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# Using Banach Fixed Point Theorem To Study The Stability Of First-Order Delay Differential Equations

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**Abstract**

In1 this1 paper1 we1 use the Banach fixed point theorem investigate 1the stability and asymptotic1 stability1 of the zero solution for the first order retarded delay differential equation
1(y(t))′=-∑_(j=1)^N▒ 〖b_j (t , 〖3y〗 _t )3y(t)+f(t , 〖3y〗 _t)〗
where the delay is constant. Also we give new conditions to ensure the stability and asymptotic stability of the zero solution of this equation.

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## 1. Introduction

In applied sciences, many practical problems concerning heat flow, species interaction microbiology neural networks, and many more are linked with delay differential equations. Burton [1,2,3] was among the first who study the stability of delay differential equation using the fixed point theory, instead of Liapunov method. Many researchers studied the stability for many types of delay differential equations, for example, in 2010, Meng Fan [6] studied the stability of the delay differential equation

$$x\acute{}(t) = -a(t,\ x_t)\ x(t) + f(t,\ x_t)$$

using fixed point theory, where $x_t(\theta) = x(t+\theta)$ and $\theta \in [-r,0]$ , $r \geq 0$ , that is with constant delay
Bo Zhang [7] in 2005 studied fixed points and stability in differential equations with variable delays

$$x\acute{}(t) = -b(t)x(t-\tau(t))$$

and its generalization

$$x\acute{}(t) = -\sum_{j=1}^{N} b_j(t)\ x(t-\tau_j(t))$$

where $b, b_j \in C(R^+, R)$ and $\tau, \tau_j \in C(R^+, R^+)$ with $t-\tau(t) \to \infty$ and
$t-\tau_j(t) \to \infty$ as $t \to \infty$.

Ramazan Yazgan [8] in 2017 studied the global asymptotic stability of solutions to neutral equations of first order

$$x\acute{}(t) = -a(t)\ x(t) + b(t)g(x(t)) + c(t)f\big(x(t-\tau_1(t)\big) + \mathrm{q}\Big(\mathrm{t}, \mathrm{x(t)}, \mathrm{x}\big(\mathrm{t}-\tau_2(\mathrm{t})\big)\Big)$$

Where $a, b, c \in C([0,\infty), R), g, f \in C(R,R), q \in C([0,\infty), R \times R, R)$ , $\tau_i \in C\big([0,\infty),(0,\infty)\big)$ with $t-\tau_i \to \infty$ as $t \to \infty$ , $(i = 1, 2)$.

In this paper we study the stability of the following retarded delay differential equation

$$y\acute{}(t) = -\sum_{j=1}^{N} b_j(t, y_t)y(t) + f(t, y_t) \qquad (1.1)$$

where $b_j(t, y_t)$ and $f(t, y_t) \in C(R^+ \times B, R)$.
Let $B = \{y_t: [-r,0] \to R$ , $y_t$ is continuous on $([-r,0], R)\}$ with the supremum norm. $\|y_t\| = sup_{t\in[-r,0]}\{y(t)\}$

## 2. Preliminaries

In this section, we give some basic concepts and retarded previous or elementary results to this work.

**Theorem 2.1: [4]**
If $T$: $X \to X$ is a contraction and $X$ is a Banach space, then there is a unique point $x^* \in X$ which is fixed by $T$ (That is $T(x) = x$). Moreover, if $x_\circ$ is any point in $X$, Then the sequence defined by $x_1 = T(x_\circ), x_2 = T(x_1), \ldots$ converge to $x^*$.

**Corollary 2.2: [4]**
If $S$ is a closed subset of the Banach space $X$, and $T: S \to S$ is a contraction, then $T$ has a unique fixed point in $S$.

**Theorem 2. 3: (Leibniz's integral rule) [5]**
Assume that $f(x,t)$ be a function such that both $f(x,t)$ and its partial derivative $f_x(x,t)$ are continuous in $t$ and $x$ in some region of the $(x,t)$ -plane, including $a(x) \le t \le b(x)$ , $x_\circ \le x \le x_1$ . Also suppose that the functions $a(x)$ 1and $b(x)$ are both continuous and both have continuous derivatives for $x_\circ \le x \le x_1$. Then, for $x_\circ \le x \le x_1$ ,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f\big(x, b(x)\big)\frac{d}{dx}b(x) - f\big(x, a(x)\big)\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

**3. Stability by using Banach fixed point:**
In this section, we will study the stability of the retarded delay differential equation (1.1).

**Theorem 3.1. :**
Assume that:
1. There exists continuous function $c_1(t), c_2(t) \in C(R, R)$ such that $c_1(t) \le b_j(t, y(t-\tau(t))) \le c_2(t)$
2. Suppose that $f(t,0) = 0$ and there exists a positive constant $D$ and a continuous function $l_1(t) \in C(R, R^+)$ such that
$$|f(t,\Phi) - f(t,\Psi)| \le l_1(t)|\Phi - \Psi|$$
For all $\Phi, \Psi \in B_D := \{y \in B : \|y\| \le D\}$.
3. If there exists a constant $\alpha \in (0,1)$ such that
$$\int_0^t e^{-\int_s^t c_1(u)du} l_1(s)ds \le \alpha \qquad , \ t \ge 0$$
4. $\lim_{t\to+\infty} \int_0^t c_1(s)ds \to +\infty$.

Then the zero solution of (1.1) is asymptotically stable.

**Proof:**
For any $t_\circ \ge 0$, there is a constant $\delta_\circ \ge 0$ such that $\delta_\circ \le D$ and $\delta_\circ k_1 + \alpha D \le D$ , where
$$k_1 := sup_{t \ge t_\circ}\left\{e^{-\int_{t_\circ}^t c_1(s)ds}\right\}$$
for any $G \in B$, it follows from (2) that there is a unique solution of (1.1) $y(t) = y(t, t_\circ, G)$ ; $y_t = G$, defined on $[t_\circ, +\infty]$. Now, we show that the zero solution of (1.1) is attractive, i.e.,
$\lim_{t\to+\infty} y(t, t_\circ, \Psi) = 0$, for all $\Psi \in B_{\delta_\circ} := \{y \in B ; \|y\| \le \delta_\circ\}$.

For any given $\Psi \in B_{\delta_\circ}$ , let $u(t) = u(t, t_\circ, \Psi)$ be the unique solution of (1.1) through $(t, \Psi)$.
Consider the initial value problem:
$$\acute{y}(t) = -\sum_{j=1}^{N} b_j(t, y_t)y(t) + f(t, y_t) \ ; \ y_t = \Psi \qquad \dots (3.1)$$
Using existence and uniqueness theorem, then (3.1) has a unique solution and using Leibniz's integral rule, one can show that this solution has the form:
$$y(t) = \Psi(0)e^{-\int_{t_\circ}^t \sum_{j=1}^N b_j(s, u_s)ds} + \int_{t_\circ}^t e^{-\int_s^t \sum_{j=1}^N b_j(h, u_h)dh} f(s, y_s)ds$$
Let $\Omega = \left\{\begin{matrix} y \in C([t_\circ - r, +\infty), R): y_{t_\circ} = \Psi , \\ |y_t| \le D , t \ge t_\circ , \lim_{t\to+\infty} y(t) = 0 \end{matrix}\right\}$
then $\Omega$ is a closed subset of a Banach space
Define the mapping $T : \Omega \to \Omega$ by
$$T_y(t) = \begin{cases} \Psi(t - t_\circ) , & t_\circ - r \le t \le t_\circ \\ \Psi(0)e^{-\int_{t_\circ}^t \sum_{j=1}^N b_j(s, u_s)ds} + \int_{t_\circ}^t e^{-\int_s^t \sum_{j=1}^N b_j(h, u_h)dh} f(s, y_s)ds , & t \ge t_\circ \end{cases}$$
We show that $T$ maps $\Omega$ into $\Omega$.
It is clear that for any $y \in \Omega$, $T_y$ is continuous, $T_y(t) = \Psi(t - t_\circ)$ for $t \in [t_\circ - r, t_\circ]$ , and
$$\begin{aligned} |T_y(t)| &= \left|\Psi(0)e^{-\int_{t_\circ}^t \sum_{j=1}^N b_j(s, u_s)ds} + \int_{t_\circ}^t e^{-\int_s^t \sum_{j=1}^N b_j(h, u_h)dh} f(s, y_s)ds\right| \\ &\le |\Psi(0)|e^{-\int_{t_\circ}^t c_1(s)ds} + \int_{t_\circ}^t e^{-\int_s^t c_1(h)dh} l_1(s)|y_s|ds \\ &\le \delta_\circ k_1 + \alpha|y_s| \\ &\le \delta_\circ k_1 + \alpha D \\ &\le D \qquad , \qquad t \ge t_\circ \end{aligned}$$
Thus, $T_y(t) \in B_D$ for $t \ge t_\circ$.
Now, we show that $T_y(t) \to 0$ as $t \to +\infty$.
By condition (4) , we have $\lim_{t\to+\infty} e^{-\int_{t_\circ}^t c_1(s)ds} = 0$. So the first term on the right-hand side of $T_y(t)$ tends to zero. Next, we will show that the second term on the right side of $T_y(t)$ tends to zero too. The fact $y \in \Omega$ implies that $|y(t)| \le D$, for all $t \ge t_\circ$

and for any given $\varepsilon > 0$, there exist $t_1 > t_\circ$ such that $|y(t)| < \varepsilon$ , for all $t \geq t_1$. It follows from (4) that there exists $t_2 > t_1$ such that;

$$e^{-\int_{t_1+r}^{t} c_1(s)ds} < \frac{\varepsilon}{\alpha D} \qquad , \qquad t > t_2$$

hence for $t > t_2$ , we have

$$\left|\int_{t_\circ}^{t} e^{-\int_s^t \sum_{j=1}^{N} b_j(h,u_h)dh} f(s, y_s)ds\right|$$

$$\leq \int_{t_\circ}^{t_1+r} e^{-\int_s^t \sum_{j=1}^{N} b_j(h,u_h)dh}|f(s, y_s)ds| + \int_{t_1+r}^{t} e^{-\int_s^t \sum_{j=1}^{N} b_j(h,u_h)dh}|f(s, y_s)ds|$$

$$\leq \int_{t_\circ}^{t_1+r} e^{-\int_s^t c_1(h)dh} l_1(s)|y_s|ds + \int_{t_1+r}^{t} e^{-\int_s^t c_1(h)dh} l_1(s)|y_s|ds$$

$$\leq e^{-\int_{t_1+r}^{t} c_1(h)dh} \int_{t_\circ}^{t_1+r} e^{-\int_s^{t_1+r} c_1(h)dh} l_1(s)|y_s|ds + \int_{t_1+r}^{t} e^{-\int_s^t c_1(h)dh} l_1(s)|y_s|ds$$

$$\leq De^{-\int_{t_1+r}^{t} c_1(h)dh} \int_{t_\circ}^{t_1+r} e^{-\int_s^{t_1+r} c_1(h)dh} l_1(s)ds + \varepsilon \int_{t_1+r}^{t} e^{-\int_s^t c_1(h)dh} l_1(s)ds$$

$$\leq \alpha De^{-\int_{t_1+r}^{t} c_1(h)dh} + \alpha\varepsilon$$
$$< \varepsilon + \alpha\varepsilon$$
$$= (1+\alpha)\varepsilon.$$

So $\lim_{t\to+\infty} T_y(t) = 0$ and hence $T_y(t) \in \Omega$.
Now, we are at the position to show that $T_y(t)$ is a contraction mapping.
For any $y, z \in \Omega$

$$\left|T_y(t) - T_z(t)\right| \leq \int_{t_\circ}^{t} e^{-\int_s^t c_1(h)dh} |f(s, y_s) - f(s, z_s)|ds$$

$$\leq \int_{t_\circ}^{t} e^{-\int_s^t c_1(h)dh} l_1(s)|y_s - z_s|ds$$

$$\leq \alpha\, |y_t - z_t|$$

By previous corollary, $T$ has unique fixed point in $\Omega$, which is the unique solution of (3. 1) and tends to zero as $t$ tends to infinity.

The unique solution of (3.1) is $u(t)$ , therefore $\lim_{t\to+\infty} u(t, t_\circ, \Psi) = 0$ , for all $\Psi \in B_{\delta_\circ}$.
To obtain the asymptotic stability, we have to show that the zero solution of (1.1) is stable. For any given $\varepsilon > 0$ , there is $\delta > 0$ such that
$\delta < \frac{(1-\alpha)\varepsilon}{k_1}$ , then we have

$$|u(t)| \leq |\Psi(0)|e^{-\int_{t_\circ}^{t} c_1(s)ds} + \int_{t_\circ}^{t} e^{-\int_s^t c_1(h)dh} l_1(s)|u_s|ds$$

$$\leq k_1|\Psi| + \alpha|u_t|$$

$$\leq k_1|\Psi| + \alpha\|u\|, \qquad t \geq t_\circ$$

Where $\|u\| = sup_{t\geq t_\circ - r}|u(t)|$ , then $|\Psi| < \delta$ implies that

$$\|u\| \leq \frac{\delta k_1}{1-\alpha} < \varepsilon \qquad , t \geq t_\circ$$

Hence, we conclude that the zero solution of (1.1) is asymptotically stable.

Next will try to answer the following question, if (1), (2) and (3) are satisfied and the solution is asymptotically stable is (4) satisfies?

**Theorem 3. 2:**
Suppose that (1) , (2) and (3) are hold for (1.1) and also the following condition is satisfied
5. $\lim_{t\to+\infty} \int_0^t c_1(s)ds > -\infty$.
If the zero solution of (1.1) is asymptotically stable, then we have
6. $\lim_{t\to+\infty} \int_0^t c_2(s)ds = +\infty$.

**Proof:**
We will prove this theorem by contradiction argument. Assume that (6) fails. Then (5) implies $\lim_{t\to+\infty} inf \int_0^t c_2(s)ds > -\infty$, and there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} \int_0^{t_n} c_2(s)ds = d$$

For some $d \in R$. We can choose a positive constant $q$ such that

$$-q \leq \int_0^{t_n} c_2(s)ds \leq q \qquad , \quad n = 1, 2, \ldots$$

since (6) fails, so (4) fails too. By (5), for the sequence $\{t_n\}$ defined above, there exists a positive constant $p$ satisfying

$$-p \leq \int_0^{t_n} c_1(s)ds \leq p$$

this yields

$$\int_0^{t_n} e^{\int_0^s c_1(u)du} l_1(s)ds \leq \alpha e^{\int_0^{t_n} c_1(u)du} < e^p$$

So the sequence $\left\{\int_0^{t_n} e^{\int_0^s c_1(u)du} l_1(s)ds\right\}$ is bounded, hence there exists a convergent subsequence. For brevity in notation, we still assume the sequence $\left\{\int_0^{t_n} e^{\int_0^s c_1(u)du} l_1(s)ds\right\}$ is convergent, then there exists a positive integer $k$ so large that

$\int_{t_k}^{t_n} e^{\int_0^s c_1(u)du} l_1(s)ds \le \frac{1-\alpha}{2k_1^2 e^{2q}}$ for all $n \ge k$

Now, we consider the solution $y(t) = y(t, t_k, \Psi)$ with $\Psi(s - t_k) \equiv \delta_\circ$ for $s \in [t_{k-r}, t_k]$, and $|y(t)| \le D$, for all $t \ge t_k$, then

$$|y(t)| \le \delta_\circ e^{-\int_{t_k}^t c_1(s)ds} + \int_{t_k}^t e^{-\int_s^t c_1(h)dh} l_1(s)|y_s|ds$$
$$\le \delta_\circ k_1 + \alpha|y_t|$$

Thus, we have

$|y_t| \le \frac{\delta_\circ k_1}{1-\alpha}$ for all $t \ge t_k$

On the other hand, for $n$ large enough, we also have

$$y(t_n) = y(t_k)e^{-\int_{t_k}^{t_n} \sum_{j=1}^N b_j(s, u_s)ds} + \int_{t_k}^{t_n} e^{-\int_s^{t_n} \sum_{j=1}^N b_j(h, u_h)dh} f(s, y_s)ds$$

So

$$|y(t_n)| \ge \delta_\circ e^{-\int_{t_k}^{t_n} c_2(s)ds} - \left|\int_{t_k}^{t_n} e^{-\int_s^{t_n} \sum_{j=1}^N b_j(h, u_h)dh} f(s, y_s)ds\right|$$
$$\ge \delta_\circ e^{-\int_{t_k}^{t_n} c_2(s)ds} - \int_{t_k}^{t_n} e^{-\int_s^{t_n} \sum_{j=1}^N b_j(h, u_h)dh} |f(s, y_s)|ds$$
$$\ge \delta_\circ e^{-\int_{t_k}^{t_n} c_2(s)ds} - \int_{t_k}^{t_n} e^{-\int_s^{t_n} \sum_{j=1}^N b_j(h, u_h)dh} l_1(s)|y_s|ds$$
$$\ge \delta_\circ e^{-\int_{t_k}^{t_n} c_2(s)ds} - \int_{t_k}^{t_k} e^{-\int_s^{t_n} c_1(h)dh} l_1|y_s|ds$$
$$\ge \delta_\circ e^{-\int_{t_k}^{t_n} c_2(s)ds} - \frac{\delta_\circ k_1}{1-\alpha} e^{-\int_0^{t_n} c_1(h)dh} \int_{t_k}^{t_n} e^{\int_0^s c_1(h)dh} l_1(s)ds$$

But

$$e^{-\int_{t_k}^{t_n} c_2(s)ds} = e^{\int_{t_n}^0 c_2(s)ds} \cdot e^{\int_0^{t_k} c_2(s)ds}$$
$$= e^{-\int_0^{t_n} c_2(s)ds} \cdot e^{\int_0^{t_k} c_2(s)ds}$$
$$\ge e^{-2q}$$

And

$e^{-\int_0^{t_n} c_1(s)ds} \le k_1$, so

$$|y(t_n)| \ge \delta_\circ e^{-2q} - \frac{\delta_\circ k_1}{1-\alpha} k_1 \frac{1-\alpha}{2k_1^2 e^{2q}}$$
$$= \frac{1}{2}\delta_\circ e^{-2q}$$

This implies

$$\lim_{t\to+\infty} y(t) \ne 0.$$

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