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Convergence and Stability for Iterative Schemes Via Integral Conditions

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Abstract

Throughout this work, we introduce some results of convergence, summable almost stability for s-iteration and Picard-Manniterationin Banach space with respect to an integral condition. Also, we study the equivalence between stability of these types of schemes.
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1. Introduction and Preliminaries

In 2007, Agarwal, O'Regan, and Sahu[1] introducedthe S-iteration scheme. Many researchers have worked on this iterative schemeto get results on convergence, equivalence and comparison with other schemes, such as [1], [2], [3], [4]. On the other hand, Khan[5] in 2013 presented the Picard-Mann iterative process and proved that it is convergence faster than Picard [6], Mann [7], and Ishikawa [8] for some types of contractions. Abed and Abbas [4] have been show that S-iteration iterative faster than Picard-Mann for general quasi multi-valued contraction mapping, and Akewe and Okeke [9] proved that convergence and stability of this iteration for a general class of contractive.

An arbitrary normed space X, let $F: X \longrightarrow X$, be a mapping, M is function and $\lambda_n, \eta_n \in (0,1)$, then the iteration scheme:

i- $w_0 \in X$,

$w_{n+1} = M(F, w_n)$

is called **s-iteration** [1] if:

$$w_{n+1} = \lambda_n F z_n + (1-\lambda_n) F w_n$$
$$z_n = \eta_n F w_n + (1-\eta_n) w_n, \forall\, n \geq 0 \quad \ldots(1.1)$$

ii- $x_0 \in X$,

$x_{n+1} = M(F, x_n)$

is called **Picard-Mann**iteration [5] if

$$x_{n+1} = F y_n$$
$$y_n = \lambda_n F x_n + (1-\lambda_n) x_n, \forall\, n \geq 0 \quad \ldots(1.2)$$

Throughout this paper, we study two things, first one is proving the convergence and equivalence of s-iteration and Picard-Mann to a fixed point of mapping satisfied an integral contraction condition. The second is studying of summable almost stability, stability of these iterative schemes for the same mapping. Our results extend, and generalize for a several research studies have been carried out on generalizing contraction conditions of integral type in [10], [11], [12].

Firstly, we recall some definitions we need them in this work:

Definition 1.1, [13]:

An arbitrary iteration scheme $x_{n+1} = M(F, x_n)$, where M is a function and a sequence $\langle x_n \rangle$ converges to a fixed point p of F. Suppose that $\langle q_n \rangle$ be a sequence in X, then $\langle x_n \rangle$ is called **Stable with respect to F (or F-stable)** if $\lim_{n\to\infty} \delta_n = 0$, implies to $\lim_{n\to\infty} q_n = p$, where:

$$\delta_n = \|q_{n+1} - M(F, x_n)\|, n \geq 0 \quad \ldots(1.3)$$

Definition 1.2, [14]:

Let X, F, $\langle x_{n+1} \rangle, \delta_n, q_n$ and p be as of definition (1.1), then the fixed point iteration procedure $\langle x_n \rangle$ is **Almost stable with respect to F (or almost F-stable)** if $\sum_{n=0}^{\infty} \delta_n < \infty$ implies that $\lim_{n\to\infty} q_n = p$.

Definition 1.3, [15]:

Suppose that X, F, $\langle x_{n+1} \rangle, \delta_n, q_n$ and p be as of definition (1.1) then a sequence $\langle x_n \rangle$ is called **Summable almost stable with respect to F (or Summable almost F-stable)** if $\sum_{n=0}^{\infty} \delta_n < \infty$ implies that $\sum_{n=0}^{\infty} \|q_n - p\| < \infty$.

For more details in this field, see [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23].

Let $\mathcal{A}$ be the calss of all monotone increasing functions, such that $\Psi(0)=0$, where $\Psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$, $\mathcal{H}$ be the class of all Lebessgue-Stieltjes integrable mappings which is summable, nonnegative and such that for each $\delta>0$, $\int_0^{\delta}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)>0$, where $\mathcal{S}:\mathbb{R}^+\longrightarrow\mathbb{R}^+$.

Lemma 1.4, [12]:

Assume that X be a complete metric space and $\mathcal{S}\in\mathcal{H}$. Let $\langle c_n\rangle,\langle d_n\rangle\subset$ X, and $\langle a_n\rangle\subset(0,1)$, for all $n\geq 0$, such that:

$$\left|\mathrm{d}(c_n,d_n)-\int_0^{\mathrm{d}(c_n,d_n)}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right|\leq a_n$$

with $\lim_{n\to\infty}a_n=0$. Then:

$$\mathrm{d}(c_n,d_n)-a_n\leq\int_0^{\mathrm{d}(c_n,d_n)}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\leq\mathrm{d}(c_n,d_n)+a_n$$

Lemma 1.5, [23]:

If γ is a real number such that $0\leq\gamma<1$, and $\langle\mu_n\rangle$ is a sequence of positive numbers, then for any sequence of positive numbers $\langle\xi_n\rangle$ holding, $\xi_{n+1}\leq\gamma\,\xi_n+\mu_n,\forall n\geq 0$, we have:

1. If $\lim_{n\to\infty}\mu_n=0$, then $\lim_{n\to\infty}\xi_n=0$.
2. If $\sum_{n=0}^{\infty}\mu_n<\infty$, then we have $\sum_{n=0}^{\infty}\xi_n<\infty$.

2. Main Results

In the following we recall the contractive condition of integral type [10] which will be used in establishing our results.

Assume that X is a complete metric space for F: X ⟶ X be a self-mapping, there exist a real number $\mathfrak{r}\in[0,1)$, $\Psi,\omega\in\mathcal{A}$, and $\mathcal{S}\in\mathcal{H}$, for all $x,y\in$ X, we obtain:

$$\int_0^{\mathrm{d}(\mathrm{F}x,\mathrm{F}y)}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\leq\Psi\left(\int_0^{\mathrm{d}(x,\mathrm{F}x)}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right)+\mathfrak{r}\int_0^{\mathrm{d}(x,y)}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)$$

Example 2.1:

Letℝ be real numbers with usual metric andF: ℝ ⟶ ℝ be a mapping, such that:

$$\mathrm{F}x=\begin{cases}1, x=1\\0,\text{otherewise}\end{cases}$$

Let $\Psi,\omega\in\mathcal{A}$ and $\mathcal{S}\in\mathcal{H}$, such that $\mathcal{S}(\tau)=\omega(\tau)=1,\forall\tau\in\mathbb{R}^+$, for each $\delta>0$, $\int_0^{\delta}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)>0$.

Let us take $x=1, \mathrm{y}=3, \mathfrak{r}=1/2$, then the condition above holding.

Theorem 2.2:

Let X be an arbitrary Banach space and let F: X ⟶ X, such that:

$$\int_0^{\|\mathrm{F}x-\mathrm{F}y\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\leq\Psi\left(\int_0^{\|x-\mathrm{F}x\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right)+\mathfrak{r}\int_0^{\|x-y\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\qquad\ldots(2.1)$$

Suppose that F has a fixed point p. For $\mathrm{w}_0\in$ X let $\langle\mathrm{w}_n\rangle$ be the S-iteration defined by (1.1), where $\lambda_n,\eta_n\in(0,1)$, $0<\lambda\leq\lambda_n$ and $0<\eta\leq\eta_n$. Let, $\Psi,\omega\in\mathcal{A}$ and $\mathcal{S}\in\mathcal{H}$. Then :

1. $\langle\mathrm{w}_n\rangle$ converge to p.
2. $\int_0^{\|\mathrm{q}_{n+1}-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\leq[(\eta\mathfrak{r}+(1-\eta)-1)\lambda+1]\mathfrak{r}\int_0^{\|\mathrm{q}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+7a_n$.

Proof: To prove part (1), let$\langle a_n\rangle\subset(0,1)$, using triangle inequality, condition (2.1) and Lemma (1.4) as follows:

$$\begin{aligned}\int_0^{\|\mathrm{w}_{n+1}-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)&\leq\|\mathrm{w}_{n+1}-\mathrm{p}\|+a_n\\
&\leq\lambda_n[\|\mathrm{F}z_n-\mathrm{p}\|-a_n]+(1-\lambda_n)[\|\mathrm{F}\mathrm{w}_n-\mathrm{p}\|-a_n]+3a_n\\
&\leq\lambda_n\int_0^{\|\mathrm{F}z_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+(1-\lambda_n)\int_0^{\|\mathrm{F}\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+3a_n\\
&\leq\lambda_n\left[\Psi\left(\int_0^{\|\mathrm{p}-\mathrm{F}\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right)+\mathfrak{r}\int_0^{\|z_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right]+(1-\lambda_n)\left[\Psi\left(\int_0^{\|\mathrm{p}-\mathrm{F}\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right)+\mathfrak{r}\int_0^{\|\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)\right]+3a_n\\
&\leq\lambda_n\mathfrak{r}\int_0^{\|z_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+(1-\lambda_n)\mathfrak{r}\int_0^{\|\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+3a_n\ (\text{since }\Psi(0)=0)\\
&\leq\lambda_n\mathfrak{r}\|\mathrm{z}_n-\mathrm{p}\|+(1-\lambda_n)\mathfrak{r}\int_0^{\|\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+4a_n\\
&\leq\lambda_n\mathfrak{r}[\eta_n(\|\mathrm{F}\mathrm{w}_n-\mathrm{p}\|-a_n)+(1-\eta_n)(\|\mathrm{w}_n-\mathrm{p}\|-a_n)]+(1-\lambda_n)\mathfrak{r}\int_0^{\|\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+6a_n\\
&\leq\lambda_n\eta_n\mathfrak{r}\int_0^{\|\mathrm{F}\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+[\lambda_n(1-\eta_n)\mathfrak{r}+(1-\lambda_n)\mathfrak{r}]\int_0^{\|\mathrm{w}_n-\mathrm{p}\|}\mathcal{S}(\tau)\mathrm{d}\omega(\tau)+6a_n\end{aligned}$$

$\leq \lambda_n \eta_n \mathfrak{r}\left[\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau)d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|w_n-p\|} \mathcal{S}(\tau)d\omega(\tau)\right] + [\lambda_n(1-\eta_n)\mathfrak{r} + (1-\lambda_n)\mathfrak{r}]\int_0^{\|w_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + 6a_n$

$= [(\eta_n \mathfrak{r} + (1-\eta_n) - 1)\lambda_n + 1]\mathfrak{r}\int_0^{\|w_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + 6a_n \int_0^{\|w_{n+1}-p\|} \mathcal{S}(\tau)d\omega(\tau)$

$\leq [(\eta\mathfrak{r} + (1-\eta) - 1)\lambda + 1]\mathfrak{r}\int_0^{\|w_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + 6a_n$

By applying Lemma (1.5), we yield $\lim_{n\to\infty} w_n = p$.

Now, to prove part (2) of above Theorem:

Let $\{q_n\}_{n=0}^{\infty}$ be a sequence in X, $\{a_n\}_{n=0}^{\infty} \subset (0,1)$, defined $\{\delta_n\}$ by:

$\delta_n = \|q_{n+1} - \lambda_n F w_n - (1-\lambda_n)Fq_n\|$

where $w_n = \eta_n F q_n + (1-\eta_n)q_n, n \geq 0$.

Using triangle inequality, condition (2.1) and Lemma (1.4), we obtain:

$\int_0^{\|q_{n+1}-p\|} \mathcal{S}(\tau)d\omega(\tau) \leq \|q_{n+1} - p\| + a_n$

$\leq [\|q_{n+1} - \lambda_n F w_n - (1-\lambda_n)Fq_n\| - a_n] + \lambda_n[\|Fw_n - p\| - a_n] + (1-\lambda_n)[\|Fq_n - p\| - a_n] + 4a_n$

$\leq \lambda_n \int_0^{\|Fw_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + (1-\lambda_n)\int_0^{\|Fq_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 4a_n$

$\leq \lambda_n\left[\Psi(\int_0^{\|p-Fp\|} \mathcal{S}(\tau)d\omega(\tau)) + \mathfrak{r}\int_0^{\|w_n-p\|} \mathcal{S}(\tau)d\omega(\tau)\right] + (1-\lambda_n)\left[\Psi(\int_0^{\|p-Fp\|} \mathcal{S}(\tau)d\omega(\tau)) + \mathfrak{r}\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau)\right] + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 4a_n$

$\leq \lambda_n\mathfrak{r}\|w_n - p\| + (1-\lambda_n)\mathfrak{r}\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 5a_n$

$\leq \lambda_n\mathfrak{r}[\eta_n(\|Fq_n - p\| - a_n) + (1-\eta_n)(\|q_n - p\| - a_n)] + (1-\lambda_n)\mathfrak{r}\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 7a_n$

$\leq \lambda_n\eta_n\mathfrak{r}\int_0^{\|Fq_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + [\lambda_n(1-\eta_n)\mathfrak{r} + (1-\lambda_n)\mathfrak{r}] \times \int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 7a_n$

$\leq \lambda_n \eta_n \mathfrak{r}\left[\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau)d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau)\right] + [\lambda_n(1-\eta_n)\mathfrak{r} + (1-\lambda_n)\mathfrak{r}]\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 7a_n$

$= [\lambda_n\eta_n\mathfrak{r}^2 + \lambda_n(1-\eta_n)\mathfrak{r} + (1-\lambda_n)\mathfrak{r}]\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 7a_n$ (since $\Psi(0) = 0$ and by using Lemma (1.4))

$= [(\eta_n \mathfrak{r} + (1-\eta_n) - 1)\lambda_n + 1]\mathfrak{r}\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 7a_n$

From $0 < \lambda \leq \lambda_n$, and $0 < \eta \leq \eta_n$, we obtain:

$\int_0^{\|q_{n+1}-p\|} \mathcal{S}(\tau)d\omega(\tau) \leq [(\eta\mathfrak{r} + (1-\eta) - 1)\lambda + 1]\mathfrak{r}\int_0^{\|q_n-p\|} \mathcal{S}(\tau)d\omega(\tau) + \int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) + 7a_n$. ■ …(2.2)

Theorem 2.3:

Assume that $X, F, p, \Psi, \omega, \mathcal{S}, \{w_n\}, \{q_n\}, \{\lambda_n\}, \{\eta_n\}$, and $\{\delta_n\}$ be as in Theorem (2.2). Let $\langle a_n \rangle$ be a nonnegative real sequence with $\sum_{n=1}^{\infty} a_n < \infty$. Then $\langle w_n \rangle$is Summable almost F-stable, moreover if $\sum_{n=1}^{\infty}\|q_n - p\| < \infty$, then $\sum_{n=1}^{\infty} \delta_n < \infty$.

Proof: Assume that $\sum_{n=1}^{\infty} \delta_n < \infty$. Then, we prove that $\sum_{n=1}^{\infty}\|q_n - p\| < \infty$.

Since $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} a_n < \infty, 0 < \lambda \leq \lambda_n, 0 < \eta \leq \eta_n$, and by applying Lemma (1.5) in (2.2) of Theorem (2.2), we get $\sum_{n=1}^{\infty}\|q_n - p\| < \infty$.

($\Leftarrow$) conversely, let $\sum_{n=1}^{\infty}\|q_n - p\| < \infty$, to show that $\sum_{n=1}^{\infty} \delta_n < \infty$.

Now using condition (2.1), triangle inequality and Lemma (1.4), we have:

$\int_0^{\delta_n} \mathcal{S}(\tau)d\omega(\tau) = \int_0^{\|q_{n+1}-\lambda_n F w_n-(1-\lambda_n)Fq_n\|} \mathcal{S}(\tau)d\omega(\tau)$

$\leq \|q_{n+1} - \lambda_n F w_n - (1-\lambda_n)Fq_n\| + a_n$

$\leq \int_0^{\|q_{n+1}-p\|} \mathcal{S}(\tau)d\omega(\tau) + \lambda_n\int_0^{\|p-Fw_n\|} \mathcal{S}(\tau)d\omega(\tau) +$

$(1-\lambda_n)\int_0^{\|p-Fq_n\|}\mathcal{S}(\tau)d\omega(\tau)+4a_n$

$\le \int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+\lambda_n\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|w_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right]+(1-\lambda_n)\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right]+4a_n$

$\le \int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+\lambda_n\mathfrak{r}\|w_n-p\|+(1-\lambda_n)\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+5a_n$

$\le \int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+\lambda_n\mathfrak{r}\,\eta_n\int_0^{\|Fq_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+[\lambda_n\mathfrak{r}(1-\eta_n)+(1-\lambda_n)\mathfrak{r}]\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+7a_n$

$\le \int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+\lambda_n\mathfrak{r}\,\eta_n\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right][\lambda_n\mathfrak{r}(1-\eta_n)+(1-\lambda_n)\mathfrak{r}]\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+7a_n$

$=\int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+[\lambda_n\mathfrak{r}^2\eta_n+\lambda_n\mathfrak{r}(1-\eta_n)+(1-\lambda_n)\mathfrak{r}]\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\,7a_n$

$=\int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+\left[\left((1-\eta_n)+\mathfrak{r}\,\eta_n-1\right)\lambda_n+1\right]\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+7a_n \quad \ldots(2.3)$

By using $(0<\lambda\le\lambda_n, 0<\eta\le\eta_n)$ and $\sum_{n=1}^{\infty}a_n<\infty$, we have $\sum_{n=1}^{\infty}\delta_n<\infty$. ■

Theorem 2.4: Let $X, F, p, \Psi, \omega, \mathcal{S}, \{w_n\}, \{q_n\}, \{\lambda_n\}, \{\eta_n\}, \{a_n\}$ and $\{\delta_n\}$ be as in Theorem (2.2). Then $\{w_n\}$is F-stable, moreover if $\lim_{n\to\infty} q_n = p$, implies to $\lim_{n\to\infty}\delta_n=0$.

Proof: Assume that $\lim_{n\to\infty}\delta_n=0$. Then, we prove that $\lim_{n\to\infty} q_n=p$.

By expresses (2.2) in the form $\xi_{n+1}\le\gamma\,\xi_n+\mu_n$

Where $0\le\gamma=[(\eta\mathfrak{r}+(1-\eta)-1)\lambda+1]\mathfrak{r}<1$, $\xi_n=\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)$ and $\mu_n=\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+7a_n$ with $\lim_{n\to\infty}\mu_n=\lim_{n\to\infty}\left(\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+7a_n\right)=0$

By Lemma (1.5) and the fact $\int_0^{\delta}\mathcal{S}(\tau)d\omega(\tau)>0$, for each$\delta>0$.

We yield:

$\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)=0$

$\Rightarrow \lim_{n\to\infty}\|q_n-p\|=0\Rightarrow\lim_{n\to\infty}q_n=p$.

($\Leftarrow$) conversely, let $\lim_{n\to\infty}q_n=p$, to show that $\lim_{n\to\infty}\delta_n=0$.

From (2.3) and by using $(0<\lambda\le\lambda_n, 0<\eta\le\eta_n)$, we have:

$\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)\le\int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)+\left[\left((1-\eta)+\mathfrak{r}\,\eta\;-1\right)\lambda+1\right]\mathfrak{r}\times\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+7a_n\longrightarrow 0$, as $n\longrightarrow\infty$.

This implies to $\lim_{n\to\infty}\delta_n=0$. ■

Theorem 2.5:

An arbitrary Banach space X, let $F: X\longrightarrow X$ satisfies the condition (2.1) and has a fixed point p. For $x_0\in X$, let $\{x_n\}_{n=0}^{\infty}$ be the Picard-Mann iteration defined by (1.2) where $\lambda_n\in(0,1)$such that $0<\lambda\le\lambda_n$. Let $\Psi,\omega\in\mathcal{A}$ and$\mathcal{S}\in\mathcal{H}$.Then:

1- $\{x_n\}_{n=0}^{\infty}$ converge to p.

2- $\int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)\le[\lambda\mathfrak{r}+(1-\lambda)]\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+5a_n$.

Proof: To prove part (1), let$\langle a_n\rangle\subset(0,1)$ and we use triangle inequality, condition (2.1) and Lemma (1.4), to obtain:

$\int_0^{\|x_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)\le\|x_{n+1}-p\|+a_n$

$=[\|Fy_n-p\|-a_n]+2a_n$

$\le\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|y_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right]+2a_n$

$=\mathfrak{r}\int_0^{\|y_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+2a_n\le\mathfrak{r}\,\|y_n-p\|+3a_n\le\lambda_n\mathfrak{r}\int_0^{\|Fx_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+(1-\lambda_n)\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+5a_n$

$\le\lambda_n\mathfrak{r}\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right]+$

$(1-\lambda_n)\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+5a_n$

$=[\lambda_n\mathfrak{r}^2+(1-\lambda_n)\mathfrak{r}]\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+5a_n$

$=[\lambda_n\mathfrak{r}+(1-\lambda_n)]\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+5a_n$

$\leq[\lambda\mathfrak{r}+(1-\lambda)]\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+5a_n.$

Now, using Lemma (1.5), to obtain $\lim_{n\to\infty}x_n=p$.

Proof part (2):

Let $\langle q_n\rangle$ be in $X,\langle a_n\rangle\subset(0,1),\{\delta_n\}$ defined by:

$\delta_n=\|q_{n+1}-Fx_n\|,$

where $x_n=\lambda_nFq_n+(1-\lambda_n)q_n, n\geq0$.

We use triangle inequality, condition (2.1) and Lemma (1.4), to obtain

$\int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)\leq\|q_{n+1}-p\|+a_n$

$\leq[\|q_{n+1}-Fx_n\|-a_n]+[\|Fx_n-p\|-a_n]+3a_n$

$\leq\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right]+3a_n$

$=\mathfrak{r}\int_0^{\|x_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+3a_n$

$\leq\mathfrak{r}\|x_n-p\|+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+4a_n$

$\leq\lambda_n\mathfrak{r}\int_0^{\|Fq_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+(1-\lambda_n)\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+6a_n$

$\leq\lambda_n\mathfrak{r}\left[\Psi\left(\int_0^{\|p-Fp\|}\mathcal{S}(\tau)d\omega(\tau)\right)+\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)\right]+(1-\lambda_n)\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+6a_n$

$=[\lambda_n\mathfrak{r}+(1-\lambda_n)]\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+6a_n$

By using $(0<\lambda\leq\lambda_n)$, we have:

$\int_0^{\|q_{n+1}-p\|}\mathcal{S}(\tau)d\omega(\tau)\leq[\lambda\mathfrak{r}+(1-\lambda)]\mathfrak{r}\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)+\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+6a_n$ (2.4). ■

Theorem 2.6:

Let $X, F, p, \Psi, \omega, \mathcal{S}, \{x_n\}, \{q_n\}, \{\lambda_n\}, \{\eta_n\}$, and $\{\delta_n\}$ be as in Theorem (2.5). Assume that $\langle a_n\rangle$ be a nonnegative real sequence with $\sum_{n=1}^{\infty}a_n<\infty$. Then $\{x_n\}$ is Summable almost F-stable, moreover if $\sum_{n=1}^{\infty}\|q_n-p\|<\infty$, then $\sum_{n=1}^{\infty}\delta_n<\infty$.

Proof: Assume that $\sum_{n=1}^{\infty}\delta_n<\infty$. Then, we prove that $\sum_{n=1}^{\infty}\|q_n-p\|<\infty$.

From (2.4), since $0\leq[\lambda\mathfrak{r}+(1-\lambda)]\mathfrak{r}<1$, and $\sum_{n=1}^{\infty}a_n<\infty$, the conclusion follows by Lemma (1.5).

($\Leftarrow$) conversely, let $\sum_{n=1}^{\infty}\|q_n-p\|<\infty$, to show that $\sum_{n=1}^{\infty}\delta_n<\infty$.

The proof completes by the same way of Theorem (2.3). ■

Theorem 2.7:

Let $X, F, p, \Psi, \omega, \mathcal{S}, \{x_n\}, \{q_n\}, \{\lambda_n\}$, and $\{\delta_n\}$ be are of Theorem (2.5). Then $\{x_n\}$ is F-stable, moreover if $\lim_{n\to\infty}q_n=p$, implies to $\lim_{n\to\infty}\delta_n=0$.

Proof: Let $\lim_{n\to\infty}\delta_n=0$. Then, we show that $\lim_{n\to\infty}q_n=p$.

Applying Lemma (1.4) in (2.4), where $0\leq\gamma=[\lambda\mathfrak{r}+(1-\lambda)]\mathfrak{r}<1$, $\xi_n=\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)$ and $\mu_n=\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+6a_n$

such that:

$\lim_{n\to\infty}\mu_n=\lim_{n\to\infty}\left(\int_0^{\delta_n}\mathcal{S}(\tau)d\omega(\tau)+6a_n\right)=0$

and the fact $\int_0^{\delta}\mathcal{S}(\tau)d\omega(\tau)>0$, $\forall\delta>0$, we get:

$\int_0^{\|q_n-p\|}\mathcal{S}(\tau)d\omega(\tau)=0$

$\Rightarrow\lim_{n\to\infty}\|q_n-p\|=0\Rightarrow\lim_{n\to\infty}q_n=p$.

($\Leftarrow$) conversely, suppose that $\lim_{n\to\infty}q_n=p$, to prove that $\lim_{n\to\infty}\delta_n=0$.

The proof completes by the same way of Theorem (2.4). ■

Remark 2.8:

By definitions (1.2), (1.3), we get the sequences $\langle w_n\rangle,\langle x_n\rangle$ in Theorems (2.3), (2.6) are also almost F-stable.

Theorem 2.9:

Suppose that X be a Banach space, $F: X\longrightarrow X$ satisfies the condition (2.1) with a fixed point p, $\Psi, \omega \in \mathcal{A}, \mathcal{S} \in \mathcal{H}$ and $\langle a_n \rangle$ be a nonnegative real sequence with $\sum_{n=1}^{\infty} a_n < \infty$. For $w_0, x_0 \in X$, let $\langle w_n \rangle, \langle x_n \rangle$ defined by (1.1), (1.2) respectively and converge to fixed point p, where $\lambda_n, \eta_n \in (0,1)$, such that $0 < \lambda \le \lambda_n, 0 < \eta \le \eta_n$. Then the following iterative schemes are equivalent:

I- The S-iteration is summable almost F-stable,

II- The Picard-Mann iteration is summable almost F-stable.

Proof: Assume that the S-iteration is summable almost F-stable, to show that Picard-Mann iteration is summable almost F-stable.

$$\|x_{n+1} - Fy_n\| \le \|x_{n+1} - \lambda_n F y_n - (1-\lambda_n)Fx_n\| + \|\lambda_n F y_n + (1-\lambda_n)Fx_n - Fy_n\|$$
$$\le \|x_{n+1} - \lambda_n F y_n - (1-\lambda_n)Fx_n\| + (1-\lambda_n)\|Fx_n - Fy_n\| \quad \ldots(2.5)$$

Hence:

$$\|Fx_n - Fy_n\| \le \|Fx_n - p\| + \|p - Fy_n\| \le \int_0^{\|Fx_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + \int_0^{\|p - Fy_n\|} \mathcal{S}(\tau) d\omega(\tau) + 2a_n$$
$$\le \left[\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau) d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + \left[\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau) d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|y_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 2a_n$$
$$= \mathfrak{r}\left[\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + \int_0^{\|y_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 2a_n \le \mathfrak{r}\left[\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + \|y_n - p\|\right] + 3a_n$$
$$\le \mathfrak{r}\left[\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + (\eta_n\|Fx_n - p\| + (1-\eta_n)\|x_n - p\|)\right] + 3a_n$$
$$\le \mathfrak{r}\left[\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + \eta_n\left(\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau) d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right) + (1-\eta_n)\int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 5a_n$$
$$= \mathfrak{r}[1 + \eta_n \mathfrak{r} + (1-\eta_n)] \int_0^{\|x_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + 5a_n$$
$$\le \mathfrak{r}[1 + \eta_n \mathfrak{r} + (1-\eta_n)]\|x_n - p\| + 6a_n \quad \ldots(2.6)$$

By butting (2.6) in (2.5), we have:

$$\|x_{n+1} - Fy_n\| \le \|x_{n+1} - \lambda_n F y_n - (1-\lambda_n)Fx_n\| + (1-\lambda_n)\mathfrak{r}[1 + \eta_n \mathfrak{r} + (1-\eta_n)]\|x_n - p\| + 6a_n$$
$$\le \|x_{n+1} - \lambda_n F y_n - (1-\lambda_n)Fx_n\| + (1-\lambda)\mathfrak{r}[1 + \eta \mathfrak{r} + (1-\eta)]\|x_n - p\| + 6a_n \quad \ldots(2.7)$$

From (2.7), (I), $0 < \lambda \le \lambda_n, 0 < \eta \le \eta_n$, $\sum_{n=1}^{\infty} a_n < \infty$ and $\langle x_n \rangle$ converge to p, we yield Picard-Mann iteration is summable almost F-stable.

($\Leftarrow$) Conversely, let Picard-Mann is summable almost F-stable, then, we prove that S-iteration issummable almost F-stable.

$$\|w_{n+1} - \lambda_n F z_n - (1-\lambda_n)Fw_n\| \le \|w_{n+1} - Fz_n\| + (1-\lambda_n)\|Fz_n - Fw_n\| \quad \ldots(2.8)$$

Since:

$$\|Fz_n - Fw_n\| \le \|Fz_n - p\| + \|p - Fw_n\| \le \int_0^{\|Fz_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + \int_0^{\|p - Fw_n\|} \mathcal{S}(\tau) d\omega(\tau) + 2a_n$$
$$\le \left[\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau) d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|z_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + \left[\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau) d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 2a_n$$
$$= \mathfrak{r}\left[\int_0^{\|z_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + \int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 2a_n \le \mathfrak{r}\left[\|z_n - p\| + \int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 3a_n$$
$$\le \mathfrak{r}\left[(\eta_n\|Fw_n - p\| + (1-\eta_n)\|w_n - p\|) + \int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + 3a_n \le$$
$$\mathfrak{r}\left[\eta_n\left(\Psi\left(\int_0^{\|p-Fp\|} \mathcal{S}(\tau) d\omega(\tau)\right) + \mathfrak{r}\int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right) + (1-\eta_n)\int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau)\right] + \mathfrak{r}\int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + 5a_n$$
$$= \mathfrak{r}[1 + \eta_n \mathfrak{r} + (1-\eta_n)] \int_0^{\|w_n - p\|} \mathcal{S}(\tau) d\omega(\tau) + 5a_n$$
$$\le \mathfrak{r}[1 + \eta_n \mathfrak{r} + (1-\eta_n)]\|w_n - p\| + 6a_n \quad \ldots(2.9)$$

By putting (2.9) in (2.8), we yield:

$$\|w_{n+1} - \lambda_n F z_n - (1-\lambda_n) F w_n\| \le \|w_{n+1} - F z_n\| + (1-\lambda_n)\mathfrak{r}[1 + \eta_n \mathfrak{r} + (1-\eta_n)]\|w_n - p\| + 6a_n$$
$$\le \|w_{n+1} - F z_n\| + (1-\lambda)\mathfrak{r}[1 + \eta\, \mathfrak{r} + (1-\eta)]\|w_n - p\| + 6a_n \quad \ldots(2.10)$$

From (2.10), (II), $0 < \lambda \le \lambda_n, 0 < \eta \le \eta_n$, $\sum_{n=1}^{\infty} a_n < \infty$ and $\langle w_n \rangle$ converge to p, we have S-iteration is summable almost F-stable. ■

Below, we shall introduce a theorem about equivalence between S-iteration and Picard-Mann iteration for stable and almost stable with respect to F without proof as it is similar to the proof of theorem (2.9).

Theorem 2.10:

Assume that X be a Banach space, $F: X \longrightarrow X$ satisfies the condition (2.1) with a fixed point p. Let $\Psi, \omega \in \mathcal{A}, \mathcal{S} \in \mathcal{H}$ and $\langle a_n \rangle$ be a nonnegative real sequence with $\sum_{n=1}^{\infty} a_n < \infty$. For $w_0, x_0 \in X$, let $\langle w_n \rangle, \langle x_n \rangle$ defined by (1.1), (1.2) respectively and converge to fixed point p, where $\lambda_n, \eta_n \in (0,1)$such that $0 < \lambda \le \lambda_n, 0 < \eta \le \eta_n$. Then the following iterative schemes are equivalent:

a- I- The S-iteration is almost F-stable,
   II- The Picard-Mann iteration is almost F-stable.

b- I- The S-iteration is F-stable,
   II- The Picard-Mann iteration is F-stable.

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