Homotopy Perturbation Method for Solving Generalized Riccati Differential Equation

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Abstract
In this work, we have studied a class of nonlinear Riccati differential equation which is used as mathematical models in many physically significant fields and applied science. The homotopy perturbation method, has been modified for solving generalized linear and nonlinear Riccati differential equation, which is a second-order ordinary differential equation. Also, we have tested the modified homotopy perturbation method on the solving of different implementations which are show the efficiency and accuracy of the proposed method. The approximated solutions are agreed well with analytical solutions for the tested problems. [DOI: 10.22401/ANJS.00.2.13]

Keyword: Riccati differential equation; Homotopy perturbation method; HPM.

1. Introduction

The most important mathematical models for physical phenomena is the differential equation. Motion of objects, Fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations. Moreover, numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of differential equations (DEs), particularly ODEs of different orders, can be found in the mathematical modeling of real life problems (Mechee et al. (2014)). Riccati differential equation is the famous first-order nonlinear differential equation with quadratic nonlinearity. Obviously, this was the reason that as soon as Newton invented differential equations, Riccati differential equation was the first one to be investigated extensively since the end of the 17th century. Some of the properties of Riccati differential equation which have been studied, are in the frame of local theory of differential equations. The homotopy perturbation method (HPM) is efficient technique to find the approximate solutions for ordinary and partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. HPM was proposed by Ji-Huan He in 1999 for solving linear and nonlinear differential equations and integral equations. Many researchers used HPM to approximate the solutions of differential equations and integral equations (Yıldırım (2010), Jalaal et al. (2010) & Ma et al. (2008)).

Many researchers published some papers in solving some classes differential equations using HPM. For example, Chun & Sakthivel (2010) used HPM for solving a linear and non-linear second-order two-point boundary value problems while Gülkaç (2010) was solved the Black-Scholes equation for a simple European option in this method to obtain a new efficient recurrent relation to solve Black-Scholes equation.

Moreover, numerous researches used HPM for solving nonlinear differential equations, Vahidi et al. (2011) was solved non-linear DEs, which yields the Maclaurin series of the exact solution, Chang & Liou (2006) developed a third-order explicit approximation to find the roots of the dispersion relation for water waves that propagate over dissipative media, Zhou &Wu (2012) solved the non-linear PB equation describing spherical and planar colloidal particles immersed in an arbitrary valence and mixed electrolyte solution, Özis, & Akçı (2011) solved certain non-linear, non-smooth oscillators, Yazdi (2013) solved nonlinear vibration analysis of functionally graded plate while He & Huan (2004) applied HPM for solving nonlinear oscillators with

Recently, we have studied approximated solutions of a wide class of nonlinear Riccati differential equation which is used as mathematical models in many physically significant fields and applied science. The approximated solutions of this class of differential equations have studied using modified HPM. The proposed method applied for solving different examples for this class of ordinary differential equations (ODEs). A comparison was made between the analytical and approximated solutions for different tested problems which showed the proposed method is more efficient in the iterations complexity and high accurate in the absolute errors. It has been highlighted that the use of modified HPM is more suitable to approximate the solutions of general Liénard and Duffing differential equations with considering the general coefficients functions.

2. Preliminaries

In this paper, we consider the Riccati differential equation as follow (File & Aga (2016), Gomanjani & Khorram (2015)):

\[ y'(t) = q_0(t) + q_1(t)y(t) + q_2(t)y^2(t), \quad t_0 \leq t \leq t_e, \ldots (1) \]

with initial condition:

\[ y(t_0) = \alpha \]

\[ \ldots (2) \]

where \( q_0(t), q_1(t) \) and \( q_2(t) \) are continuous
functions and $t_0, t_e$ and $\alpha$ are an arbitrary constants. If $q_1(t) = 0$ the equation reduces to a Bernoulli equation, while if $q_2(t) = 0$ the equation becomes first-order linear ODE. The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The fundamental theories of Riccati differential equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics, the solution of this equation can be achieved using HPM to solve the nonlinear Riccati in an analytic form. Also, Riccati differential equation is an important equation in the optimal control literature. Solution of this equation can be obtained using classical numerical method as Runge-Kutta method or the forward Euler method (Batiha (2015), Abbasbandy (2006b)).

### 2.1 Homotopy Perturbation Method:

In this section, we present a brief description of the HPM, to illustrate the basic ideas of this method, we consider the following differential equation (Neamaty & Darzi (2010), Chun & Sakhthivel (2010), Batiha (2015) & Abbasbandy (2006b)):

$$A(u) - f(\tau) = 0, \; \tau \in \partial \Omega$$  

with boundary conditions:

$$B \left( u, \frac{\partial u}{\partial \tau} \right), \; \tau \in \partial \Omega$$  

where $A$ is general differential operator, $B$ is a boundary operator, $f(\tau)$ a known analytic function and $\partial \Omega$ is the boundary of the domain $\Omega$. The operator $A$ can be generally divided into two parts of $L$ and $N$ where $L$ is linear part, while $N$ is the nonlinear part in the differential equations. Therefore Eq.(3) can be rewritten as follows (He (1999)):

$$L(u) + N(u) - f(\tau) = 0$$  

By using homotopy technique, one can construct a homotopy $V(\tau, p): \Omega \times [0,1] \longrightarrow \mathbb{R}$, which satisfies:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + pL(v) + N(v) - f(\tau)) = 0$$  

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\tau))] = 0$$  

where $p \in [0, 1]$, $\Omega \times p$ is called homotopy parameter and $u_0$ is an initial approximation for the solution of equation (3) which satisfies the boundary conditions obviously, Using equation (6) or (7), we have the following equation:

$$H(\tau, 0) = L(v) - L(u_0) = 0$$  

and

$$H(\tau, 1) = L(v) + N(v) - f(\tau) = 0$$

Assume that the solution of (6) or (7) can be expressed as a series in $p$ as follows:

$$V = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots$$

$$= \sum_{i=0}^{\infty} p^i v_i$$

set $p = 1$, resulting the approximate solution of (3). Consequently:

$$u(\tau) = \lim_{p \to 1} V$$

$$= v_0 + v_1 + v_2 + v_3 + \ldots$$

$$= \sum_{i=0}^{\infty} v_i$$

It is worth to note that the major advantage of He’s method is that the perturbation equation can be freely constructed in many ways and approximation can also be freely selected.

### 3. Analysis of the HPM for Solving Riccati Differential Equation

In this section, we will present a review of the HPM to solve Riccati differential equations, we consider the following equation (File & Aga (2016)):

$$y''(t) = q_0(t) + q_1(t)y(t) + q_2(t)y^2(t)$$  

with initial condition:

$$y(t_0) = \alpha$$

where $t_0 \leq t \leq t_e$, $y(t) \in \mathbb{R}$ and $\alpha$ is an arbitrary constant.

The implementation of the proposed modified HPM for solving Riccati differential Equation (12) according the following algorithm:

#### 3.1 The Proposed Method:

Firstly, we start with the initial approximation $y_0(t) = \alpha$.

Secondly, we can construct a homotopy for the second-order ODE (12) as follows:
Then the solution of equation (14) is:

\[ y(t) = y_0(t) + \int y_0(t) \, dt \]

Fourthly, suppose that the solution of equation (15) is in the form:

\[ p^0: \quad \dot{y}_1(t) - q_0(t) = 0 \]

Finally, using the equations (17) with some simplifications, we get the following terms of the solution:

\[ y_0(t) = \alpha \]

\[ y_1(t) = \int (-\dot{y}_0(t) + q_0(t) + q_1(t)y_0(t) + q_2(t)y_0^2(t)) \, dt \]

\[ y_2(t) = \int (q_1(t)y_1(t) - q_2(t)(2y_0(t)y_1(t)) \, dt \]

\[ y_3(t) = \int (q_1(t)y_2(t) - q_2(t)(2y_0(t)y_2(t)) + y_0^2(t)) \, dt \]

and

\[ y_4(t) = \int (q_1(t)y_3(t) + q_2(t)(2y_0(t)y_3(t) + 2y_1(t)y_2(t)) \, dt \]

Hence, the general term has the following form:

\[ y_n(t) = \int (q_1(t)y_{n-1}(t) + q_2(t)) \sum_{i=0}^{n-1} y_i(t)y_{n-i-1}(t) \, dt, \quad n = 2, 3, \ldots \]

Then the solution of equation (12) is:

\[ y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \ldots \]

\[ y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \ldots \]

4. Implementations

In order to assess the accuracy of the solving generalized Riccati differential equation by HPM we will introduce some different examples in general and to compare the approximated solution with the exact solutions for these problems, we will consider the following five problems.

4.1 Problem 1:

Consider the following ODE (Ghomanjani & Khorram (2015)):

\[ y(t) + y(t) + \frac{d}{dt}y(t) = 0, \quad 0 \leq t \leq 1 \quad \ldots (20) \]

subject to the initial condition:

\[ y(0) = \lambda, \quad \text{with the exact solution:} \]

\[ y(t) = \frac{1}{e^{\lambda t} + 1} \]

Comparing equation (20), we have \( q_0(t) = 0, \quad q_1(t) = 1, \quad \text{and} \quad q_2(t) = 1 \)

The initial approximation has the form

\[ y_0(0) = \lambda, \quad \text{substituting} \ (18) \text{into} \ (20), \text{we have:} \]

\[ y_1(t) = \int (-\dot{y}_0(t) - y_0(t) + y_0^2(t)) \, dt \]

\[ y(0) = \lambda, \quad \text{and} \]

\[ y_n(t) = \int (-y_{n-1}(t) + \sum_{i=0}^{n-1} y_i(t)y_{n-i-1}(t)) \, dt, \quad \ldots (22) \]

for \( k = 2, 3, \ldots \). So, simplification of equation (22) lends to the following solutions:

\[ y_2(t) = \frac{1}{2} \lambda^2 \left( 2\lambda - 1 \right)(\lambda - 1), \]

\[ y_3(t) = \left( -\frac{1}{3} \lambda^4 + \frac{1}{2} \lambda^2 - \frac{1}{6} \lambda \right) \tau^3 \]

\[ y_4(t) = \frac{1}{24} \lambda^6 \left( 2\lambda - 1 \right)(\lambda - 1)(4\lambda^2 - 8\lambda + 1), \]

\[ y_5(t) = \frac{1}{120} \lambda^8 \left( \lambda - 1 \right)(24\lambda^4 - 96\lambda^3 + 94\lambda^2 - 26\lambda + 1), \]

\[ y_6(t) = \frac{1}{720} \lambda^{10} \left( 2\lambda - 1 \right)(\lambda - 1)(24\lambda^4 + 24\lambda^3 - 56\lambda^2 + 4\lambda + 1) \]

Then, the general solution of equation (20) is written as follows:

\[ y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + \ldots \]

\[ = \lambda + \lambda(\lambda - 1)t + \frac{1}{2} \lambda^2(2\lambda - 1)(\lambda - 1) + \left( -\frac{1}{3} \lambda^4 + \frac{1}{2} \lambda^2 + \frac{1}{2} \lambda \right) \tau^3(2\lambda - 1)(\lambda - 1) + \left( 4\lambda^2 - 8\lambda + 1 \right) - \frac{1}{120} \lambda^8 \left( \lambda - 1 \right)(24\lambda^4 - 96\lambda^3 + 94\lambda^2 - 26\lambda + 1) + \ldots \]
4.2 Problem 2:

Consider the following nonlinear first-order ODE (Abbasbandy (2007), Abbasbandy (2006a) & Abbasbandy (2006b)):

\[ y'(t) = 1 + 2y(t) - y^2(t), \quad -1 \leq t \leq 1 \quad \ldots(23) \]

subject to the initial condition:

\[ y(0) = \alpha \]

with the exact solution:

\[ y(t) = \frac{\sqrt{2} \tan^{-1} \left( \sqrt{\frac{2t}{2t+1}} \right)}{\sqrt{2} \tan^{-1} \left( \sqrt{\frac{2t}{2t+1}} \right)} \]

Comparing equation (23) we have

\[ q_0(t) = 1, \quad q_1(t) = 2 \quad \& \quad q_2(t) = -1 \]

The initial approximation has the form

\[ y_0(t) = \alpha \quad \ldots(24) \]

substituting (18) into (23), we have

\[ y_1(t) = \int (-y_0(t)) + 1 + 2y_0(t) - y_0^2(t) dt \]

\[ = (-\alpha^2 + 2\alpha + 1)t \]

and

\[ y_n(t) = \int (2y_{n-1}(t) \sum_{i=0}^{n-1} y_i y_{n-i-1}(t)) dt \]

for \( k = 2, 3, \ldots \) So, simplification of equation (26) lends to the following solutions:

\[ y_2(t) = \frac{1}{45}t^6(\alpha - 1)(\alpha^2 - 2\alpha - 1)(45\alpha^4 - 180\alpha^3 + 150\alpha^2 + 60\alpha - 7) \]

\[ y_3(t) = \frac{1}{315}t^7(\alpha^2 - 2\alpha - 1)(315\alpha^6 - 1890\alpha^5 + 3675\alpha^4 - 2100\alpha^3 - 651\alpha^2 + 462\alpha + 53) \]

\[ y_4(t) = \frac{1}{315}t^7(\alpha - 1)(\alpha^2 - 2\alpha - 1)(315\alpha^6 - 1890\alpha^5 + 3465\alpha^4 - 2160\alpha^3 - 1323\alpha^2 + 126\alpha + 71) \]

Hence, the general solution of equation (23) is given as follow:

\[ y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + y_7(t) + \ldots \]

\[ = \alpha + (-\alpha^2 + 2\alpha + 1)t + \frac{t^2(\alpha^2 - 2\alpha - 1) - 135t^6(\alpha^2 - 2\alpha - 1)}{315} \]

\[ + \frac{t^7(\alpha - 1)(\alpha^2 - 2\alpha - 1)(315\alpha^6 - 1890\alpha^5 + 3465\alpha^4 - 2160\alpha^3 - 1323\alpha^2 + 126\alpha + 71)}{315} \]

4.3 Problem 3:

Consider the following ODE:(File & Aga...
subject to the initial condition \( u(0) = a \), with the exact solution \( u(t) = e^t \).

Comparing equation (27), we have:
\[
q_0(t) = e^t - e^{3t}, \quad q_1(t) = 2e^{2t} \text{ and } q_2(t) = -e^t
\]
The initial approximation has the form \( u_0(t) = e^t \). Substituting (18) into (27), we have:
\[
\begin{align*}
\dot{u}_1(t) &= \int (-\dot{u}_0(t) + e^t - e^{3t} + 2e^{2t}u_0(t) - e^tu_0^2(t)) \, dt \\
&= e^t - \frac{1}{3}e^{3t} + 4e^{2t} - a^2e^t 
\end{align*}
\]
and
\[
\begin{align*}
u_n(t) &= \int (2e^{2t}y_{n-1}(t) - e^t \sum_{i=0}^{n-1} u_i(t)u_{n-i-1}(t)) \, dt, \quad k = 2, 3, \ldots 
\end{align*}
\]
So, the general solution of equation (27) is written as follows:
\[
y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + \ldots
\]
Therefore, the general solution of equation (27) is written as follows:
\[
y(t) = e^{t} - e^{3t} + 2e^{2t}u(t) - e^{3t} + 2e^{2t}u(t) - e^{t}u_0^2(t), \quad -5 \leq t \leq -1 \quad (27)
\]

Figure 3: Comparison of proposed HPM and exact solutions of Problem 3 at \( c = 2 \), \( \lambda = 1 \) and \( a = 0.5 \).

4.4 Problem 4:
Consider the following ODE:
\[
y'(t) = 16t^2 - 5 + 8ty(t) + y^2(t), \quad -0.5 \leq t \leq 1 \quad (30)
\]
subject to the initial condition \( y(0) = 1 \) with the analytic solution:
\[
y(t) = \frac{(4e^{-2t-4})t + ce^{-2t-1}}{ce^{-2t}}
\]
Comparing equation (30), we have \( q_0(t) = 16t^2 - 5, q_1(t) = 8t \) and \( q_2(t) = 1 \).

The initial approximation has the form \( y_0(t) = 1 \). Substituting (18) into (30), we have:
\[
y_1(t) = \int (-\dot{y}_0(t) + 16t^2 - 5 + 8ty_0(t) + y_0^2(t)) \, dt = 16\frac{t^3}{3} + 4t^2 - 4t
\]
and
\[
y_n(t) = \int (8ty_{n-1}(t) + \sum_{i=0}^{n-1} y_iy_{n-i-1}(t)) \, dt, \quad k = 2, 3, \ldots
\]
So, the general solution of equation (32) is written as follows:
\[
y_2(t) = 128\frac{t^5}{15} + 32\frac{t^4}{3} - 8t^3 - 4t^2
\]
\[
y_3(t) = 4352\frac{t^7}{315} + \frac{1088}{15}t^6 - \frac{208}{15}t^5 - 20t^4 + \frac{8}{3}t^3
\]
Then, the general solution of equation (30) is written as follow:
\[
y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \ldots
\]
4.5 Problem 5:

Consider the following ODE:
\[ y'(t) = t^2 + y^2(t) - 1, -4 \leq t \leq 4 \quad \text{...(33)} \]
subject to the initial condition \( y(0) = 0 \).

Comparing equation (33), we have:
\[ q_0(t) = t^2 - 1, q_1(t) = 0 \quad \text{and} \quad q_2(t) = 1 \]
The initial approximation has the form, \( y_0(t) = \omega \).

Substituting (18) into (33), we have:
\[
y_1(t) = \int (-\dot{y}_0(t) + t^2 - 1 + y_0(t)) dt = \frac{1}{3} t^3 + (\omega^2 - 1) t \quad \text{...(34)}
\]
and
\[
y_n(t) = (\int \sum_{i=0}^{n-1} y_i Y_{n-i-1}(t)) dt; k = 2, 3, \ldots \quad \text{...(35)}
\]
So, simplification of equation (35) leads to the following solutions:
\[
y_2(t) = \frac{1}{6} \omega t^4 + (\omega^3 - \omega)t^2
\]
\[
y_3(t) = \frac{1}{63} t^7 + \left(\frac{4}{9} \omega^2 - \frac{2}{15} \right) t^5 + (\omega^4 - \frac{4}{3} \omega^2 + \frac{1}{3}) t^3
\]
\[
y_4(t) = \frac{1}{56} \omega t^6 + \left(\frac{2}{21} \omega^3 - \frac{19}{90} \right) t^4 + (-\frac{5}{3} \omega^2 + \frac{2}{3} \omega) t^2
\]
\[
y_5(t) = \frac{2}{2079} t^{11} + \left(\frac{8}{315} \omega^2 - \frac{38}{2835} \right) t^9 + \left(\frac{4}{15} \omega^4 - \frac{104}{315} \omega^2 + \frac{22}{315} \right) t^7 + (\omega^6 - 2 \omega^4 + \frac{17}{15} \omega^2 - \frac{2}{15}) t^5
\]
\[
y_6(t) = \frac{53}{33264} \omega t^{12} + \frac{1}{124740} \omega(\frac{42471}{40} - \frac{34859}{10}) t^{10} + \frac{1}{124740} \omega(37422 \omega^4 - 58806 \omega^2 + 21582) t^8
\]

Consequently, the general solution of equation ODE (33) is written as follow:
\[
y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) + y_6(t) + \ldots
\]
\[
= \omega + \left(\frac{1}{3} t^3 + (\omega^2 - 1) t \right) + \left(\frac{1}{6} \omega t^4 + \omega^3 - \omega \right) t^2 + \left(\frac{1}{15} \omega^2 - \frac{2}{45} \omega^4 + \frac{1}{3} \omega^6 \right) t^4 + \left(\frac{1}{10} \omega^3 - \omega^5 + \frac{2}{30} \omega^7 \right) t^6 + \left(\frac{1}{90} \omega^4 - \frac{19}{90} \omega^6 - \frac{2}{15} \omega^8 \right) t^8 + \left(\frac{42471}{40} \omega^2 - \frac{34859}{10} \omega^4 \right) t^{10} + \left(\frac{42471}{10} \omega^4 - \frac{34859}{10} \omega^6 \right) t^{12} + \left(\frac{42471}{40} \omega^2 - \frac{34859}{10} \omega^4 \right) t^{14} + \left(\frac{42471}{10} \omega^4 - \frac{34859}{10} \omega^6 \right) t^{16} + \ldots
\]

5. Discussion and Conclusion

In this paper, HPM, has been modified for solving generalized linear & nonlinear Riccati differential equation. The approximated solution of a class of nonlinear Riccati differential equation has been studied. Also, we have tested the modified HPM on the solving of different implementations which show the efficiency and accuracy of...
the proposed method. The approximated solutions are agree well with analytical solutions for the tested problems. Moreover, the approximated solutions using the proposed method proved to be more accurate.

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References
Jalaal, M., Ganji, D., & Mohammadi, F. 2010, He’s homotopy perturbation method for two-dimensional heat conduction equation: Comparison with finite element
Ozis, T., & Akçi, C. 2011, Periodic solutions for certain non-smooth oscillators by iterated homotopy perturbation method combined with modified Lindstedt-Poincare technique, Meccanica, 46, 341.
Yusufoglu, E. 2009, Improved homotopy perturbation method for solving Fredholm type integro-differential equations, Chaos,
Solitons & Fractals, 41, 28.