3}((1),(3))=\dim_{F_3}F_3\,W_4\;f_{(3)}\left[Z_1^{((1),(3))}\right]=0,$$

hence the minimum distance d_3 does not exist. Therefore, the subspace $U_{(3)}^{((1),(3))}$ (which represents the submodule $N_{F_3}((1),(3))$) of the vector space F_3^4 (which represents the module $M_{F_3}((1),(3))$) is a linear (4,0,–,3)-code.

(ii) If $p\ge 5,$ then by theorem 2.9 (1), we have that $S_{F_p}((1),(3))$ is irreducible F_pW_4-module. Hence:

$$N_{F_p}((1),(3))=F_pW_4\,f_{(p)}\left[Z_1^{((1),(3))}\right]=S_{F_p}((1),(3)),$$

since $f_{(p)}\left[Z_1^{((1),(3))}\right]=3!\,(\mathrm{mod}\,p)\cdot f_{(p)}\left(Z_1^{((1),(3))}\right)=6\,(\mathrm{mod}\,p)\;x_1\ne 0$ (since $6\,(\mathrm{mod}\,p)\ne 0$).

Thus:

$$k_p=\dim_{F_p}N_{F_p}((1),(3))=\dim_{F_p}S_{F_p}((1),(3))=4,$$

and the minimum distance $d_p=1,$ since the Specht polynomial $f_{(p)}\left(Z_1^{((1),(3))}\right)$ consists of only one monomial.

Therefore for $p\ge 5,$ the subspace $U_{(p)}^{((1),(3))}$ (which represents the submodule $N_{F_p}((1),(3))$) of the vector space F_p^4 (which represents the module $M_{F_p}((1),(3))$) is a linear $(4,4,1,p)$-code.

13) For the pair of partitions $((1),(2,1))$, we have that $m(M)=\dim_{F_p}M_{F_p}((1),(2,1))=\frac{4!}{1!\cdot 2!\cdot 1!}=12$, and we have $\dim_{F_p}S_{F_p}((1),(2,1))=\frac{4!}{1\cdot 3\cdot 1\cdot 1}=8.$

(i) If $p=3,$ then:

$f_{(3)}\left(Z_1^{((1),(2,1))}\right)=x_1x_4^2+2x_1x_2^2,$

$f_{(3)}\left(Z_2^{((1),(2,1))}\right)=x_1x_3^2+2x_1x_2^2,$

$f_{(3)}\left(Z_3^{((1),(2,1))}\right)=x_2x_4^2+2x_1x_2^2,$

$f_{(3)}\left(Z_4^{((1),(2,1))}\right)=x_2x_3^2+2x_1^2x_2,$

$f_{(3)}\left(Z_5^{((1),(2,1))}\right)=x_3x_4^2+2x_1^2x_3,$

$f_{(3)}\left(Z_6^{((1),(2,1))}\right)=x_2^2x_3+2x_1^2x_3,$

$f_{(3)}\left(Z_7^{((1),(2,1))}\right)=x_3^2x_4+2x_1^2x_4,$

$f_{(3)}\left(Z_8^{((1),(2,1))}\right)=x_2^2x_4+2x_1^2x_4,$

and the 3-reduced symmetrized Specht polynomials are:

$f_{(3)}\left[Z_1^{((1),(2,1))}\right]=2x_1x_2^2+2x_1x_3^2+2x_1x_4^2,$

$f_{(3)}\left[Z_2^{((1),(2,1))}\right]=2x_1x_2^2+2x_1x_3^2+2x_1x_4^2,$

$f_{(3)}\left[Z_3^{((1),(2,1))}\right]=2x_1^2x_2+2x_2x_3^2+2x_2x_4^2,$

$f_{(3)}\left[Z_4^{((1),(2,1))}\right]=2x_1^2x_2+2x_2x_3^2+2x_2x_4^2,$

$f_{(3)}\left[Z_5^{((1),(2,1))}\right]=2x_1^2x_3+2x_2^2x_3+2x_3x_4^2,$

$f_{(3)}\left[Z_6^{((1),(2,1))}\right]=2x_1^2x_3+2x_2^2x_3+2x_3x_4^2,$

$f_{(3)}\left[Z_7^{((1),(2,1))}\right]=2x_1^2x_4+2x_2^2x_4+2x_3^2x_4,$

$f_{(3)}\left[Z_8^{((1),(2,1))}\right]=2x_1^2x_4+2x_2^2x_4+2x_3^2x_4.$

The above polynomials $f_{(3)}\left[Z_1^{((1),(2,1))}\right],\ldots,$ $f_{(3)}\left[Z_8^{((1),(2,1))}\right]$ give the following matrix:

$$\begin{bmatrix} 2&2&2&0&0&0&0&0&0&0&0&0\\ 2&2&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&2&2&2\\ 0&0&0&0&0&0&0&0&0&2&2&2 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2+2R_1\to R_5\\ R_4+2R_3\to R_6\\ R_6+2R_5\to R_7\\ R_8+2R_7\to R_8}]{\substack{R_1\to R_1\\ R_3\to R_2\\ R_5\to R_3\\ R_7\to R_4}}$$

$$\begin{bmatrix} 2&2&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&2&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&2&2&2\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The first 4 rows of the matrix above form a basis of the subspace $U_{(3)}^{((1),(2,1))}$ (which represents the submodule $N_{F_3}((1),(2,1))$).

Hence $k_3=\dim_{F_3} N_{F_3}((1),(2,1))=4,$ and the minimum distance $d_3=3.$

Therefore, the subspace $U_{(3)}^{((1),(2,1))}$ which represents the submodule $N_{F_3}((1),(2,1))$ of the vector space F_3^{12} (which represents the module $M_{F_3}((1),(2,1))$) is a linear $(12, 4, 3, 3)$- code.

(ii) If $p\geq 5,$ then by theorem 2.9 (1), we have that $S_{F_p}((1),(2,1))$ is irreducible F_pW_4-module. Hence:

$$N_{F_p}((1),(2,1))=F_p\,W_4\,f_{(p)}\left[Z_1^{((1),(2,1))}\right]$$
$$=S_{F_p}((1),(2,1)),$$

Since $f_{(p)}\left[Z_1^{((1),(2,1))}\right]=(p-1)x_1x_2^2+(p-1)x_1x_3^2$ $+2x_1x_4^2\neq 0.$ Thus:

$$k_p=\dim_{F_p} N_{F_p}((1),(2,1))$$
$$=\dim_{F_p} S_{F_p}((1),(2,1))=8,$$

and the minimum distance $d_p=2,$ since the Specht polynomial $f_{(p)}\left(Z_1^{((1),(2,1))}\right)=x_1x_4^2+$ $(p-1)\,x_1x_2^2,$ which consists of 2 monomials.

Therefore for each $p\geq 5,$ the subspace $U_{(p)}^{((1),(2,1))}$ (which represents the submodule $N_{F_p}((1),(2,1))$) of the vector space F_p^{12} (which represents the module $M_{F_p}((1),(2,1))$) is a linear $(12, 8, 2, p)$-code.

14) For the pair of partitions $((1),(1,1,1))$, we have that $m(M)=\dim_{F_p} M_{F_p}((1),(1,1,1))=\dfrac{4!}{1!\cdot 1!\cdot 1!\cdot 1!}=24,$ and we have $\dim_{F_p} S_{F_p}((1),(1,1,1))=\dfrac{4!}{1\cdot 3\cdot 2\cdot 1}=4.$

Thus we have 4 standard $((1),(1,1,1))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((1),(1,1,1))}\right)=x_1x_3^2x_4^4+(p-1)\,x_1x_2^2x_4^4+(p-1)\,x_1x_2^4x_3^2$$
$$+(p-1)\,x_1x_3^4x_4^2+x_1x_2^4x_4^2+\ x_1x_2^2x_3^4,$$

$$f_{(p)}\left(Z_2^{((1),(1,1,1))}\right)=x_2x_3^2x_4^4+(p-1)\,x_1^2x_2x_4^4+(p-1)\,x_1^4x_2x_3^2$$
$$+\ (p-1)\,x_2x_3^4x_4^2+\ x_1^4x_2x_4^2+\ x_1^2x_2x_3^4,$$

$$f_{(p)}\left(Z_3^{((1),(1,1,1))}\right)=x_2^2x_3x_4^4+(p-1)\,x_1^2x_3x_4^4+(p-1)\,x_1^4x_2^2x_3$$
$$+\ (p-1)\,x_2^4x_3x_4^2+x_1^4x_3x_4^2+x_1^2x_2^4x_3,$$

$$f_{(p)}\left(Z_4^{((1),(1,1,1))}\right)=\ x_2^2x_3^4x_4+\ (p-1)\cdot x_1^2x_3^4x_4+$$
$$(p-1)\,x_1^4x_2^2x_4+(p-1)\cdot x_2^4x_3^2x_4+$$
$$x_1^4x_3^2x_4+\ x_1^2x_2^4x_4,$$

and $f_{(p)}\left[Z_1^{((1),(1,1,1))}\right]=f_{(p)}\left(i\,Z_1^{((1),(1,1,1))}\right)$ $f_{(p)}\left(Z_1^{((1),(1,1,1))}\right)$, for each $p\geq 3.$ Thus:

$$k_p=\dim_{F_p} N_{F_p}((1),(1,1,1))$$
$$=\dim_{F_p} S_{F_p}((1),(1,1,1))=4,$$

and the minimum distance $d_p=6,$ since each Specht polynomial (which we give above) consists of 6 monomials.

Therefore for each $p\geq 3,$ the subspace $U_{(p)}^{((1),(1,1,1))}$ (which represents the submodule $N_{F_p}((1),(1,1,1))$) of the vector space F_p^{24} (which represents the module $M_{F_p}((1),(1,1,1))$) is a linear $(24, 4, 6, p)$- code.

15) For the pair of partitions $((\ ),(4)),$ we have that $m(M)=\dim_{F_p} M_{F_p}((\ ),(4))=\dfrac{4!}{4!}=1,$ and we have $\dim_{F_p} S_{F_p}((\ ),(4))=\dfrac{4!}{4\cdot 3\cdot 2\cdot 1}=1,$ and thus we have only one standard $((\ ),(4))$-tableau $Z_1^{((\ ),(4))}$ whose Specht polynomial is $f_{(p)}\left(Z_1^{((\ ),(4))}\right)=1.$

(i) If $p=3,$ then $f_{(3)}\left[Z_1^{((\ ),(4))}\right]=4!\,(\bmod 3)\cdot$ $f_{(3)}\left(Z_1^{((\ ),(4))}\right)=24\,(\bmod 3)\cdot 1=0$ (since 24 (mod 3) = 0). Thus:

$k_3 = \dim_{F_3} N_{F_3}((\ ),(4))$

$= \dim_{F_3} F_3 W_4 f_{(3)}\left[Z_1^{((\ ),(4))}\right] = 0,$

hence the minimum distance d_3 does not exist. Therefore, the subspace $U_{(3)}^{((\ ),(4))}$ (which represents the submodule $N_{F_3}((\ ),(4))$) of the vector space F_3 (which represents the module $M_{F_3}((\ ),(4))$) is a linear (1, 0, – , 3)-code.

(ii) If $p \geq 5$, then $f_{(p)}\left[Z_1^{((\ ),(4))}\right] = 4! \pmod p \cdot f_{(p)}\left(Z_1^{((\ ),(4))}\right) = 24 \pmod p \cdot 1 \neq 0$ (since $24 \pmod p \neq 0$), where $Z_1^{((\ ),(4))}$ is the standard $((\ ),(4))$-tableau.

Thus:

$k_p = \dim_{F_p} N_{F_p}((\ ),(4))$

$= \dim_{F_p} F_p W_4 f_{(p)}\left[Z_1^{((\ ),(4))}\right] = 1,$

since $N_{F_p}((\ ),(4))$ is a nontrivial submodule of the Specht module $S_{F_p}((\ ),(4))$, hence the minimum distance $d_p = 1$.

Therefore, the subspace $U_{(p)}^{((\ ),(4))}$ (which represents the submodule $N_{F_p}((\ ),(4))$) of the vector space F_p (which represents the module $M_{F_p}((\ ),(4))$) is a linear $(1, 1, 1, p)$-code.

16) For the pair of partitions $((\ ),(3,1))$, we have that $m(M) = \dim_{F_p} M_{F_p}((\ ),(3,1)) = \frac{4!}{3! \cdot 1!} = 4$, and we have $\dim_{F_p} S_{F_p}((\ ),(3,1)) = \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1} = 3$, and thus we have 3 standard $((\ ),(3,1))$-tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((\ ),(3,1))}\right) = x_4^2 + (p-1)\, x_1^2,$

$f_{(p)}\left(Z_2^{((\ ),(3,1))}\right) = x_3^2 + (p-1)\, x_1^2,$

$f_{(p)}\left(Z_3^{((\ ),(3,1))}\right) = x_2^2 + (p-1)\, x_1^2.$

If $p \geq 3$, then by theorem 2.9 (1), we have that $S_{F_p}((\ ),(3,1))$ is irreducible $F_p W_4$-module. Hence $N_{F_p}((\ ),(3,1)) = F_p W_4\, f_{(p)}\left[Z_1^{((\ ),(3,1))}\right] = S_{F_p}((\ ),(3,1))$ since $f_{(p)}\left[Z_1^{((\ ),(3,1))}\right] = 6 \pmod p)x_4^2 + (p-2)\, x_3^2 + (p-2)\, x_2^2 + (p-2)\, x_1^2 \neq 0$. Thus:

$k_p = \dim_{F_p} N_{F_p}((\ ),(3,1)) = \dim_{F_p} S_{F_p}((\ ),(3,1)) = 3,$

and the minimum distance $d_p = 2$, since each Specht polynomial (which we give above) consists of 2 monomials.

Therefore for each $p \geq 3$, the subspace $U_{(p)}^{((\ ),(3,1))}$ (which represents the submodule $N_{F_p}((\ ),(3,1))$) of the vector space F_p^4 (which represents the module $M_{F_p}((\ ),(3,1))$) is a linear $(4, 3, 2, p)$-code.

17) For the pair of partitions $((\ ),(2,2))$, we have that $m(M) = \dim_{F_p} M_{F_p}((\ ),(2,2)) = \frac{4!}{2! \cdot 2!} = 6$, and we have $\dim_{F_p} S_{F_p}((\ ),(2,2)) = \frac{4!}{3 \cdot 2 \cdot 2 \cdot 1} = 2$, and thus we have 2 standard $((\ ),(2,2))$-tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((\ ),(2,2))}\right) = x_3^2 x_4^2 + (p-1)x_2^2 x_3^2 + (p-1)x_1^2 x_4^2 + x_1^2 x_2^2,$

$f_{(p)}\left(Z_2^{((\ ),(2,2))}\right) = x_2^2 x_4^2 + (p-1)x_2^2 x_3^2 + (p-1)x_1^2 x_4^2 + x_1^2 x_3^2.$

(i) If $p = 3$, then:

$f_{(3)}\left[Z_1^{((\ ),(2,2))}\right] = x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_3^2 + x_2^2 x_4^2 + x_3^2 x_4^2,$

$f_{(3)}\left[Z_2^{((\ ),(2,2))}\right] = x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_3^2 + x_2^2 x_4^2 + x_3^2 x_4^2.$

The above polynomials modulo 3 give the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 + 2R_1 \to R_2]{R_1 \to R_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first row of the matrix above form a basis of the subspace $U_{(3)}^{((\ ),(2,2))}$. Hence $k_3 = \dim_{F_3} N_{F_3}((\ ),(2,2)) = 1$, and the minimum distance $d_3 = 6$.

Therefore, the subspace $U_{(3)}^{((\ ),(2,2))}$. (which represents the submodule $N_{F_3}((\ ),(2,2))$) of the vector space F_3^6 (which represents the module $M_{F_3}((\ ),(2,2))$) is a linear $(6, 1, 6, 3)$- code.

(ii) If $p \geq 5$, then:

$f_{(p)}\left[Z_1^{((\ ),(2,2))}\right] = 4\, x_1^2 x_2^2 + (p-2) \cdot x_1^2 x_3^2 + (p-2)\, x_1^2 x_4^2 + (p-2)\, x_2^2 x_3^2 + (p-2)\, x_2^2 x_4^2 + 4 x_3^2 x_4^2,$

$f_{(p)}\left[Z_2^{((\ ),(2,2))}\right] = (p-2) x_1^2 x_2^2 + 4 x_1^2 x_3^2 + (p-2)\, x_1^2 x_4^2 + (p-2)\, x_2^2 x_3^2 + 4 x_2^2 x_4^2 + (p-2) x_3^2 x_4^2.$

The above polynomials modulo p give the following matrix:

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$$\begin{bmatrix} 4 & p-2 & p-2 & p-2 & p-2 & 4 \\ p-2 & 4 & p-2 & p-2 & 4 & p-2 \end{bmatrix}$$

$$\xrightarrow[R_2+(p-1)R_1 \to R_2]{R_1 \to R_1}$$

$$\begin{bmatrix} 4 & p-2 & p-2 & p-2 & p-2 & 4 \\ -6\ (\bmod p) & 6\ (\bmod p) & 0 & 0 & 6\ (\bmod p) & -6\ (\bmod p) \end{bmatrix}$$

The rows of the above matrix form a basis of the subspace $U_{(p)}^{((\ ),(2,2))}$. Hence:

$k_p = \dim_{F_p} N_{F_p}((\ ),(2,2)) = 2,$

and the minimum distance $d_p = 4$. Therefore, for each $p \geq 5$, the subspace $U_{(p)}^{((\ ),(2,2))}$ (which represents the submodule $N_{F_p}((\ ),(2,2))$) of the vector space F_p^6 (which represents the module $M_{F_p}((\ ),(2,2))$) is a linear $(6, 2, 4, p)$-code.

18) For the pair of partitions $((\ ),(2,1,1))$, we have that $m(M) = \dim_{F_p} M_{F_p}((\ ),(2,1,1)) = \dfrac{4!}{2!\cdot 1!\cdot 1!} = 12$,

and we have $\dim_{F_p} S_{F_p}((\ ),(2,1,1)) = \dfrac{4!}{4\cdot 1\cdot 2\cdot 1} = 3$, and thus we have 3 standard $((\ ),(2,1,1))$-tableaux whose Specht polynomials are:

$f_{(p)}\left(Z_1^{((\ ),(2,1,1))}\right) = x_3^2x_4^4 + (p-1)x_1^2x_4^4 + (p-1)x_4^4x_3^2 + (p-1)x_3^4x_4^2 + x_1^4x_4^2 + x_1^2x_3^4,$

$f_{(p)}\left(Z_2^{((\ ),(2,1,1))}\right) = x_2^2x_4^4 + (p-1)x_1^2x_4^4 + (p-1)x_1^4x_2^2 + (p-1)x_2^4x_4^2 + x_1^4x_4^2 + x_1^2x_2^4,$

$f_{(p)}\left(Z_3^{((\ ),(2,1,1))}\right) = x_2^2x_3^4 + (p-1)x_1^2x_3^4 + (p-1)x_1^4x_2^2 + (p-1)x_2^4x_3^2 + x_1^4x_3^2 + x_1^2x_2^4.$

If $p \geq 3$, then by theorem 2.9 (1), we have that $S_{F_p}((\ ),(2,1,1))$ is irreducible F_pW_4-module. Hence:

$N_{F_p}((\ ),(2,1,1)) = F_p W_4\ f_{(p)}\left[Z_1^{((\ ),(2,1,1))}\right]$
$= S_{F_p}((\ ),(2,1,1)),$

Since:

$f_{(p)}\left[Z_1^{((\ ),(2,1,1))}\right] = (p-1)x_1^4x_3^2 + (p-1)x_2^4x_3^2 + x_1^2x_3^4 + x_2^2x_3^4 + x_1^4x_4^2 + x_2^4x_4^2 + (p-2)\,x_3^4x_4^2 + (p-1)\,x_1^2x_4^4 + (p-1)\,x_2^2x_4^4 + 2x_3^2x_4^4 \neq 0.$

Thus:

$k_p = \dim_{F_p} N_{F_p}((\ ),(2,1,1))$
$= \dim_{F_p} S_{F_p}((\ ),(2,1,1)) = 3,$

and the minimum distance $d_p = 6$, since each Specht polynomial (which we give above) consists of 6 monomials.

Therefore for each $p \geq 3$, the subspace $U_{(p)}^{((\ ),(2,1,1))}$ (which represents the submodule $N_{F_p}((\ ),(2,1,1))$) of the vector space F_p^{12} (which represents the module $M_{F_p}((\ ),(2,1,1))$) is a linear $(12, 3, 6, p)$-code.

19) For the pair of partitions $((\ ),(1,1,1,1))$, we have that $m(M) = \dim_{F_p} M_{F_p}((\ ),(1,1,1,1)) = \dfrac{4!}{1!\cdot 1!\cdot 1!\cdot 1!} = 24,$

and we have $\dim_{F_p} S_{F_p}((\ ),(1,1,1,1)) = \dfrac{4!}{4\cdot 3\cdot 2\cdot 1} = 1.$

Thus we have only one standard $((\ ),(1,1,1,1))$-tableau whose Specht polynomial is:

$f_{(p)}\left(Z_1^{((\ ),(1,1,1,1))}\right) = x_2^2x_3^4x_4^6 + (p-1)\cdot x_1^2x_3^4x_4^6 + (p-1)\,x_1^4x_2^2x_4^6 + (p-1)\cdot x_1^6x_2^2x_3^4 + (p-1)\,x_2^4x_3^2x_4^6 + (p-1)\cdot x_2^6x_3^4x_4^2 + (p-1)\,x_2^2x_3^6x_4^4 + x_1^4x_3^2x_4^6 + x_1^2x_2^4x_4^6 + x_1^6x_3^4x_4^2 + x_1^2x_2^6x_3^4 + x_1^6x_2^2x_4^4 + x_1^4x_2^2x_3^6 + x_2^6x_3^2x_4^4 + x_2^4x_3^6x_4^2 + x_1^2x_3^6x_4^4 + x_1^4x_2^6x_4^2 + x_1^6x_2^4x_3^2 + (p-1)\,x_1^6x_3^2x_4^4 + (p-1)\cdot x_1^4x_3^6x_4^2 + (p-1)\,x_1^6x_2^4x_4^2 + p-1)\cdot x_1^2x_2^6x_4^4 + (p-1)x_1^4x_2^6x_3^2 + (p-1)\cdot x_1^2x_2^4x_3^6,$

and $f_{(p)}\left[Z_1^{((\ ),(1,1,1,1))}\right] = f_{(p)}\left(i\ \ Z_1^{((\ ),(1,1,1,1))}\right) = f_{(p)}\left(Z_1^{((\ ),(1,1,1,1))}\right)$, for each $p \geq 3$.

Hence $N_{F_p}((\ ),(1,1,1,1)) = S_{F_p}((\ ),(1,1,1,1)).$

Thus:

$k_p = \dim_{F_p} N_{F_p}((\ ),(1,1,1,1))$
$= \dim_{F_p} S_{F_p}((\ ),(1,1,1,1)) = 1,$

and the minimum distance $d_p = 24$, since the Specht polynomial $f_{(p)}(Z_1^{((\ ),(1,1,1,1))})$ (which we give above) consists of 24 monomials.

Therefore for each $p \geq 3$, the subspace $U_{(p)}^{((\ ),(1,1,1,1))}$ (which represents the submodule $N_{F_p}((\ ),(1,1,1,1))$) of the vector space F_p^{24} (which represents the module $M_{F_p}((\ ),(1,1,1,1))$) is a linear $(24, 1, 24, p)$-code.

Finally, we summarize the above linear codes in the following Table 1:

Table 1

No.	(λ,μ) of $n=4$	$m(M)$	k_3	d_3	$k_p, p\geq 5$	$d_p, p\geq 5$
1	((4),( ))	1	0	–	1	1
2	((3,1),( ))	4	3	2	3	2
3	((2,2),( ))	6	1	6	2	4
4	((2,1,1),( ))	12	3	6	3	6
5	((1,1,1,1),( ))	24	1	24	1	24
6	((3),(1))	4	0	–	4	1
7	((2,1),(1))	12	4	3	8	2
8	((1,1,1),(1))	24	4	6	4	6
9	((2),(2))	6	6	1	6	1
10	((1,1),(2))	12	6	2	6	2
11	((2),(1,1))	12	6	2	6	2
12	((1,1),(1,1))	24	6	4	6	4
13	((1),(3))	4	0	–	4	1
14	((1),(2,1))	12	4	3	8	2
15	((1),(1,1,1))	24	4	6	4	6
16	(( ),(4))	1	0	–	1	1
17	(( ),(3,1))	4	3	2	3	2
18	(( ),(2,2))	6	1	6	2	4
19	(( ),(2,1,1))	12	3	6	3	6
20	(( ),(1,1,1,1))	24	1	24	1	24

where $m(M)$ is the dimension of the vector space $F_p^{m(M)}$ which represents the F_pW_4 -module $M_{F_p}(\lambda,\mu)=F_pW_4\ g_{(p)}(Z^{(\lambda,\mu)})$, k_p is the dimension of the subspace $U_{(p)}^{(\lambda,\mu)}$ of $F_p^{m(M)}$, where $U_{(p)}^{(\lambda,\mu)}$ represents the irreducible F_pW_4-submodule $N_{F_p}(\lambda,\mu)$ of $M_{F_p}(\lambda,\mu)$, and d_p is the minimum distance, which is the least number of the nonzero coordinates in any nonzero vector of the subspace $U_{(p)}^{(\lambda,\mu)}$.

8. Conclusions

When p is a prime number greater than or equal to 3 and $n=4$, we conclude the following:

1) If (λ,μ) and $(\bar{\lambda},\bar{\mu})$ are two pairs of partitions of 4, such that $\lambda=\bar{\mu}$, $\mu=\bar{\lambda}$ and $U_{(p)}^{(\lambda,\mu)}$ is a linear $(m(M),k_p,d_p,p)$-code, then $U_{(p)}^{(\bar{\lambda},\bar{\mu})}$ is the same linear $(m(M),k_p,d_p,p)$- code.
2) If $(\lambda,\mu)=((\lambda_1,\ldots,\lambda_s),(\mu_1,\ldots,\mu_t))$ be a pair of partitions of 4 and p divides $(\lambda_1-\lambda_2)!$ or $(\mu_1-\mu_2)!$ then $U_{(p)}^{(\lambda,\mu)}$ is a linear $(m(M),0,-,p)$- code.
3) If $U_{(p)}^{(\lambda,\mu)}$ is a linear $(m(M),k_p,d_p,p)$- code and $k_p=m(M)$, then $d_p=1$.
4) If $U_{(p)}^{(\lambda,\mu)}$ is a linear $(m(M),k_p,d_p,p)$-code and $k_p=1$, then $d_p=m(M)$.

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