Homotopy Perturbation Method for Solving Generalized Riccati Differential Equation

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Abstract

In this work, we have studied a class of nonlinear Riccati differential equation which is used as mathematical models in many physically significant fields and applied science. The homotopy perturbation method, has been modified for solving generalized linear and nonlinear Riccati differential equation, which is a second-order ordinary differential equation. Also, we have tested the modified homotopy perturbation method on the solving of different implementations which are show the efficiency and accuracy of the proposed method. The approximated solutions are agreed well with analytical solutions for the tested problems. [DOI: 10.22401/ANJS.00.2.13]

Keyword: Riccati differential equation; Homotopy perturbation method; HPM.

1. Introduction

The most important mathematical models for physical phenomena is the differential equation. Motion of objects, Fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations. Moreover, numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of differential equations (DEs), particularly ODEs of different orders, can be found in the mathematical modeling of real life problems (Mechee et al. (2014)). Riccati differential equation is the famous first-order nonlinear differential equation with quadratic nonlinearity. Obviously, this was the reason that as soon as Newton invented differential equations, Riccati differential equation was the first one to be investigated extensively since the end of the 17th century. Some of the properties of Riccati differential equation which have been studied, are in the frame of local theory of differential equations. The homotopy perturbation method (HPM) is efficient technique to find the approximate solutions for ordinary and partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. HPM was proposed by Ji-Huan He in 1999 for solving linear and nonlinear differential equations and integral equations. Many researchers used HPM to approximate the solutions of differential equations and integral equations (Yıldırım (2010), Jalaal et al. (2010) & Ma et al. (2008)).

Many researchers published some papers in solving some classes differential equations using HPM. For example, Chun & Sakthivel (2010) used HPM for solving a linear and non-linear second-order two-point boundary value problems while Gülkaç (2010) was solved the Black-Scholes equation for a simple European option in this method to obtain a new efficient recurrent relation to solve Black-Scholes equation.

Moreover, numerous researches used HPM for solving nonlinear differential equations, Vahidi et al. (2011) was solved non-linear DEs, which yields the Maclaurin series of the exact solution, Chang & Liou (2006) developed a third-order explicit approximation to find the roots of the dispersion relation for water waves that propagate over dissipative media, Zhou &Wu (2012) solved the non-linear PB equation describing spherical and planar colloidal particles immersed in an arbitrary valence and mixed electrolyte solution, Öziş & Akçı (2011) solved certain non-linear, non-smooth oscillators, Yazdi (2013) solved nonlinear vibration analysis of functionally graded plate while He & Huan (2004) applied HPM for solving nonlinear oscillators with discontinuities, nonlinear Duffing equation and some nonlinear ODEs. For class of linear partial differential equations, Al-Saif & Abood (2011) solved the Kortewegde Vries (KdV) equation and convergence study of HPM, Babolian et al. (2009) used HPM to solve time-dependent differential equations, Aswhad & Jaddoa (2016) solved advection Prob- lem, vibrating beam equation linear and nonlinear PDEs and the system of nonlinear PDEs and Babolian et al. (2009) used HPM to solve time-dependent differential. Also, many researchers used HPM for solving the class of non-linear PDEs, Yazdi (2013) was approximated solution for free nonlinear vibration of thin rectangular laminated FGM plates, Liao (2004) has solved nonlinear PDEs, Yildirim (2009) was used to implement the nonlinear Korteweg-de Vries equation, Md Nasrudin et al. (2014) combined HPM-Padé approximant to acquire the approximate analytical solution of the KdV equation, Taghipour (2010) solved Parabolic equations and Periodic equation linear and nonlinear PDEs, Janalizadeh et al. (2008) obtained the solution of a second-order non-linear wave equation, Fereidoon et al. (2011) utilized to derive approximate explicit analytical solution for the nonlinear foam drainage equation, Momani & Odibat (2007) modified the algorithm which provides approximate solutions in the form of convergent series with easily computable components, Babolian et al. (2009) solved time-dependent differential equations while He (2000), solved non-linear problems using the homotopy technique. However, for the system of DEs, Bataineh et al. (2009) solved systems of second-order BVPs, Javidi (2009) solved SEIR model, Wang & Song (2007) solution of a model for HIV infection of $CD4 + T$ cells, Rafei et al. (2007) solution of the system of nonlinear ordinary differential equations governing on the problem, Noor et al. (2013) solved the system of linear equations. Noor & Mohyud-Din (2008) solution of linear and non-linear sixth-order boundary value problems and system of differential equations, Javidi (2009) solved system of linear Fredholm integral equations (LFIEs). Yusufoğlu (2009) has solved a linear Fredholm type integro-differential equations with separable kernel. Javidi (2009) solved non-linear Fredholm integral equations, Saberi-Nadjafi & Tamamgar (2008) used modified HPM for solving the system linear and nonlinear of Volterra integral equations, Kumar et al. (2011) solved generalized Abel integral equation. For the differential equations of fractional type, Odibat & Momani (2008) solved nonlinear differential equations of fractional order, Jafari et al. (2010) solved non- linear problems of fractional Riccati differential equation & Yildirim & Agirseven (2009) solved the space-time fractional advection-dispersion equation. Lastly, some researchers published papers for solving Ricatti equation, File & Aga (2016) introduced the classical fourth order Runge-Kutta method (RK4) for solving the numerical solution of the quadratic Riccati differential equations. Abbasbandy (2007) solved the quadratic Riccati differential equation by He's variational iteration method with considering Adomian's polynomials.

Recently, we have studied approximated solutions of a wide class of nonlinear Riccati differential equation which is used as mathematical models in many physically significant fields and applied science. The approximated solutions of this class of differential equations have studied using modified HPM. The proposed method applied for solving different examples for this class of ordinary differential equations (ODEs). A comparison was made between the analytical and approximated solutions for different tested problems which showed the proposed method is more efficient in the iterations complexity and high accurate in the absolute errors. It has been highlighted that the use of modified HPM is more suitable to approximate the solutions of general Liénard and Duffing differential equations with considering the general coefficients functions.

2. Preliminaries

In this paper, we consider the Riccati differential equation as follow (File & Aga (2016), Ghomanjani & Khorram (2015)):

$$y'(t) = q_0(t)+q_1(t)y(t)+q_2(t)y^2(t),\ t_0 \leq t \leq t_e \ldots(1)$$

with initial condition:

$$y(t_0) = \alpha \qquad \ldots(2)$$

where $q_0(t)$, $q_1(t)$ and $q_2(t)$ are continuous functions and t_0, te and α are an arbitrary constants. If $q_0(t) = 0$ the equation reduces to a Bernoulli equation, while if $q_2(t) = 0$ the equation becomes first-order linear ODE. The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The fundamental theories of Riccati differential equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics, the solution of this equation can be reached using HPM to solve the nonlinear Riccati in an analytic form. Also, Riccati differential equation is an important equation in the optimal control literature. Solution of this equation can be obtained using classical numerical method as Runge-Kutta method or the forward Euler method (Batiha (2015), Abbasbandy (2006b)).

2.1 Homotopy Perturbation Method:

In this section, we present a brief description of the HPM, to illustrate the basic ideas of this method, we consider the following differential equation (Neamaty & Darzi (2010), Chun & Sakthivel (2010), Batiha (2015) & Abbasbandy (2006b)):

$$A(u) - f(\tau) = 0,\ \tau \in \partial\Omega \quad \dots(3)$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial \tau}\right),\ \tau \in \partial\Omega \quad \dots(4)$$

where A is general differential operator, B is a boundary operator, $f(\tau)$ a known analytic function and $\partial\Omega$ is the boundary of the domain Ω. The operator A can be generally divided into two parts of L and N where L is linear part, while N is the nonlinear part in the differential equations. Therefore Eq.(3) can be rewritten as follows (He (1999)):

$$L(u) + N(u) - f(\tau) = 0 \quad \dots(5)$$

By using homotopy technique, one can construct a homotopy $V(\tau,p):\Omega\times[0,1] \longrightarrow R$, which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(\tau)] = 0 \quad \dots(6)$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0 + p[N(v) - f(\tau)]) = 0 \quad \dots(7)$$

where p [0, 1], $\tau\Omega$ & p is called homotopy parameter and u_0 is an initial approximation for the solution of equation (3) which satisfies the boundary conditions obviously, Using equation (6) or (7), we have the following equation:

$$H(v,0)= L(v)-L(u0)=0 \quad \dots(8)$$

and

$$H(v,1) = L(v) + N(v)-f(\tau) = 0 \quad \dots(9)$$

Assume that the solution of (6) or (7) can be expressed as a series in p as follows:

$$V = v_0+pv_1+p^2v_2+p^3v_3+ \dots = \sum_{i=0}^{\infty} p^i v_i \quad \dots(10)$$

set $p = 1$, resulting the approximate solution of (3). Consequently:

$$u(\tau) = \lim_{p\to 1} V = v_0+v_2+v_3 + \dots = \sum_{i=0}^{\infty} v_i \quad \dots(11)$$

It is worth to note that the major advantage of He's method is that the perturbation equation can be freely constructed in many ways and approximation can also be freely selected.

3. Analysis of the HPM for Solving Riccati Differential Equation

In this section, we will present a review of the HPM to solve Riccati differential equations, we consider the following equation (File & Aga (2016)):

$$y'(t) = q_0(t) + q_1(t)y(t) + q_2(t)y^2(t) \quad \dots(12)$$

with initial condition:

$$y(t_0) = \alpha \quad \dots(13)$$

where $t_0 \le t \le t_e$, $y(t) \in R$ and α is an arbitrary constant.

The implementation of the proposed modified HPM for solving Riccati differential Equation (12) according the following algorithm:

3.1 The Proposed Method:

Firstly, we start with the initial approximation $y_0(t) = \alpha$.

Secondly, we can construct a homotopy for the second-order ODE (12) as follows:

$H(y(t),p) = (1-p)\,(y'(t) - \dot{y}_0(t)) + p(y'(t) - q_0(t) - q_1(t)y(t) - q_2(t)y^2(t)) = 0$ …(14)

Thirdly, using Taylor expansion about $t = 0$, by substituting the Taylor expansions for the coefficients functions. However:

$H(y(t),p) = (1-p)(\sum_{i=0}^{\infty} p^i \dot{y}_i(t) - \dot{y}_0(t)) + p(\sum_{i=0}^{\infty} p^i \dot{y}_i(t) - q_0(t) - q_1(t)\sum_{i=0}^{\infty} p^i y_i(t)$
$q_2(t)\sum_{i=0}^{\infty} p^i y_i(t)) = 0$ …(15)

Fourthly, suppose that the solution of equation (15) is in the form:

$P^0 : \dot{y}_0(t) - \dot{y}_0(t) = 0$ …(17)

$P^1 : \dot{y}_1(t) + \dot{y}_0(t) - q_0(t) - q_1(t)y_0(t) - q_2(t)y_0^2(t) = 0$

$P^2 : \dot{y}_2(t) - q_1(t)y_1(t) - q_2(t)(2y_0(t)y_1(t)) = 0,$

$P^3 : \dot{y}_3(t) - q_1(t)y_2(t) - q_2(t)(2y_0(t)y_2(t) + y_0^2(t)) = 0,$

$P^4 : \dot{y}_4(t) - q_1(t)y_3(t) - q_2(t)(2y_0(t)y_3(t) + 2y_1(t)y_2(t)) = 0,$

$P^5 : \dot{y}_5(t) - q_1(t)y_4(t) - q_2(t)(2y_0(t)y_4(t) + 2y_1(t)y_3(t) + y_0^2(t) = 0$

Hence, for $n = 2, 3, 4,\ldots$; we have:

$p^n : \dot{y}(t) - q_1(t)\, y_{n-1}(t) - q_2(t) \sum_{i=0}^{n-1} y_i(t) y_{n-i-1}(t) = 0$

Finally, using the equations (17) with some simplifications, we get the following terms of the solution:

$y_0(t) = \alpha$

$y_1(t) = \int(-\dot{y}_0(t) + q_0(t) + q_1(t)y_0(t) + q_2(t)y_0^2(t))dt$

$y_2(t) = \int(q_1(t)y_1(t) - q_2(t)(2y_0(t)y_1(t))\,dt$

$y_3(t) = \int(q_1(t)y_2(t) - q_2(t)\big(2y_0(t)y_2(t)\big) + y_0^2(t))\,dt$

$y_4(t) = \int(q_1(t)y_3(t) + q_2(t)\big(2y_0(t)y_3(t) + 2y_1(t)y_2(t)\big)\,dt$

and

$y_5(t) = \int(q_2(t)2y_0(t)y_4(t) + 2y_1(t)\big(y_3(t)\big) + y_0^2(t)\,dt$

Hence, the general term has the following form:

$y_n(t) = \int(q_1(t)y_{n-1}(t) + q_2(t)\sum_{i=0}^{n-1} y_i(t)y_{n-i-1}(t))dt, n = 2,3,\ldots$

Then the solution of equation (12) is:

$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \ldots$ …(19)

4. Implementations

In order to assess the accuracy of the solving generalized Riccati differential equation by HPM we will introduce some different examples in general and to compare the approximated solution with the exact solutions for these problems, we will consider the following five problems.

4.1 Problem 1:

Consider the following ODE (Ghomanjani & Khorram (2015)):

$y'(t) - y(t) + y^2(t) = 0,\ 0 \le t \le 1$ …(20)

subject to the initial condition:

$y(0) = \lambda$

with the exact solution:

$y(t) = \frac{1}{ce^t+1}$

Comparing equation (20), we have $q_0(t) = 0$, $q_1(t) = 1$, and $q_2(t) = 1$

The initial approximation has the form $y_0(t) = \lambda$ substituting (18) into (20), we have:

$y_1(t) = \int(-\dot{y}_0(t) - y_0(t) + y_0^2(t))dt$ …(21)
$= \lambda(\lambda - 1)t$

and

$y_n(t) = \int(-y_{n-1}(t) + \sum_{i=0}^{n-1} y_i y_{n-i-1}(t))dt$, …(22)

for $k = 2, 3, \ldots$. So, simplification of equation (22) lends to the following solutions:

$y_2(t) = \frac{1}{2}\lambda t^2 (2\lambda-1)(\lambda-1),$

$y_3(t) = (-\frac{1}{3}\lambda^4 + \frac{1}{2}\lambda^2 - \frac{1}{6}\lambda)\tau^3$

$y_4(t) = \frac{1}{24}\lambda t^4(2\lambda-1)(\lambda-1)(4\lambda^2-8\lambda + 1),$

$y_5(t) = \frac{1}{120}\lambda t^5(\lambda- 1)(24\lambda^4 - 96\lambda^3 + 94\lambda^2 - 26\lambda+ 1),$

$y_6(t) = \frac{1}{720}\lambda t^6 (2\lambda-1)(\lambda-1)(24\lambda^4 + 24\lambda^3 - 56\lambda^2 + 4\lambda+1)$

Then, the general solution of equation (20) is written as follows:

$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + \ldots$
$= \lambda + \lambda(\lambda- 1)t + \frac{1}{2}\lambda t^2(2\lambda- 1)(\lambda-1) + (-\frac{1}{3}\lambda^4 + \frac{1}{2}\lambda^2 - \frac{1}{2}\lambda t)\ \frac{1}{24}\lambda t^4(2\lambda-1)\ (\lambda-1)(4\lambda^2-8\lambda+1) - \frac{1}{120}\lambda t^5(\lambda-1)(24\lambda^4 - 96\lambda^3+94\lambda^2-26\lambda+1) + \frac{1}{720}\lambda t^6(2\lambda-1)(\lambda-1)(24\lambda^4+24\lambda^3-56\lambda^2+4\lambda+1) + \ldots$

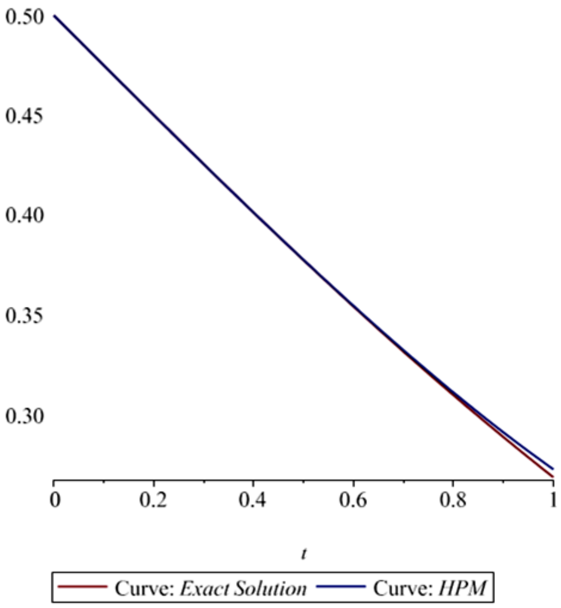

Figure 1: Comparison of proposed HPM and exact solutions of Problem 1 at $c = 2, \lambda = 1$.

4.2 Problem 2:

Consider the following nonlinear first-order ODE (Abbasbandy (2007), Abbasbandy (2006a) & Abbasbandy(2006b)):

$y'(t) = 1 + 2y(t) - y^2(t), -1 \leq t \leq 1$ …(23)

subject to the initial condition:

$y(0) = \alpha$

with the exact solution:

$y(t) = 1 + \sqrt{2} tanh\, (\sqrt{2}t + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})$

Comparing equation (23) we have

$q_0(t) = 1$, $q_1(t) = 2$ & $q_2(t) = -1$

The initial approximation has the form

$y_0(t) = \alpha$ ...(24)

substituting (18) into (23), we have

$y_1(t) = \int(-\dot{y}_0(t) + 1 + 2y_0(t) - y_0^2(t))dt$ ...(25)

$= (-\alpha^2 + 2\alpha + 1)t$

and

$y_n(t) = \int(2y_{n-1}(t) \sum_{i=0}^{n-1} yiy_{n-i-1}(t))dt$ ...(26)

for $k = 2, 3, \ldots$ So, simplification of equation (26) lends to the following solutions:

$y_2(t) = t^2(\alpha-1)(\alpha^2-2\alpha-1)$

$y_3(t) = -\frac{1}{3}t^3(3\alpha^2-6\alpha+1)(\alpha^2-2\alpha-1)$

$y_4(t) = \frac{1}{3}t^4(\alpha-1)(3\alpha^2-6\alpha-1)(\alpha^2-2\alpha-1)$

$y_5(t) = \frac{1}{15}t^5(\alpha^2-2\alpha-1)(15\alpha^4-60\alpha^3+60\alpha^2-7)$

$y_6(t) = \frac{1}{45}t^6(\alpha-1)(\alpha^2-2\alpha-1)(45\alpha^4-180\alpha^3+150\alpha^2+60\alpha-7)$

$y_7(t) = -\frac{1}{315}t^7(\alpha^2 - 2\alpha - 1)(315\alpha^6 - 1890\alpha^5 + 3675\alpha^4 - 2100\alpha^3 - 651\alpha^2 + 462\alpha + 53)$,

$y_8(t) = \frac{1}{315}t^8(\alpha-1)(\alpha^2-2\alpha-1)(315\alpha^6-1890\alpha^5+3465\alpha^4-2160\alpha^3-1323\alpha^2)+126\alpha+71)$

Hence, the general solution of equation (23) is given as follow:

$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + y_7(t) + y_8(t) + \ldots$

$= \alpha + (-\alpha^2 + 2\alpha + 1)t + t^2(\alpha - 1)(\alpha^2 - 2\alpha - 1) - \frac{1}{3}t^3(3\alpha^2 - 6\alpha + 1)(\alpha^2 - 2\alpha$□$- 1\frac{1}{3}$□□□$t^4$ $(\alpha-1)(3\alpha^2-6\alpha-1)(\alpha^2-2\alpha-1) - \frac{1}{15}t^5(\alpha^2-2\alpha-1)(15\alpha^4-60\alpha^3+60\alpha^2-7) + \frac{1}{45}t^6(\alpha-1)(\alpha^2-2\alpha-1)(45\alpha^4-180\alpha^3+150\alpha^2+60\alpha-7) - \frac{1}{315}t^7(\alpha^2-1)(315\alpha^6-1890\alpha^5+3675\alpha^4-2100\alpha^3-651\alpha^2+462\alpha+53) + \frac{1}{315}t^8(\alpha-1)(\alpha^2-2\alpha-1)(315\alpha^6-1890\alpha^5+3465\alpha^4-2160\alpha^3-1323\alpha^2+126\alpha+71)\ldots$

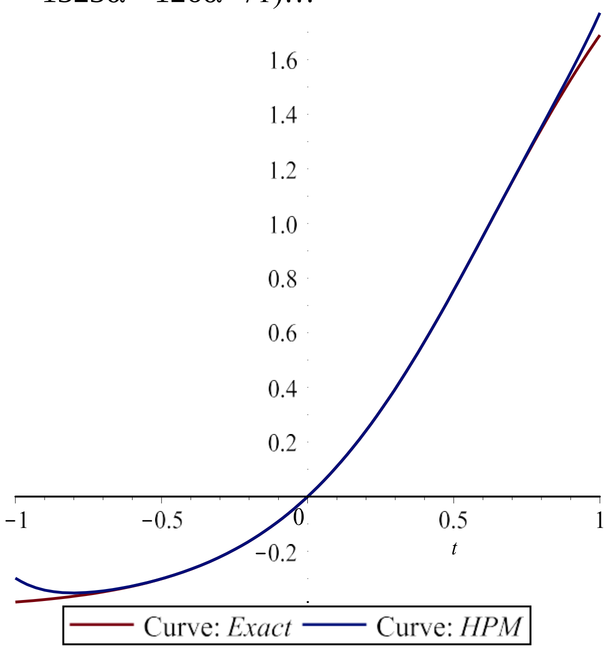

Figure 2: Comparison of proposed HPM and exact solutions of Problem2 at $\alpha = 0$.

4.3 Problem 3:

Consider the following ODE:(File & Aga

(2016))(Ghomanjani & Khorram (2015))

$\dot{u}(t) = e^t - e^{3t} + 2e^{2t}u(t) - e^t u^2(t),\ -5 \le t \le -1$ …(27)

subject to the initial condition $u(0)=a$, with the exact solution $u(t) = e^t$.

Comparing equation (27), we have:

$q_0(t) = e^t - e^{3t}$, $q_1(t) = 2e^{2t}$ and $q_2(t) = -e^t$

The initial approximation has the form $u_0(t) = e^t$ Substituting (18) into (27), we have:

$$u_1(t) = \int(-\dot{u}_0(t) + e^t - e^{3t} + 2e^{2t}u_0(t) - e^t u_0^2(t))dt$$
$$= e^t - \frac{1}{3}e^{3t} + ae^{2t} - a^2e^t \quad \text{...(28)}$$

and

$$u_n(t) = \int(2e^{2t}y_{n-1}(t) - e^t \sum_{i=0}^{n-1} u_i(t)u_{n-i-1}(t))dt,\ \text{k=2,3…} \quad \text{…(29)}$$

So, simplification of equation (29) leads to the following solutions:

$$u_2(t) = \frac{2}{3}e^{3t} - \frac{2}{15}e^{5t} + \frac{2}{3}ae^{4t} - \frac{4}{3}a^2e^{3t} - ae^{2t} + a^3e^{2t}$$

$$u_3(t) = \frac{8}{15}e^{5t} - \frac{22}{315}e^{7t} + \frac{22}{45}ae^{6t} - \frac{22}{15}a^2e^{5t} - \frac{11}{6}ae^{4t} + \frac{13}{6}a^3e^{4t}$$

$$u_4(t) = -\frac{8}{15}e^{5t} + \frac{16}{63}e^{7t} - \frac{8}{315}e^{9t} + \frac{5}{6}ae^{4t} - \frac{121}{90}ae^{6t} + \frac{41}{15}a^2e^{5t} - 2a^3e^{4t} + \frac{7}{6}a^5e^{4t} + \frac{8}{35}ae^{8t} + \frac{179}{90}a^3e^{6t} - \frac{32}{35}a^2e^{7t} - \frac{5}{6}a^4e^{5t}$$

$$u_5(t) = \frac{4}{15}e^{5t} - \frac{136}{315}e^{7t} + \frac{124}{945}e^{9t} - \frac{544}{51975}e^{11t} + \frac{68}{45}ae^{6t} - \frac{8}{5}a^2e^{5t} - \frac{20}{21}ae^{8t} - \frac{191}{45}a^3e^{6t} + \frac{304}{105}a^2e^{7t} + \frac{138}{15}a^4e^{5t} + \frac{544}{4725}ae^{10t} - \frac{544}{945}a^2e^{9t} + \frac{104}{63}a^3e^{8t} - \frac{128}{45}a^4e^{7t} + \frac{25}{9}a^5e^{6t} - \frac{6}{5}a^6e^5$$

Therefore, general solution of equation (27) is written, as follows:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + \ldots$$
$$= -\frac{1}{3}e^{3t} + e^t + \frac{2}{3}ae^{5t} - \frac{26}{105}e^{7t} + \frac{20}{189}e^{9t} - \frac{22}{315} - \frac{544}{51975}e^{11t} - a^2e^t + \frac{2}{3}a^2e^{3t} + a^3e^{2t} - \frac{1}{3}ae^{4t} + \frac{59}{90}ae^{6t}\frac{1}{3}a^2e^{5t} - \frac{1}{6}a^3e^{4t} - \frac{4}{3}a^4e^{3t} + \frac{7}{6}a^5e^{4t} - 2a^3e^{4t} + \frac{7}{6}a^5e^{4t} + \frac{8}{35}ae^{8t} + \frac{179}{90}a^3e^{6t} - \frac{32}{35}a^2e^{7t} - \frac{7}{3}a^4e^{5t} - \frac{76}{105}ae^{8t} - \frac{203}{90}a^3e^{6t} + \frac{208}{105}a^2e^{7t} + \frac{1}{5}a^4e^{5t} + \frac{544}{4725}ae^{10t}\frac{544}{945}a^2e^{9t} + \frac{104}{63}a^3e^{8t} - \frac{128}{45}a^4e^{7t} + \frac{25}{9}a^5e^{6t} - \frac{6}{5}a^6e^{5t} + \ldots$$

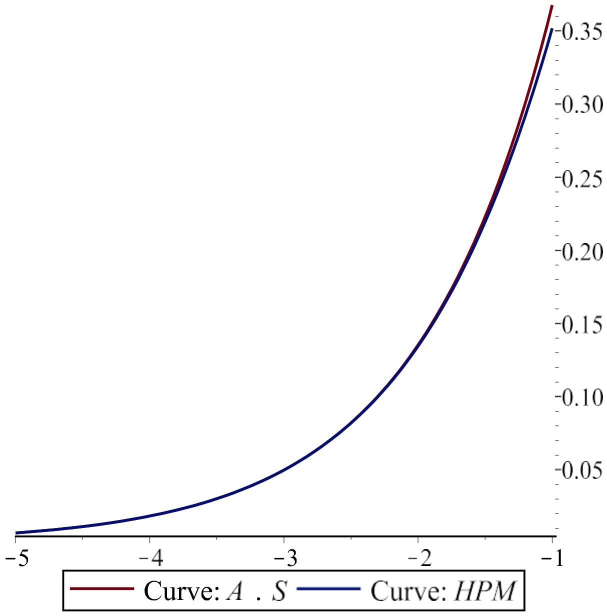

Figure 3: Comparison of proposed HPM and exact solutions of Problem3 at c = 2, λ = 1 and a = 0.5.

4.4 Problem 4:

Consider the following ODE:(File & Aga (2016), Ghomanjani & Khorram (2015))

$y'(t) = 16t^2 - 5 + 8ty(t) + y^2(t),\ -0.5 \le t \le 1$ …(30)

subject to the initial condition $y(0)=1$ with the analytic solution:

$$y(t) = \frac{(4ce^{-2t}-4)t + ce^{-2t} - 1}{ce^{-2t} - 1}$$

Comparing equation(30), we have $q_0(t) = 16t^2 - 5$, $q_1(t) = 8t$ *and* $q_2(t) = 1$.

The initial approximation has the form $y_0(t) = 1$. Substituting (18) into (30), we have:

$$y_1(t) = \int(-\dot{y}_0(t) + 16t^2 - 5 + 8ty_0(t) + y_0^2(t))dt$$
$$= \frac{16}{3}t^3 + 4t^2 - 4t \quad \text{…(31)}$$

and

$$y_n(t) = \int(8ty_{n-1}(t) + \sum_{i=0}^{n-1} y_i y_{n-i-1}(t))dt;\ k=2,3,\ldots \quad \text{…(32)}$$

So, simplification of equation (32) lends to the following solutions:

$$y_2(t) = \frac{128}{15}t^5 + \frac{32}{3}t^4 - 8t^3 - 4t^2$$

$$y_3(t) = \frac{4352}{315}t^7 + \frac{1088}{45}t^6 - \frac{208}{15}t^5 - 20t^4 + \frac{8}{3}t^3$$

Then, the general solution of equation (30) is written as follow:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + \ldots$$
$$= 1 - 4t + \frac{28}{3}t^4 - \frac{16}{3}t^5 + \frac{4352}{315}t^7 + \frac{1088}{45}t^6 + \ldots$$

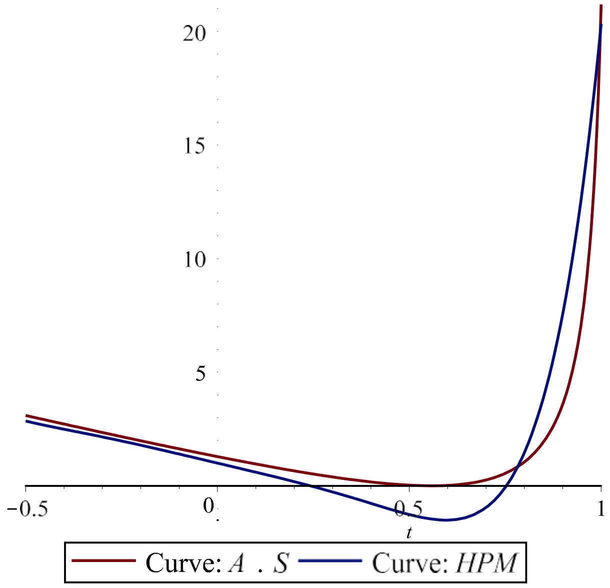

Figure 4: Comparison of proposed HPM and exact solutions of Problem4 at c = 2, λ = 14, and c = 8.

4.5 Problem 5:

Consider the following ODE:

$$y'(t) = t^2 + y^2(t) - 1, \ -4 \le t \le 4 \qquad \ldots(33)$$

subject to the initial condition $y(0) = □$.

Comparing equation (33), we have:

$q_0(t) = t^2 - 1$, $q_1(t) = 0$ and $q_2(t) = 1$

The initial approximation has the form, $y_0(t) = \omega$.

Substituting (18) into (33), we have:

$$y_1(t) = \int(-\dot{y}_0(t) + t^2 - 1 + y_0(t))dt$$

$$= \frac{1}{3}t^3 + (\omega^2 - 1)t \qquad \ldots(34)$$

and

$$y_n(t) = \left(\int \sum_{i=0}^{n=1} y_i y_{n-i-1}(t)\right)dt; k=2,3,\ldots \qquad \ldots(35)$$

So, simplification of equation (35) lends to the following solutions:

$$y_2(t) = \frac{1}{6}\omega t^4 + (\omega^3 - \omega)t^2$$

$$y_3(t) = \frac{1}{63}t^7 + \left(\frac{1}{5}\omega^2 - \frac{2}{15}\right)t^5 + \left(\omega^4 - \frac{4}{3} + \frac{1}{3}\right)t^3$$

$$y_4(t) = \frac{1}{56}\omega t^8 + \left(\frac{7}{30}\omega^3 - \frac{19}{90}\omega\right)t^6 + \left(-□\frac{5}{3}\omega^3 + \omega^5 + \frac{2}{3}\omega\right)t^4$$

$$y_5(t) = \frac{2}{2079}t^{11} + \left(\frac{8}{315}\omega^2 \frac{38}{2835}\right)t^9 + \left(\frac{4}{15}\omega^4 - \frac{104}{315}\omega^2 + \frac{22}{315}\right)t^7 + \left(\omega^6 - 2\omega^4 + \frac{17}{15}\omega^2 - \frac{2}{15}\right)t^5$$

$$y_6(t) = \frac{53}{33264}\omega t^12 + \frac{1}{124740}\omega\left(\frac{42471}{10}\omega^2 - \frac{34859}{10}\right)t^{10} + \frac{1}{124740}\omega(37422\omega^4 - 58806\omega^2 + 21582)t^8 + \frac{1}{124740}\omega(124740\omega^6 - 9\omega^4 + 213444\omega^2 - 47124)t^6$$

Consequently, the general solution of equation ODE (33) is written as follow:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + \ldots$$

$$= \omega + \frac{1}{3}t^3 + (\omega^2 - 1)t + \frac{1}{6}\omega t^4 + (\omega^3 - \omega)t^2 + \frac{1}{63}t^7 + \left(\frac{1}{5}\omega^2 - \frac{2}{15}\right)t^5 + \left(\omega^4 - \frac{4}{3} + \frac{1}{3}\right)t^3 + \frac{1}{56}\omega t^8 + \left(\frac{7}{30}\omega^3 - \frac{19}{90}\omega\right)t^6 + \left(-\frac{5}{3}\omega^3 + \omega^5 + \frac{2}{3}\omega\right)t^4 + \frac{2}{2079}t^{11} + \left(\frac{8}{315}\omega^2 - \frac{38}{2835}\right)t^9 + \left(\frac{4}{15}\omega^4 - \frac{104}{315}\omega^2 + \frac{22}{315}\right)t^7 + \left(\omega^6 - 2\omega^4 + \frac{17}{15}\omega^2 - \frac{2}{15}\right)t^5 + \frac{53}{33264}\omega t^{12} + \frac{1}{124740}\omega\left(\frac{42471}{10}\omega^2 - \frac{34859}{10}\right)t^{10} + \frac{1}{124740}\omega(37422\omega^4 - 58806\omega^2 + 21582)t^8 + \frac{1}{124740}\omega(124740\omega^6 - 29\omega^4 + 213444\omega^2 - 47124)t^6 + \ldots$$

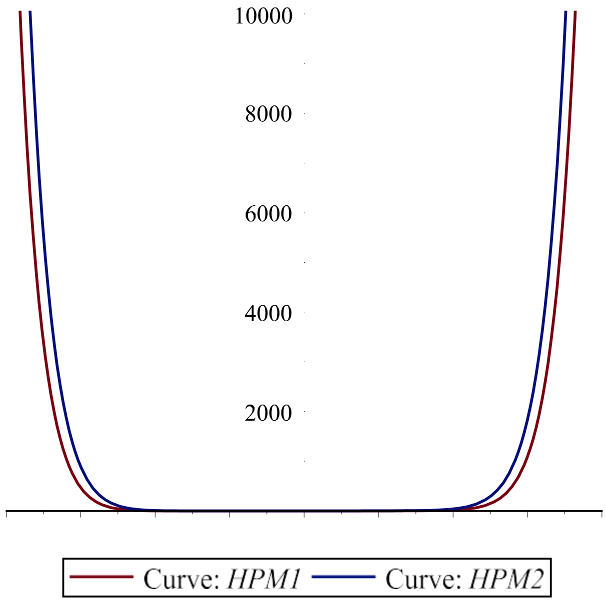

Figure5: Comparison of proposed hereditary property m and exact solutions of Problem 5 at ω = 0.9 and w = 1.

5. Discussion and Conclusion

In this paper, HPM, has been modified for solving generalized linear & nonlinear Riccati differential equation. The approximated solution of a class of nonlinear Riccati differential equation has been studied. Also, we have tested the modified HPM on the solving of different implementations which are show the efficiency and accuracy of the proposed method. The approximated solutions are agree well with analytical solutions for the tested problems Moreover, the approximated solutions using the proposed method proved to be more accurate.

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