When Compact sets are $\alpha$-Closed

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Abstract
This paper is devoted to introduce new concepts so called $K(\alpha c)$-space several various theorems about these concepts are provided In addition, further properties are studied such as the relationships between those concepts and other types of $K(\alpha c)$-spaces are investigated.

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1. Introduction
It is known that there is no relationship between compact sets and closed sets, so this motivates the author [1] to introduce a concept of $Kc$ – spaces; these are the spaces in which every compact subset is closed, and we know that there is no relationship between compact sets and semi closed sets so in 2015 the researcher [6] studied another type of $Kc$-space namely $K(sc)$-space which is every compact are semi-closed sets.

And the researcher [6] has been studied by another type of $Kc$-space namely $K(\theta c)$-space which is every compact space are $\theta$ – closed sets and this space $(K(\theta c)$-space) will be weaker than $Kc$-space.

Also the same researcher [6] introduced another type of $Kc$-space namely $\theta K(\theta c)$-space which is weaker than $Kc$-space.

After studying all the above of these space, we studied the relationship compact and $\alpha$ – closed and we found that there is no relationship between compact and $\alpha$ – closed, as we explained the examples during the course of the research, so we studied another type of $Kc$-space namely $K(\alpha c)$-space which is every compact sets are $\alpha$-closed. The aim of this paper is to continuous study $Kc$-spaces.

The basic definitions that needed in this work are recalled. In this work, a subset $A$ of a topological space $(X, \tau)$ is called $\alpha$ – open if $A \subset int cl int(A)$ and $\alpha$ – closed set if $cl int cl(A) \subset A$ [2], as the following example $(R, \tau_{Ind})$ be an $\alpha$ – open, where $\tau_{Ind} = \{ U \subset X, x_0 \notin U, \text{for some x_0} \in X \}$ Every open is $\alpha$ – open, but the converse may be not true [3], as the following example shows:

$X = \{1,2,3\},\ \tau = \{ \emptyset, X, \{ 1 \} \}$ let $A = \{1,2\}$ therefore $A^0 = \{1,2\}^0 = \{1\}$ and hence $\overline{A^0} = \overline{\{1\}} = X \Rightarrow \overline{A^0} = \overline{X^0} = X$

This implies to $\{1,2\} \subseteq \{1,2\}^0$ is $\alpha$ – open but $\{1,2\}$ is not open, a subset $A$ of topological space $X$ is an open if and only if there exists an open set $U$, such that $U \subset A \subset int cl(A)$ [4], let $U$ be an open set a space $X$, then every subset $B$ of $X$, $U \cap \overline{B} \subseteq \overline{U \cap B}$ [5].

2. On $k(\alpha c)$ – Space
Since we provided the fact that there is no relationship between compact and closed $(\alpha$ – closed) sets, as in the following example shows:

(1) In $(R, \tau_{Ind})$ the subset $Q$ in $R$ is compact, but not $\alpha$ – closed, in the fact thenon empty subsets of $R$ is compact, but not $\alpha$ – closed.

(2) In $(R, \tau_{Coc})$ the set $R$ is $\alpha$ – closed, which is not compact.

(3) In $(R, \tau_D)$ the set $R$ is closed $(\alpha$ – closed), but not compact.

Now we introduce the following concept:

Definition (1): A space $X$ is called $k(\alpha c)$-space if every compact subset of $X$ is $\alpha$-closed.

Example (1): $(X, \tau_D)$, where $\tau_D$ be a discrete space is $k(\alpha c)$ – space

Definition: A nonempty set $X$ with a topological space $\tau$ is said to be $\alpha$ – compact, if every cover of $X$ with $\alpha$ – open sets has finite subcover.
**Definition:** A nonempty set $X$ with a topological space $\tau$ is said to be $\alpha$–Lindelof, if every cover of $X$ with $\alpha$–open sets has a countable subcover.

**Example:** $(\mathbb{R}, \tau_{\text{Excluded}})$ is $\alpha$–compact and $\alpha$–Lindelof.

**Remark:**
(i) Every $\alpha$–compact ($\alpha$–Lindelof) is compact (Lindelof).
(ii) Every $\alpha$–compact is $\alpha$–Lindelof, but the converses may be not true.

**Examples (a):** In $(\mathbb{Z}, \tau_D)$ the set $\mathbb{Z}$ is $\alpha$–Lindelof (Lindelof), but not $\alpha$–compact (compact).

In fact every infinite countable set with discrete topology satisfy the above results.

(b) $(\mathbb{R}, \tau_{cc})$ is $\alpha$–Lindelof, but not $\alpha$–compact

**Proposition (1):** Every $\alpha$–closed set in $\alpha$–compact ($\alpha$–Lindelof) is $\alpha$–compact ($\alpha$–Lindelof).

**Proof:** Let $(X, \tau)$ be an $\alpha$–compact space and $Y$ is $\alpha$–closed in $X$, then $Y$ is $\alpha$–compact. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an $\alpha$–open cover for $Y$, and $Y \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$, where $U_\alpha$ is $\alpha$–open in $X$. Since $Y = \bigcup_{\alpha \in \Lambda} U_\alpha$, $Y$ is $\alpha$–closed in $X$ if and only if $Y^c$ is $\alpha$–open in $X$.

**Definition:** A space $X$ is said to be $\alpha T_1$–space if every two distinct points $x, y$ in $X$, there exists two $\alpha$–open sets $U, V$, such that $x \in U$, but $x \notin V$ and $y \in V$ but $y \notin U$.

**Example:** $(\mathbb{R}, \tau_{cof})$ is $\alpha T_1$–space.

**Definition:** A space $X$ is said to be $\alpha T_2$–space if every two distinct points $x, y$ in $X$, there exists two $\alpha$–open sets $U, V$, such that $x \in U$, but $x \notin V$ and $y \in V$ but $y \notin U$ and $U \cap V = \emptyset$.

**Example:** $(\mathbb{R}, \tau_D)$ is $\alpha T_2$–space.

**Theorem:** Every $\alpha$–compact set of $\alpha T_2$–space is $\alpha$–closed.

**Proof:** Let $A$ be an $\alpha$–compact set in a topological space $X$, to prove that $A$ is $\alpha$–closed, let $p \in A^c$ since $X$ is $\alpha T_2$–space, so $\forall q \in A$, there exists $U, V \in \tau^a$ (the set of all $\alpha$–open sets) with $q \in U, p \in V$ such that $U \cap V = \emptyset$, now the collection $\{U(q): q \in A\}$ is an $\alpha$–open cover of $A$, which is $\alpha$–compact, then there exist finite subcover of $A$, that is $A \subseteq \bigcup_{i=1}^n U(q_i)$, let $V^* = \bigcap_{i=1}^n V_i(p)$, then $V^*$ is an $\alpha$–open set containing $p$ (finite intersection of $\alpha$–open sets), we claim that $U^* \cap V^* = \emptyset$, let $x \in U^*$, therefore $x \in U(q_i)$ and suppose $x \notin V^*$ and hence $A \cap V^* = \emptyset$, $V^* \subseteq A^c \Rightarrow A^c$ is $\alpha$–open in $X$. This implies to $A$ is $\alpha$–closed.

**Lemma:** Let $(X, \tau)$ and $(Y, \tau)$ be two topological spaces, let $f: X \to Y$ be a homeomorphism from a space $X$ into a space $Y$ if $F$ is $\alpha$–closed in $X$, then $f(F)$ is $\alpha$–closed in $Y$.  

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**Proof:** Let F be an α – closed in X, that is; \( M^0 \subseteq F \subseteq M \), to prove that \( f(F) \) is \( \alpha \)-closed in Y.

Since \( M^0 \subseteq F \subseteq M \), therefore

\[
\begin{align*}
(f(M))^0 &= f(M^0) \quad \text{(i)} \\
f(M^0) &= f(M) \quad \text{(ii)} \\
f(M^0) &= f(M) \quad \text{(iii)}
\end{align*}
\]

Since \( M \) is closed set in X and \( f \) is closed function (since \( f \) is homeomorphism), so \( f(M) \) is closed set in Y, since \( M^0 \subseteq M \), so \( f(M^0) \subseteq f(M) \). But \( f(M^0) = f(M^0) \subseteq f(M) \) (\( f \) is a homeomorphism). Also:

\[
\begin{align*}
(f(M))^0 &= f(M^0) \quad \text{(iv)} \\
f(M^0) &= f(M^0) \quad \text{(v)} \\
f(M) &= f(M) \quad \text{(vi)}
\end{align*}
\]

Whenever \( f \) is home & \( M^0 \) is a subset of X, then by (ii) and (iii), we get that:

\[
f(M) = f(M) \quad \text{(vii)}
\]

And by (i) and (iv) we get that \( f(M) \subseteq f(F) \subseteq f(M) \), so \( f(F) \) is an \( \alpha \)-closed in Y.

**Definition [5]:** Let \( f: X \to Y \) be a function of a space, then:

(i) \( f \) is called a continuous function if \( f^{-1}(A) \) is an open set in X for every open set A in Y.

(ii) \( f \) is called an \( \alpha \)-open function if \( f(A) \) is an open set in Y, for every \( \alpha \)-open set A in X.

**Definition [4]:** Let \( f: X \to Y \) be a function of a space, then \( f \) is called \( \alpha \)-continuous function, if \( f^{-1}(A) \) is an \( \alpha \)-open set in X for every open set A in Y.

**Definitions:** Let \( f: X \to Y \) be a function of a space, then:

(i) \( f \) is called an \( \alpha \)-open function if \( f(A) \) is an \( \alpha \)-open in Y, for every open set A in X.

(ii) \( f \) is called an \( \alpha^* \)-open function, if \( f(A) \) is an open in Y, for every \( \alpha \)-open set A in X.

(iii) \( f \) is called an \( \alpha^* \)-open function, if \( f(A) \) is an \( \alpha \)-open in Y, for every \( \alpha \)-open set A in X.

(iv) \( f \) is called an \( \alpha \)-closed function if \( f(A) \) is an \( \alpha \)-closed in Y, for every \( \alpha \)-closed set A in X.

(v) \( f \) is called an \( \alpha^* \)-closed function, if \( f(A) \) is an \( \alpha \)-closed in Y, for every \( \alpha \)-closed set A in X.

(vi) \( f \) is called an \( \alpha^* \)-closed function, if \( f(A) \) is an \( \alpha \)-closed in Y, for every \( \alpha \)-closed set A in X.

(vii) \( f \) is called \( \alpha^* \)-continuous function if \( f^{-1}(A) \) is an open set in X for every \( \alpha \)-open set A in Y.

(viii) \( f \) is \( \alpha^* \)-continuous function if \( f^{-1}(A) \) is an \( \alpha \)-open set in X for every \( \alpha \)-open set A in Y.

**Definition:** A space X is called \((\alpha K)\)-c-space, if every \( \alpha \)-compact subset of X is closed.

**Example:** the discrete space \((X, \tau_D)\) satisfy the above definition.

**Theorem:** Let \( f: X \to Y \) be bijective open and \( \alpha^* \)-closed (\( \alpha^* \)-closed) function, then Y is \((\alpha K)\)-c-space, whenever X is \((\alpha K)\)-c-space.

**Proof:** Let A be \( \alpha \)-compact in Y, to prove that \( f^{-1}(A) \) is \( \alpha \)-compact in X, let \( \{U_\alpha\}_{\alpha \in \mathcal{A}} \) be an \( \alpha \)-open cover to \( f^{-1}(A) \) (means that)

\[
f^{-1}(A) = \bigcup_{\alpha \in \mathcal{A}} U_\alpha
\]

Therefore

\[
f^{-1}(A) = f(U_{\alpha \in \mathcal{A}}) = U_{\alpha \in \mathcal{A}} f(U_\alpha) \quad \text{and hence} \quad A \subseteq U_{\alpha \in \mathcal{A}} f(U_\alpha),
\]

since \( f(U_\alpha) \) is \( \alpha \)-open in Y \( \forall \alpha \in \Lambda \) therefore but A is \( \alpha \)-compact, so:

\[
A \subseteq \bigcup_{n=1}^{\infty} f(U_{\alpha_i}) \implies f^{-1}(A) \subseteq f^{-1} \bigcup_{n=1}^{\infty} f(U_{\alpha_i}) = \bigcup_{n=1}^{\infty} f^{-1}(U_{\alpha_i})
\]

Therefore

\[
f^{-1}(A) \subseteq \bigcup_{n=1}^{\infty} U_{\alpha_i}, \text{so } f^{-1}(A) \text{ is } \alpha \text{-compact in } X, \text{ which is } (\alpha K)\text{-c-space, so } f^{-1}(A) \text{ is closed } \implies f \left( f^{-1}(A) \right) = A \text{ is closed in } Y, \text{ which implies } X \text{ is } (\alpha K)\text{-c-space}. \blacksquare
\]

**Proposition:** Let \( f: X \to Y \) be bijective open and \( \alpha^* \)-closed (\( \alpha^* \)-closed) function, then Y is \((K(\alpha c))\)-c-space, whenever X is \((\alpha c)\)-c-space.

**Proof:** Let A be compact in Y, to prove that \( f^{-1}(A) \) is compact in X, let \( \{U_\alpha\}_{\alpha \in \mathcal{A}} \) be an open cover to \( f^{-1}(A) \) (means that)

\[
f^{-1}(A) = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \quad \implies f^{-1}(A) = \bigcup_{\alpha \in \mathcal{A}} f(U_\alpha) \quad \Rightarrow \quad A \subseteq \bigcup_{\alpha \in \mathcal{A}} f(U_\alpha),
\]

since \( f(U_\alpha) \) is open in Y \( \forall \alpha \in \Lambda \) therefore but A is compact, so

\[
A \subseteq \bigcup_{n=1}^{\infty} f(U_{\alpha_i}) \quad \implies f^{-1}(A) \subseteq f^{-1} \bigcup_{n=1}^{\infty} f(U_{\alpha_i}) = \bigcup_{n=1}^{\infty} f^{-1}(U_{\alpha_i})
\]

Therefore

\[
f^{-1}(A) \subseteq \bigcup_{n=1}^{\infty} U_{\alpha_i}, \text{so } f^{-1}(A) \text{ is } \alpha \text{-compact in } X, \text{ which is } (K(\alpha c))\text{-c-space, so } f^{-1}(A) \text{ is closed } \implies f \left( f^{-1}(A) \right) = A \text{ is closed in } Y, \text{ which implies it is } \alpha \text{-closed in } Y, \text{ which A is compact}
\]

Hence it is \((K(\alpha c))\)-c-space. \blacksquare
Proposition: Let \( f: X \to Y \) be bijective \( \alpha^* \)-open and \( \alpha^{**} \)-closed function, then \( Y \) is \( K(\alpha c) \)-space, whenever \( X \) is \( K(\alpha c) \)-space.

Proof: Let \( A \) be \( \alpha \)-compact in \( Y \), to prove that \( f^{-1}(A) \) is \( \alpha \)-compact in \( X \), let \( \{ U_{\alpha} \}_{\alpha \in \Lambda} \) be an \( \alpha \)-open cover to \( f^{-1}(A) \) (means that) \( f^{-1}(A) = \bigcup_{\alpha \in \Lambda} U_{\alpha} \Rightarrow f(f^{-1}(A)) = f(U_{\alpha} \in \Lambda) = U_{\alpha \in \Lambda} f(U_{\alpha}) \Rightarrow A \subseteq U_{\alpha \in \Lambda} f(U_{\alpha}) \). Since \( f(U_{\alpha}) \) is \( \alpha \)-open in \( Y \) \( \forall \alpha \in \Lambda \) \( \Rightarrow \) but \( A \) is \( \alpha \)-compact, so \( A \subseteq U_{i=1}^n f(U_{\alpha_i}) = U_{i=1}^n f^{-1}(U_{\alpha_i}) = U_{i=1}^n U_{\alpha_i} \) (since \( f \) is one to one) \( \Rightarrow f^{-1}(A) \) \( \subseteq U_{i=1}^n U_{\alpha_i} \), so \( f^{-1}(A) \) is \( \alpha \)-compact in \( X \), which is \( K(\alpha c) \)-space, so \( f^{-1}(A) \) is \( \alpha \)-closed, \( \text{sof}(f^{-1}(A)) = A \) is \( \alpha \)-closed in \( Y \), which \( A \) is \( \alpha \)-compact, then \( X \) is \( K(\alpha c) \)-space. ■

Proposition: The \( \alpha^* \)-continuous image and serjective of \( \alpha \)-compact (\( \alpha \)-Lindelof) set is \( \alpha \)-compact (\( \alpha \)-Lindelof).

Proof: Let \( \{ U_{\alpha} \}_{\alpha \in \Lambda} \) be an \( \alpha \)-open cover for \( Y \) (means that) \( Y = U_{\alpha \in \Lambda} U_{\alpha} \) but \( f \) is continuous, so \( f^{-1}(U_{\alpha}) \) is \( \alpha \)-open in \( X \) \( \forall \alpha \in \Lambda \)

\[
f^{-1}(Y) = f^{-1}(U_{\alpha \in \Lambda} U_{\alpha}) = U_{\alpha \in \Lambda} f^{-1}(U_{\alpha}) \text{ also } f^{-1}(Y) = X \Rightarrow X = U_{\alpha \in \Lambda} f^{-1}(U_{\alpha}) \Rightarrow \{ f^{-1}(U_{\alpha}) \}_{\alpha \in \Lambda} \text{ is an } \alpha \)-open cover for \( X \), which is \( \alpha \)-compact \\
\Rightarrow \text{there exists } \alpha_1, \ldots, \alpha_n \in \Lambda, \text{ such that } X = U_{i=1}^n f^{-1}(U_{\alpha_i})
\]

\[
f^{-1}(U_{i=1}^n(U_{\alpha_i})) \text{ therefore } f(X) = f(f^{-1}(U_{i=1}^n(U_{\alpha_i}))) = U_{i=1}^n(U_{\alpha_i}) \text{ (since } f \text{ is on to) and hence } f(X) = Y \Rightarrow Y = U_{i=1}^n U_{\alpha_i} \text{ which implies } Y \text{ is } \alpha \text{ – compact. ■}

Proposition: Every finite space is \( \alpha \)-compact.

Proof: Let \((X, \tau)\) be a finite topological space say \( X = \{ a_1, \ldots, a_n \} \), to prove that \( X \) is \( \alpha \)-compact, let \( \{ U_{\alpha} \}_{\alpha \in \Lambda} \) be an \( \alpha \)-open cover to \( X \) (means that) \( X = U_{\alpha \in \Lambda} U_{\alpha} \) since \( a_i \in X = U_{\alpha \in \Lambda} U_{\alpha}, X = U_{i=1}^n(a_i) = U_{i=1}^n U_{\alpha_i} \to X \) is \( \alpha \)-compact. ■

Proposition: The property of being a \( K(\alpha c) \)-space is a topological property.

Proof: Let \( f: X \to Y \) be a homeomorphism function from an \( K(\alpha c) \)-space \( X \) in to a space \( Y \) to show that \( Y \) is also \( K(\alpha c) \)-space

Let \( A \) be a compact subset of \( Y \), but \( f^{-1}(A) \) is also compact of \( X \), which is \( K(\alpha c) \)-space, then \( f^{-1}(A) \) is \( \alpha \)-closed in \( X \), since \( f \) is onto, then \( A = f(f^{-1}(A)) \), then \( A \) is \( \alpha \)-closed of \( Y \), therefore \( Y \) is an \( K(\alpha c) \)-space.

References:


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