Results Concerning the Trace of Some Biadditive Mappings of Prime and Semiprime Rings

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Abstract

The purpose of this paper is to present some results concerning the trace of symmetric (α, α)-Biderivation and symmetric left α-Bimultiplier on prime rings. In these results we investigate commutativity of rings, further some certain identities satisfying by symmetric (α, α)-Biderivation and biadditive mappings that make these mapping α-commuting.

Keywords**:** Prime rings, Trace of biadditive mappings, Symmetric (α, α)-Biderivation, Symmetric left α-Bimultiplier, α-commuting mappings.

1. Introduction

Throughout this discussion, unless otherwise mentioned R will represent an associative prime ring with center $Z(R)$ and α, $\tau \in Aut(R)$. For $x,y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$. A ring R is called 2-torsion free, if $2x=0$, $x \in R$, implies $x=0$. Recall that R is prime if for any $a,b \in R$, $aRb =\{0\}$ implies $a=0$ or $b=0$ and semiprime if for any $a \in R$, $aRa =\{0\}$ implies $a=0$.

In [1], T. K. Lee introduce the notion of α-commuting mappings in the following way: A mapping $\varphi: R \longrightarrow R$ is said to be α-centralizing on R if $[\varphi(x), \alpha(x)] \in Z(R)$, for all $x \in R$. In special case when $[\varphi(x), \alpha(x)] =0$, for all $x \in R$, the mapping φ is called α-commuting. If $\varphi(x)\alpha(x) + \alpha(x)\varphi(x)= 0$ holds for all $x \in U$, then φ is said to be skew α-commuting

A mapping $\mathcal{B}: R\times R\to R$ is called symmetric if $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all pairs $x,y \in R$. A mapping $f: R \longrightarrow R$ defined by $f(x) = \mathcal{B}(x, x)$, where $\mathcal{B}$ is a symmetric mapping will be called the trace of $\mathcal{B}$. It obvious that in case $\mathcal{B}$ is a symmetric mapping which is also biadditive (i.e., additive in both arguments), the trace of $\mathcal{B}$ satisfies $f(x+y)= f(x) +2\mathcal{B}(x,y)+ f(y)$, for all $x,y \in R$. The notion of symmetric Biderivation was introduced by Maksa in [2]. A symmetric biadditive mapping $D(. , .): R\times R\longrightarrow R$ is called symmetric Biderivation if $D(xy, z)=D(x, z)\, y + xD(y, z)$ holds all $x,y,z \in R$. If D satisfies that $D(x^2, z) =D(x, z)\, x + xD(x, z)$ for all $x,y \in R$, then D is said to be symmetric Jordan Biderivation. In 2007 Y. Ceven, and M. A. Öztürk in [3] introduce the concept of symmetric (α, τ)-Biderivation as follows: A symmetric biadditive mappings $F(.,.)$: $R\times R\longrightarrow R$ is called said to be a symmetric (α, τ)-Biderivation if $F(xy, z) = F(x, z)\,\alpha(y)+ \tau(x)\,F(y, z)$, for all $x,y,z \in R$. Obviously, in this case the relation $F(x,yz) = F(x, y)\alpha(z)+ \tau(y)\,F(x, y)$ is also satisfied for all $x,y,z \in R$. M. Ashraf in 2010 [4], introduced the notion of symmetric generalized (α, τ)-Biderivation as follows: A symmetric biadditive mapping $G(.,.):R\times R \longrightarrow R$ is symmetric generalized (α,τ)-Biderivation if there exist symmetric (α, τ)-Biderivation D such that $G(xz, y)=G(x, y)\,\alpha(z) + \tau(x)D(z, y)$, for all $x, y, z \in R$. In case $\alpha = \tau$ the mappings F and G are said to be a symmetric (α, α)-Biderivation and symmetric generalized (α, α)-Biderivation respectively. A Symmetric biadditive mapping $\mathcal{T}: R\times R\longrightarrow R$ is called a Symmetric left (right) α-Bimultiplier where is a homomorphism of R if:

$$\mathcal{T}(xz, y) = \mathcal{T}(x, y)\,(z)\ \ (\mathcal{T}(xz, y) = \alpha(x)\,\mathcal{T}(z, y)),$$
holds for all $x,y,z \in R$.

The mapping $\mathcal{T}$ is called a Symmetric α-Bimultiplier if it is both Symmetric left and right α- Bimultiplier (see [5]).

Over the last five decades, many authors [6, 7, 8] present several results concerning the relationship between the commutativity of prime and semiprime rings and the existence of specific types of a nonzero symmetric generalized (α, τ)-Biderivation and affiliated mappings. In this paper many results of this kind was presented. We shall also briefly discuses of the notion of α-commuting mappings.

2. Some Preliminaries

We shall do a great of calculations with commutators, routinely using the following basic identities (see [2]):

$[xy, z] = [x, z]y + x[y, z]$ &
$[x, yz] = [x, y]z + y[x, z]$, for all $x,y,z \in R$.

We state the following well-known results which will be useful in the sequel.

Lemma (2.1): [9]

Let R be a prime ring of characteristic different from 2 and $\mathcal{I}$ be a nonzero ideal of R. let a, b be fixed elements of R. if $axb+bxa=0$ is fulfilled for all $x \in \mathcal{I}$, then either $a=0$ or $b=0$.

Lemma (2.2): [10]

Let R be semiprime ring, $\mathcal{I}$ a right ideal of R. If $\mathcal{I}$ is a commutative as a ring, then $\mathcal{I} \subset Z(R)$.In addition if R is a prime, then R must be commutative.

Lemma (2.3): [11]

Let R be a prime ring, and $\mathcal{I}$ be a nonzero left ideal of R. If a (σ, τ)-Biderivation $D: R\times R \longrightarrow R$ satisfies that $D(\mathcal{I}, \mathcal{I}) = 0$, then $D = 0$.

Also, we need to prove the following lemma.

Lemma (2.4):

Let U be a nonzero left ideal in a 2-torsion free prime ring R. If a symmetric (α, α)-Biderivation $F: R\times R \rightarrow R$ has a zero Trace on U, then R is commutative or F is zero on R.

Proof:

Let f be the Trace of F, then

$f(u)=0$, for all $u \in U$.

The linearization of above relation leads because of the 2-torsionity free of R to:

$F(u, \omega)=0$, for all $u, \omega \in U$.

Consequently, for any $r,s \in R$, we have:

$F(ru, s\omega)=0$, for all $u, \omega \in U$. .... (1)

We shall compute (1) in two different ways to get:

$F(s, r)(\omega)\alpha(u)=0$, for $u, \omega \in U$ and $r,s \in R$. . (2)

$F(s, r)(u)\alpha(\omega)=0$, for $u, \omega \in U$ and $r,s \in R$. . (3)

Subtracting (2) from (3) implies that:

$F(s, r)[\alpha(u), \alpha(\omega)]=0$, for $u,\omega \in U$, $r,s \in R$. .. (4)

Putting st instead of s in (4), using (4), we arrive at:

$F(s, r)(t)\alpha([u, \omega])=0$, for $u, \omega \in U$, $r,s,t \in R$.

By primeness of R yields that either $F(s, r) = 0$, for all $r,s \in R$, that is F is zero on R or $\alpha([u, \omega]) = 0$ and consequently $[u, \omega]=0$, for all $u, \omega \in U$.

If $[u,\omega]=0$, for all $u, \omega \in U$ then an application of Lemma (2.2) yields that R is commutative.

3. The Main Results

We start our main results with following theorem which looking for the conditions that forces the prime ring R to be commutative.

Theorem (3.1):

Let R be a 2-torsion free prime ring and $D: R\times R \longrightarrow R$ be a nonzero Symmetric Jordan Biderivation such that $xy - yd(x) = yx - xd(y)$, for all $x,y \in R$, where d is the Trace of D, then R is commutative.

Proof:

Form our hypothesis, we see**:**

$[x, y] = yd(x) - xd(y)$, for all $x,y \in R$. .... (1)

The linearization of above relation with respect x, we get:

$[x, y] + [z, y] = yd(x) + yd(z) + 2yD(x, z) - xd(y) - zd(y)$, for all $x,y,z \in R$.

In view of (1), and 2-torsionity free of R, the above relation reduces to:

$yD(x, z) = 0$, for all $x,y,z \in R$. .... (2)

Now, the substitution x^2 for x leads to:

$yD(x, z)x + yxD(x, z)=0$, for all $x,y,z \in R$.

According to (2), we have:

$yxD(x, z)=0$, for all $x,y,z \in R$. .... (3)

Also, the left multiplication of (2) by x, we get:

$xyD(x, z) = 0$, for all $x,y,z \in R$. .... (4)

Combining (3) and (4), implies to:

$[x, y]\, D(x, z) = 0$, for all $x,y,z \in R$. .... (5)

Replacing y by yr in (5), using (5), leads to:

$[x, y]\, r\, D(x, z) = 0$, for all $x,y,r,z \in R$.

Now, define

$\mathcal{K} = \{x \in R : D(x, z) = 0$, for all $z \in R\}$
$\mathcal{H} = \{x \in R : [x, y] = 0$, for all $y \in R\}$

Then $\mathcal{K}$ and $\mathcal{H}$ are two disjoint sub group of R satisfies that there union equal to R, which contradicts Brauer's trick. Since D is a nonzero Jordan Biderivation, we conclude that:

$[x, y] = 0$, for all $x,y \in R$.

Hence R is a commutative ring.

Theorem (3.2):
Let R be a 2-torsion free ring and α be an automorphism on R. if a symmetric (α,α)-Biderivation $F: R\times R \longrightarrow R$ satisfies $(xy)-f(xy) = \alpha(yx)- f(yx)$, for all $x,y \in R$, where f is the Trace of F, then R is commutative.

Proof:
For any $x,y \in R$, we have:

$[\alpha(x), \alpha(y)] = f(xy) - f(yx)$
$= [\alpha(x)^2, f(y)] + [f(x), \alpha(y)^2] + 2\alpha(x) F(x, y)\, \alpha(y) - 2\alpha(y)\, F(x, y)\, \alpha(x)$. .... (1)

The substitution $x+y$ for x in (1), we get:

$[\alpha(x), \alpha(y)] = [\alpha(x)^2, f(y)] + [\alpha(x)\alpha(y), f(y)] + [\alpha(y)\alpha(x), f(y)] + [f(x), \alpha(y)^2] + 2[F(x, y),\alpha(y)^2] + 2\alpha(x)F(x, y)\alpha(y) + 2\alpha(x)\, f(y)\alpha(y) - 2\alpha(y)\, F(x, y)\, \alpha(x) - 2\alpha(y)\, f(y)\alpha(x)$, for all $x,y \in R$.

In view of (1), the above relation reduces to:

$[\alpha(x)(y), f(y)] + [\alpha(y)\alpha(x), f(y)] + 2[F(x, y), \alpha(y)^2] + 2\alpha(x)\, f(y)\alpha(y) - 2\alpha(y)\, f(y)\alpha(x) = 0$.
.... (2)

Again, taking $x+y$ instead of x in (2) and using (2) imply that:

$2([\alpha(x)^2, f(y)] + [f(x), \alpha(y)^2] + 2\alpha(x)F(x, y)\alpha(y) - 2\alpha(y)F(x, y)\alpha(x)) = 0$, for all $x,y \in R$.

Using the 2-torsionity free of R and relation (1), we arrive at:
$[\alpha(x), \alpha(y)] = \alpha([x, y]) = 0$, for all $x,y \in R$.

Using the fact that α is an automorphism on R, we see:

$[x, y] = 0$, for all $x,y \in R$.

Hence R is commutative.
In similar manner we can prove the following theorem.

Theorem (3.3):
Let R be a 2-torsion free ring and α be an automorphism on R. if a symmetric (α, α)-Biderivation $F: R\times R \rightarrow R$ satisfies $(xy)+ f(xy) = \alpha(yx)+ f(yx)$, for all $x,y \in R$, where f is the Trace of F, then R is a commutative ring.

Theorem (3.4):
Let R be a non-commutative 2-torsion free prime ring and $F: R\times R \longrightarrow R$ be a symmetric (α, α)-Biderivation. If the Trace f of F is skew α-commuting on a nonzero ideal U of R, then R is a commutative ring or F is zero on R.

Proof:
According to our hypothesis, we have:

$f(x)\alpha(x) + \alpha(x)\, f(x) = 0$, for all $x \in U$. .... (1)

The linearization of (1) with respect x, we get:

$f(x)\alpha(\omega) + f(\omega)\alpha(x) + 2F(x, \omega)\alpha(x) + 2F(x, \omega)(\omega) + \alpha(x)f(\omega) + 2\alpha(x)F(x, \omega) + (\omega)f(x) + 2(\omega)F(x, \omega) = 0$, for all $x, \omega \in U$. .... (2)

Putting $2x$ instead of x imply that:

$2f(x)\alpha(\omega) + 4f(\omega)\alpha(x) + 4F(x, \omega)\alpha(x) + 8F(x, \omega)\alpha(\omega) + 4\alpha(x)f(\omega) + 4\alpha(x)F(x, \omega) + 2\alpha(\omega)f(x) + 8(\omega)F(x, \omega) = 0$, for all $x, \omega \in U$. .... (3)

Comparing (2) with (3), we arrive because of the 2-torsinity free of R at:

$f(x)\alpha(\omega) + \alpha(\omega)f(x) + 2F(x, \omega)\alpha(x) + 2\alpha(x)F(x, \omega) = 0$, for all $x, \omega \in U$. .... (4)

Replacing ω by $x\omega$ in (4) leads to:

$f(x)\alpha(x)\alpha(\omega) + \alpha(x)\alpha(\omega)f(x) + 2f(x)\alpha(\omega)\alpha(x) + 2\alpha(x)F(x, \omega)\alpha(x) + 2\alpha(x)f(x)\alpha(\omega) + 2\alpha(x^2)\, F(x, \omega) = 0$, for all $x, \omega \in U$.

Equivalently

$\alpha(x)\big(f(x)\alpha(\omega) +\alpha(\omega)f(x)+2F(x, \omega)\alpha(x)+ 2\alpha(x) F(x, \omega)\big) + \big(f(x)\alpha(x) +\alpha(x)f(x)\big)\alpha(\omega) +2 f(x) (\omega)(x)= 0$, for all $x, \omega \in U$. .... (5)

In view of (1) and (4), the relation (5) reduces because of the *2*-torsinity free of *R* to:

$f(x)\alpha(\omega)\alpha(x)= 0$, for all $x, \omega \in U$.

The substitution $r\omega$ for ω in (4), we see:

$f(x)\ \alpha(r)\alpha(\omega)\alpha(x)= 0$, for all $x, \omega \in U, r \in R$. .... (6)

Recall that *(U)* is a nonzero ideal of *R*, also by the primeness of *R* we can get some $\omega_0 \in U$ such that $(\omega_0 I)\neq\{0\}$, moreover, the automorphisms. It y of α leads to $\omega_0 I\neq\{0\}$. So there exist $x_0 \in U$ satisfies that $\omega_0 x_0 \neq 0$.

Now, putting ω_0 for ω and x_0 for x in (6) gives:

$f(x_0)\ \alpha(r)\alpha(\omega_0 x_0)= 0$, for some $x_0, \omega_0 \in U$ and all $r \in R$.

Using the primeness of *R*, since $(\omega_0 x_0) \neq 0$, we conclude that $f(x_0)= 0$.

Therefore

$f(x)= 0$, for all x satisfies that $\omega_0 x \neq 0$. .... (7)

Our next task is to prove that $f(x)= 0$, for all $x \in U$.

Choose $x \in U$ such that $\omega_0 x=0$, then

$\omega_0(x+ x_0) \neq 0$ and $\omega_0(x - x_0) \neq 0$, then an application of (7), we have:

$0=f (x + x_0)= f(x)+ f(x_0)+2F(x, x_0)$
$= f(x) + 2F(x, x_0)$ .... (8)

$0=f (x - x_0) = f(x) - f(x_0)- 2F(x, x_0)$
$= f(x) - 2F(x, x_0)$ .... (9)

Combining (8) with (9), we conclude because of the *2*-torsinity free of *R* that $f(x)=0$.

Hence $f(x)=0$, for all $x \in U$. So an application of Lemma (2.4) we get the assertion of the theorem.

<u>Theorem (3.5)</u>:

Let *R* be a prime ring of characteristic different from *2* and *3* and $U\neq\{0\}$ be an ideal of *R*. if $D_1, D_2:R\times R\longrightarrow R$ are nonzero symmetric (α, α)-Biderivations with trace f_1, f_2 respectively satisfies that $f_1(u) f_2(u)=0$ for all $u \in U$, then either f_2 is α-commuting on *U* or *R* is a commutative ring.

Proof:

By hypothesis, we have:

$f_1(u) f_2(u)=0$, for all $u,\omega \in U$. .... (1)

The linearization of (1) leads to:

$f_1(u)\ f_2(\omega)+ f_1(\omega)\ f_2(u)+2f_1(u)D_2(u, \omega) + 2f_1(\omega)\ D_2(u, \omega)+2D_1(u, \omega)\ f_2(\omega)+2D_1(u, \omega)\ f_2(u)+ 4D_1(u, \omega)\ D_2(u, \omega) =0$, for all $u,\omega \in U$.

Putting *-u* instead of *u* in above relation gives:

$f_1(u)\ f_2(\omega)+ f_1(\omega)\ f_2(u)+ 4D_1(u, \omega)\ D_2(u, \omega) =0$, for all $u,\omega \in U$. .... (2)

The linearization of (1) with respect to ω, we find:

$f_1(u)\ f_2(\omega)+ f_1(u)\ f_2(z)+ 2f_1(u)D_2(\omega, z)+f_1(\omega)\ f_2(u)+ f_1(z)\ f_2(u)+ 2D_1(\omega, z)\ f_2(u)+4D_1(u, \omega)\ D_2(u, \omega)+4D_1(u, z)D_2(u, \omega)+4D_1(u, \omega)D_2(u, z) +4D_1(u, z)\ D_2(u, z)=0$, for all $u,\omega \in U$.

According to (2), the last relation reduces to:

$2f_1(u)D_2(\omega, z)+2D_1(\omega, z)\ f_2(u)+4D_1(u, z)\ D_2(u, \omega)+4D_1(u, \omega)\ D_2(u, z) =0$, for $u,\omega \in U$.

Replacing *u* by ω in above relation, we find:

$6f_1(\omega)D_2(\omega, z)+6D_1(\omega, z)f_2(\omega)=0$. .... (3)

The substitution *zv* for *z* in (3) gives:

$f_1(\omega)D_2(\omega, z)\alpha(v)+ f_1(\omega)\alpha(z)D_2(\omega, v) +D_1(\omega, z)\ \alpha(v)f_2(\omega) + \alpha(z)D_1(\omega, v)\ f_2(\omega)=0$.

In view of (3), the above relation can be written as:

$D_1(\omega, z)[\alpha(v), f_2(\omega)]+ [f_1(\omega), \alpha(z)]D_2(\omega, v) =0$, for all $v,z, \omega \in U$.

Putting $(z)f_1(\omega)$ instead of $\alpha(z)$ yields that:

$D_1(\omega, z)[\alpha(v), f_2(\omega)]+[f_1(\omega), \alpha(z)]\ f_1(\omega)\ D_2(\omega, v) =0$, for all $v,z, \omega \in U$.

The substitution ω for *v* and using (1) leads to:

$D_1(\omega, z)[\alpha(\omega), f_2(\omega)] =0$, for all $z,\omega \in U$. .... (4)

Putting uz for z in (4), using (4) implies that:

$D_1(\omega, u)\alpha(z)\ [\alpha(\omega), f_2(\omega)] = 0$, for all $z,u,\omega \in U$.

Again, replace z by zr in the last relation leads to:

$D_1(\omega, u)\alpha(z)\ \alpha(r)\ [\alpha(\omega), f_2(\omega)] = 0$, for all $z,u,\omega \in U$ and $r \in R$.

Now, define

$\mathcal{H}=\{\omega \in U: [\alpha(\omega), f_2(\omega)] = 0\}$
$\mathcal{K}=\{\omega \in U: D_1(\omega, u)\alpha(z) = 0$, for all $u, z \in U\}$

Since a group cannot be the set theoretic union of two it's proper subgroups, hence either $U=\mathcal{H}$ or $U=\mathcal{K}$. If $U=\mathcal{H}$, this leads that f_2 is α-commuting on U. Otherwise, $U=\mathcal{K}$, that is:

$D_1(\omega, u)\ \alpha(z) = 0$, for all $z,u,\omega \in U$.

Putting sz instead of z, we find:

$D_1(\omega, u)\alpha(s)\alpha(z)=0$, for all $z,u,\omega \in U$ and $s \in R$.

By the primeness of R, we have either f_2 is α-commuting on U or:

$D_1(\omega, u)\alpha(s)\alpha(z)=0$, for all $z,u,\omega \in U$ and $s \in R$.

Since α is an automorphisms, then by the primeness of R (Recall that (U) is a nonzero ideal of R), we find that $D_1(\omega, u)=0$, for all $u,\omega \in U$. Consequently by Lemma (2.3) we conclude that R is commutative.

In similar manner we can prove:

Theorem (3.6):

Let R be a non-commutative prime ring of characteristic different from 2 and 3 and $U\neq\{0\}$ be an ideal of R. if D_1, D_2: $R\times R\longrightarrow R$ are nonzero symmetric (α, α)-Biderivations with trace f_1, f_2 respectively satisfies that $f_1(u)f_2(u)=0$ for all $u \in U$, then either f_1 is α-commuting or D_2 is a zero mapping on R.

Theorem (3.7):

Let R be a semiprime ring of characteristic different from 2, 3 and α is an automorphism on R. if a symmetric left α-Bimultiplier $F: R\times R\longrightarrow R$ satisfies that $[[f(x), \alpha(x)], \alpha(x)]$ is a central, where f is the Trace of F, then f is α-commuting on R.

Proof:

For any $x \in R$, we have:

$[[f(x), \alpha(x)], \alpha(x)] \in Z(R)$. .... (1)

The linearization of (1) leads to:

$[[f(\omega), \alpha(x)], \alpha(x)]+2[[F(x,\omega), \alpha(x)], \alpha(x)]+ [[f(\omega), \alpha(\omega)], \alpha(x)]+ 2[[F(x, \omega), \alpha(\omega)], \alpha(x)] + [[f(x), \alpha(x)], \alpha(\omega)]+ [[f(\omega), \alpha(x)], \alpha(\omega)]+ 2[[F(x, \omega), \alpha(x)], \alpha(\omega)] + [[f(x), \alpha(\omega)], \alpha(\omega)]+ [[f(x), (\omega)], \alpha(x)] +2[[F(x, \omega), \alpha(\omega)], (\omega)] \in Z(R)$, for all $x, \omega \in R$. .... (2)

The substitution $-x$ for x in (2), then combining the relation so obtained with (2), we arrive because of the 2-torsionity free of R at:

$2[[F(x, \omega), \alpha(x)], \alpha(x)] + [[f(x), \alpha(\omega)], \alpha(x)] + [[f(\omega), \alpha(\omega)], \alpha(x)] + [[f(x), \alpha(x)], \alpha(\omega)] + [[f(\omega), \alpha(x)], \alpha(\omega)] + 2[[F(x, \omega), \alpha(\omega)], (\omega)] \in Z(R)$, for all $x, \omega \in R$. .... (3)

Also, putting $2x$ instead of x in (3), we get:

$16[[F(x, \omega), \alpha(x)], \alpha(x)]+ 8[[f(x), \alpha(\omega)], \alpha(x)] + 2[[f(\omega), \alpha(\omega)], \alpha(x)] +8[[f(x), \alpha(x)], \alpha(\omega)] + 2[[f(\omega), \alpha(x)], \alpha(\omega)] + 4[[F(x, \omega), \alpha(\omega)], (\omega)] \in Z(R)$, for all $x, \omega \in R$. .... (4)

Comparing (4) with (3), leads because of the 2-torsinity free of R to:

$2[[F(x, \omega), \alpha(x)], \alpha(x)] +[[f(x), \alpha(\omega)], \alpha(x)]+ [[f(x), \alpha(x)], \alpha(\omega)] \in Z(R)$, for all $x, \omega \in R$. .... (5)

Replacing ω by x^2 in (5) and using the commutator identity, we see:

$[[f(x), \alpha(x)], \alpha(x)]\ \alpha(x)+ \alpha(x)[[f(x), \alpha(x)], \alpha(x)] + \alpha(x)[[f(x), \alpha(x)], \alpha(x)]+ [[f(x), \alpha(x)], \alpha(x)]\ \alpha(x) + [[f(x), \alpha(x)], \alpha(x)]\ \alpha(x) + \alpha(x)[[f(x), \alpha(x)], \alpha(x)] \in Z(R)$, for all $x \in R$.

In view of (1), since R is of characteristic different from 2 and 3, we can get:

$[[f(x), \alpha(x)], \alpha(x)]\alpha(x) \in Z(R)$, for all $x \in R$.

So for any $u \in R$, we have:

$(u)[[f(x), \alpha(x)], \alpha(x)]\alpha(x) - [[f(x), \alpha(x)], \alpha(x)]\ \alpha(x)(u)= 0$, for all $x \in R$.

According to (1), the above relation can be written as:

$[[f(x), \alpha(x)],(x)][(u), \alpha(x)]= 0$, for $x,u \in R$. .... (6)

Putting $(u)[f(x), \alpha(x)]$ instead of $\alpha(u)$ in (6), using (6), leads to:

$[[f(x), \alpha(x)], \alpha(x)] (u) [[f(x), \alpha(x)], \alpha(x)]= 0$, for all $x,u \in R$.

Using the semiprimeness of R and automorphismity of α, we conclude that:

$[[f(x), \alpha(x)], \alpha(x)]= 0$, for all $x \in R$. .... (7)

Now, using a same argument on (7) as used to get (5) from (1), we can see:

$[[f(x), (\omega)], \alpha(x)]+ [[f(x), \alpha(x)], (\omega)] + 2[[F(x, \omega), \alpha(x)], \alpha(x)] =0$, for $x, \omega \in R$. ..... (8)

Replacing ω by $v\omega$ in (8) gives:

$[(\omega)[f(x), \alpha(v)], \alpha(x)]+[[f(x), \alpha(\omega)] \alpha(v), \alpha(x)] +[[f(x), \alpha(x)], \alpha(\omega)]\alpha(v) +\alpha(\omega) [[f(x), \alpha(x)], \alpha(v)] + 2[[F(x, \omega) \alpha(v), \alpha(x)], \alpha(x)]=0$.

That is for all $x,v,\omega \in R$, we have:

$[(\omega), \alpha(x)] [f(x), \alpha(v)]+(\omega)[[f(x), \alpha(v)], \alpha(x)]+[f(x), \alpha(\omega)] [\alpha(v), \alpha(x)]+ [[f(x), \alpha(\omega)], \alpha(x)]\alpha(v)+[[f(x), \alpha(x)], \alpha(\omega)]\alpha(v)+\alpha(\omega)[f(x), \alpha(x)], \alpha(v)]+2[F(x, \omega), \alpha(x)] [\alpha(v), \alpha(x)] + 2F(x,\omega)[[\alpha(v), \alpha(x)], \alpha(x)]+ 2[[F(x, \omega), \alpha(x)], \alpha(x)]\alpha(v) +2[F(x, \omega), \alpha(x)] [\alpha(v), \alpha(x)]=0$,

An application of (8), the above relation reduces to:

$[f(x), \alpha(\omega)] [(v), \alpha(x)]+ [(\omega), \alpha(x)] [f(x), (v)] + 4[F(x,\omega), \alpha(x)] [(v), \alpha(x)] +2F(x, \omega) [[\alpha(v), \alpha(x)], \alpha(x)] =0$, for all $x,v,\omega \in R$. .... (9)

The substitution x for v in (9) imply that:

$[(\omega), \alpha(x)] [f(x), \alpha(x)] =0$, for all $x,\omega \in R$. .... (10)

Putting $f(x)(\omega)$ instead of $\alpha(\omega)$ in (10), then using (10) gives:

$[f(x), \alpha(x)] (\omega) [f(x), \alpha(x)] =0$, for all $x,\omega \in R$.

The semiprimeness of R leads to:

$[f(x), \alpha(x)] =0$, for all $x,\omega \in R$.

Hence f is an α-commuting mapping on R.

We end this paper with the following result which gives a suitable condition on asymmetric generalized (α, α)-Biderivation G$: R\times R\rightarrow R$ that makes the ring R is a commutative.

<u>Theorem (3.8)</u>:

Let R be a 2-torision free prime ring and U be a nonzero ideal of R. if a symmetric generalized (α, α)-Biderivation $G:R\times R\longrightarrow R$ with associated (α, α)-Biderivation D satisfies that $G(d(u), v) =0$ for all $u,v \in U$ where d is the Trace of D, then D is a zero mapping on R.

Proof:

By hypothesis, we have:

$G(d(u), v) =0$, for all $u,v \in U$. .... (1)

Replacing v by vz in above relation implies that:

$G(d(u), v)\alpha(z)+ \alpha(v)D(d(u), z) =0$, for $u,v,z \in U$.

According to (1), the above relation reduces to:

$D(d(u), z) =0$, for all $u,z \in U$. .... (2)

The substitution $u+v$ for u in (2) give:

$D(d(u), z)+ D(d(v), z)+ 2D(D(u, v), z) =0$, for all $u,v \in U$.

According to (2), the last relation becomes:

$2D(D(u, v), z) =0$, for all $u,v,z \in U$. .... (3)

Putting $v\omega$ instead of v in (3), we get:

$2D(D(u, v)\alpha(\omega)+ \alpha(v)D(u, \omega), z) =0$, for all $u,v,z, \omega \in U$.

Equivalently

$2D(D(u, v), z)\alpha^{2}(\omega)+2\alpha(D(u, v)D(\alpha(\omega), z) +2D(\alpha(v), z)\alpha(D(u, \omega)+2\alpha^{2}(v)D(D(u, \omega), z) =0$, for all $u,v,z, \omega \in U$.

An application of (3) on above relation leads to:

$D(\alpha(v), z)\alpha(D(u, \omega)+ \alpha(D(u, v)) D(\alpha(\omega), z) =0$, for all $u,v,z, \omega \in U$. .... (4)

Replacing v by vk in (4) implies that:

$\alpha(D(u, v)\alpha^2(k)D(\alpha(\omega), z)+ \alpha^2(v)\alpha(D(u, k)D(\alpha(\omega), z)+ \alpha^2(v)D(\alpha(k), z)\alpha(D(u, \omega))+ D(\alpha(v), z)\alpha^2(k)\alpha(D(u, \omega) =0$, for $u,v,z,k, \omega \in U$.

In view of (4), the above relation becomes:

$\alpha(D(u, v))\alpha^2(k)D(\alpha(\omega), z)+ D(\alpha(v), z)\alpha^2(k) \alpha(D(u, \omega)) =0$, for all $u,v,z,k, \omega \in U$.

Putting u for z and ω for v in above relation, we find:

$\alpha(D(u, \omega))\alpha^2(k)D(\alpha(\omega), u)+ D(\alpha(\omega), u)\alpha^2(k) \alpha(D(u, \omega)) =0$, for all $u,k,\omega \in U$.

Recall that (U) is an ideal of R, replace (ω) by ω, then an application of Lemma (2.1) on above relation yields because of automorphismity of α and symmetry of D that:

$D(u, \omega) =0$, for all $u,\omega \in U$.

Using Lemma (3.2), we get the requirements of the theorem. .... ■

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الخلاصة

الهدف من هذا البحث تقديم بعض النتائج المتعلقة بالدوال ثنائية المشتقات-(α,α) المتناظرة والدوال ثنائية المضروبات-α المتناظرة المعرفة على الحلقات الأولية. بحثنا في هذه النتائج عن إبدالية الحلقات إضافة إلى ذلك بحثنا في بعض المتطابقات التي تحققها دوال ثنائية المشتقات-(α, α) المتناظرة وبعض الدوال ثنائية الخطية التي تعطي لهذه الدوال الخاصية الإبدالية-α.