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Mixture of Two Inverse Exponential Distributions Based on Fuzzy Data

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Abstract

In this paper we will discuss the mixture distribution consisting of two Inverse Exponential Distributions (MTIED) based on fuzzy data. We will study the Maximum Likelihood Estimator (MLE) via the Newton Raphson (NR) algorithm and Bayes estimation under square error loss and quadratic loss functions for the unknown parameters of the distribution, and reliability function. The obtained estimates of the unknown parameters and reliability function are compared numerically through Monte-Carlo simulation study in term of the mean square error (MSE) values and (IMSE) respectively. [DOI: 10.22401/ANJS.00.2.11]

Keywords: Finite Mixtures; Reliability; Inverse Exponential distribution; NR algorithm; Lindley's approximation; Mean square error; quadratic loss function.

1. Introduction

Finite mixture models play an important role in many applicable fields, such as economics, medicine, psychology, life testing, reliability analysis and etc., [12]. Several researches have assumed that the underlying population is a homogeneous one with the failure time distribution given by$F(x,\theta)$, where the form F is known but the parameter θ is unknown, [11].

Many authors interested with inferences on mixtures of exponential distributions and among them Jaheen [7] and Everitt and Hand [6]. Also, Elsherpieny [5] estimated the parameters of mixed generalized exponentially distributions.

The past experience as well as experimental constraints may suggest that the assumption of homogeneity may not hold and the underlying population may consist of several subpopulation, say $sp_1, sp_2, \dots, sp_k$ mixed in proportion; $p_1, p_2, \dots, p_k$.

Further, cumulative distribution function in each subpopulation is given by$F_j(x,\theta_j), j = 1, \dots, k$ with p.d.f.'s $f_j(x,\theta_j)$ respectively [10].

Inverse Exponential distribution is also known as reciprocal exponential distribution finds use in the analysis of fading wireless communication systems.

The mixture of two Inverse Exponential distributions (MTIES) has its p.d.f. as:

$$f(x;\theta_1,\theta_2) = p_1 f_1(x;\theta_1) + p_2 f_2(x;\theta_2) \quad \dots(1)$$

$$p_1 + p_2 = 1, 0 < p_1, p_2 < 1, (i.e., k = 2)$$

where$f_i(x;\theta_i)$, the density function of the ith component (inverse exponential), is given by:

$$f_i(x,\theta_i) = \frac{\theta_i}{x^2} e^{-\theta_i/x}, x > 0, \theta_i > 0, i = 1,2 \quad \dots(2)$$

The cumulative distribution function (CDF) of the MTIED is given by:

$$F(x;\theta_1,\theta_2) = p_1 F_1(x,\theta_1) + p_2 F_2(x;\theta_2) \quad \dots(3)$$

where $F_i(t,\theta_i)$, the c.d.f. of the ith component, is given by:

$$F_i(x;\theta_i) = e^{-\theta_i/x}, x > 0, \theta_i > 0, i = 1,2 \quad \dots(4)$$

The reliability function at time t is given by:

$$R(t) = p_1\left(1 - e^{-\theta_1/t}\right) + p_2\left(1 - e^{-\theta_2/t}\right), t \ge 0 \quad \dots(5)$$

Note that the mean of the p.d.f. of the MTIED given in (1) and (2) does not exist. The inverse Generalized Exponential and the inverse Weibull distributions are both the generalization of an inverse exponential distribution [8].

Usually, it is assumed that an observed data are precise (exact) numbers. However, in real world situations, some collected data might be imprecise and are represented in the form of fuzzy numbers. The first publications in fuzzy set theory by Zadeh [13].

Thus, this paper focused on evaluate the estimating the unknown parameters and reliability function of MTIED through the methods (MLE, Bayes) based on fuzzy data and presenting a comparative study for estimating.

2.Maximum Likelihood Estimators

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be an i.i.d. random vector of a random sample of size n from MTIED with p.d.f. given by (1).If a realization of $\underline{x}$ was known exactly, then the complete data likelihood function is:

$$L(\theta_1, \theta_2 | \underline{x}) = \prod_{i=1}^{n} f_x(x_i, \theta_1, \theta_2)$$
$$= \prod_{i=1}^{n} \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \quad \ldots(6)$$

Now, suppose that $\underline{x}$ is not observed precisely and only partial informations about $\underline{x}$ are available in the form of fuzzy subset $\underline{\tilde{x}}$ with the Borel measurable membership function $\mu_{\tilde{x}}(x)$, so we can compute its probability according to Zadeh's definition of the probability of a fuzzy event $\tilde{A}$ in R^n, which is defined as the expectation of the membership function$\mu_{\tilde{A}}$ with respect to p, [14]:

$$P(\tilde{A}) = \int \mu_{\tilde{A}}(x) dp; \ \forall x \in R^n$$

The observed-data likelihood function can then be obtained as:

$$L(\theta_1, \theta_2 | \underline{\tilde{x}}) = \prod_{i=1}^{n} \int \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx \quad \ldots(7)$$

and the observed –data natural log-likelihood function will be:

$$\ln L(\theta_1, \theta_2 | \underline{\tilde{x}}) = \sum_{i=1}^{n} \ln \int \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx \quad ..(8)$$

assuming that the parameters θ_1 and θ_2 are unknown and P is known.

Differentiating the natural Log-likelihood function $\ell(\theta_1, \theta_2 | \underline{\tilde{x}})$, given by equation (8), partially with respect to θ_1, θ_2 and then equating to zero, we have:

$$\frac{\partial \ln L(\theta_1, \theta_2 | \underline{\tilde{x}})}{\partial \theta_1} = \sum_{i=1}^{n} \frac{\int \left[\frac{p_1}{x^2} e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3} e^{-\theta_1/x} \right] \mu_{\tilde{x}_i}(x) dx}{\int \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx} = 0$$

$$\frac{\partial \ln L(\theta_1, \theta_2 | \underline{\tilde{x}})}{\partial \theta_2} = \sum_{i=1}^{n} \frac{\int \left[\frac{(1-p_1)}{x^2} e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^3} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx}{\int \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx} = 0 \quad \ldots(10)$$

The solution of the two nonlinear Likelihood equations (9) and (10) yields the MLEs of θ_1 and θ_2 respectively.Since there are no closed forms of the solutions, iterative approximation techniques can be used to obtain the MLEs.

In the following, we consider iterative approximation techniques namely Newton-Raphson (NR) algorithm to determine the MLEs of the parameters θ_1 and θ_2.

Newton-Raphson (NR) Algorithm:

In this algorithm, the solution of the Likelihood equation is obtained through an iterative procedure.

<u>Step (1):</u> Let $\theta_1^{(0)}$ and $\theta_2^{(0)}$ starting values of θ_1 and $\theta_2$when $h = 0$.

<u>Step (2):</u> At iteration $(h+1)$, estimate the new value of θ_1 and θ_2, as:

$$\begin{bmatrix} \hat{\theta}_1^{(h+1)} \\ \hat{\theta}_2^{(h+1)} \end{bmatrix} = \begin{bmatrix} \hat{\theta}_1^{(h)} \\ \hat{\theta}_2^{(h)} \end{bmatrix} - \begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta_1^2} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell}{\partial \theta^2} \end{bmatrix}^{-1}_{\substack{\theta_1 = \hat{\theta}_1^{(h)} \\ \theta_2 = \hat{\theta}_2^{(h)}}} \begin{bmatrix} \frac{\partial \ell}{\partial \theta_1} \\ \frac{\partial \ell}{\partial \theta_2} \end{bmatrix} \quad \ldots(11)$$

where the first-order derivatives of the natural Log-Likelihood with respect to the parameters θ_1 and θ_2, required for proceeding with the NR-algorithm, are obtained as in the equation (9) and (10) and the second-order derivatives are obtained, as follows:

$$\frac{\partial^2 \ell(\theta_1, \theta_2 | \underline{\tilde{x}})}{\partial \theta_1^2} = \sum_{i=1}^{n} \frac{\int \left[\frac{-2p_1}{x^3} e^{-\theta_1/x} + \frac{p_1\theta_1}{x^4} e^{-\theta_1/x} \right] \mu_{\tilde{x}_i}(x) dx}{\int \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx} \left(\frac{\int \left[\frac{p_1}{x^2} e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3} e^{-\theta_1/x} \right] \mu_{\tilde{x}_i}(x) dx}{\int \left[\frac{p_1\theta_1}{x^2} e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2} e^{-\theta_2/x} \right] \mu_{\tilde{x}_i}(x) dx} \right)^2 \quad \ldots(12)$$

$$\frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_1\partial\theta_2}=\sum_{i=1}^{n}\frac{\left(\int\left[\frac{p_1}{x^2}e^{-\theta_1/x}-\frac{p_1\theta_1}{x^3}e^{-\theta_1/x}\right]\mu_{\tilde{x}_i}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x}+\frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right)^2}\times$$
$$\left(\int\left[\frac{1-p_1}{x^2}e^{-\theta_2/x}-\frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right) \quad \ldots(13)$$

$$\frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_2^2}=\sum_{i=1}^{n}\frac{\left(\int\left[\frac{-2(1-p_1)}{x^3}e^{-\theta_2/x}+\frac{(1-p_1)\theta_2}{x^4}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x}+\frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right)}$$
$$(-)\left(\frac{\int\left[\frac{(1-p_1)}{x^2}e^{-\theta_2/x}-\frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x}+\frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right)}\right)^2 \quad \ldots(14)$$

Step (3): Repeat step (2) until convergence occurs, i.e., $\left|\hat{\theta}_1^{(h+1)}-\hat{\theta}_1^{(h)}\right|+\left|\hat{\theta}_2^{(h+1)}-\hat{\theta}_2^{(h)}\right|<\varepsilon$, for some pre-fixed $\varepsilon>0$. When the convergence occurs then the current $\hat{\theta}_1^{(h+1)}$ and $\hat{\theta}_2^{(h+1)}$ represent the maximum Likelihood estimate of θ_1 and θ_2 via NR algorithm which we referred to as $\left(\hat{\theta}_{1\,ML}^{NR},\hat{\theta}_{2\,ML}^{NR}\right)$.

In the following, we provide Bayesian estimations of the parameters of MTIED when the available data are in the form of fuzzy numbers.

3. Bayes Estimations of the Parameters

For a Bayesian estimation of the unknown parameters, we need prior distributions for these parameters. Consider the prior distributions of θ_1 and θ_2 of MTIED are taken to be independent Gamma$(a,b)$and Gamma (c,d) respectively with p.d.fs.

$$\pi_1(\theta_1)=\frac{b^a}{\Gamma(a)}\theta_1^{a-1}e^{-b\theta_1};\ \theta_1>0,a,b>0 \quad \ldots(15)$$

$$\pi_2(\theta_2)=\frac{d^c}{\Gamma(c)}\theta_2^{c-1}e^{-d\theta_2};\ \theta_2>0,c,d>0 \quad \ldots(16)$$

leads to a joint prior distribution of θ_1 and θ_2 of the form:

$$\pi(\theta_1,\theta_2)=\pi_1(\theta_1).\pi_2(\theta_2)=\frac{b^a d^c}{\Gamma(a)\Gamma(c)}\theta_1^{(a-1)}\theta_2^{(c-1)}e^{-(b\theta_1+d\theta_2)} \quad \ldots(17)$$

The joint posterior density function of θ_1 and θ_2 given fuzzy data can be obtained by combining (7) and (17)

$$\prod(\theta_1,\theta_2|\underline{\tilde{x}})=\frac{\pi(\theta_1,\theta_2|\underline{\tilde{x}})}{\int_{\theta_2}\int_{\theta_1}\pi(\theta_1,\theta_2|\underline{\tilde{x}})\,d\theta_1 d\theta_2}$$

where:

$$\pi(\theta_1,\theta_2|\underline{\tilde{x}})=L(\theta_1,\theta_2|\underline{\tilde{x}})\pi(\theta_1,\theta_2)$$
$$=\frac{b^a d^c}{\Gamma(a)\Gamma(c)}\theta_1^{(a-1)}\theta_2^{(c-1)}e^{-(b\theta_1+d\theta_2)}\times\prod_{i=1}^{n}\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x}+(1-p_1)\frac{\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx \quad \ldots(18)$$

The squared error loss function (SELF) was proposed by Legendre (1805) and Gauss (1810) in order to develop least square theory. The formula of this loss function for θ is, [1]

$$L(\theta,\hat{\theta})=(\hat{\theta}\theta)^2 \quad \ldots(19)$$

according to eq.(19), Bayes estimator of θ based on SELF is obtained by:

$$\hat{\theta}=E(\theta|\underline{\tilde{x}}) \quad \ldots(20)$$

So, Bayes estimation of any function of the parameters, say $w(\theta_1,\theta_2)$, under a squared error loss function, $\hat{w}_{BS}(\theta_1,\theta_2)$, can be written as:

$$\hat{w}_{BS}(\theta_1,\theta_2)=E\left(w(\theta_1,\theta_2)|\underline{\tilde{x}}\right)=\frac{\int_0^\infty\int_0^\infty w(\theta_1,\theta_2)\pi(\theta_1,\theta_2|\underline{\tilde{x}})\,d\theta_1 d\theta_2}{\int_0^\infty\int_0^\infty \pi(\theta_1,\theta_2|\underline{\tilde{x}})\,d\theta_1 d\theta_2} \quad \ldots(21)$$

Note that, Bayes estimators in (21) is of the form of ratio of two integrals, which cannot be simplified in to a closed form. However, we can approximate this Bayes estimator into a form containing no integrals by using the Lindley's approximation form.

<u>Lindley's Approximation:</u>

$$E\left(w(\theta_1,\theta_2)|\underline{\tilde{x}}\right)=\frac{\int_0^\infty\int_0^\infty w(\theta_1,\theta_2)e^{\ell(\theta_1,\theta_2)|\underline{\tilde{x}})+\rho(\theta_1,\theta_2)}\,d\theta_1 d\theta_2}{\int_0^\infty\int_0^\infty e^{\ell(\theta_1,\theta_2)|\underline{\tilde{x}})+\rho(\theta_1,\theta_2)}\,d\theta_1 d\theta_2} \quad \ldots(22)$$

where:

$w(\theta_1,\theta_2)$ is a function of θ_1 and θ_2 only, $\ell(\theta_1,\theta_2|\underline{\tilde{x}})$ is natural Log-Likelihood function defined by (8), $\rho(\theta_1,\theta_2)$ is natural Log-joint prior density function.

Now, according to Lindley, for sufficiently Large sample size, the ratio of integral $I(\underline{\tilde{x}}) = E\left(w(\theta_1,\theta_2)|\underline{\tilde{x}}\right)$ appears in equation (22) can be written as, (see [2], [9])

$$I(\underline{\tilde{x}}) = \hat{w} + \frac{1}{2}\Big[\left(\hat{w}_{\theta_2\theta_2} + 2\hat{w}_{\theta_2}\hat{\rho}_{\theta_2}\right)\hat{\sigma}_{\theta_2\theta_2} + \left(\hat{w}_{\theta_2\theta_1} + 2\hat{w}_{\theta_2}\hat{\rho}_{\theta_1}\right)\hat{\sigma}_{\theta_2\theta_1} + \left(\hat{w}_{\theta_1\theta_2} + 2\hat{w}_{\theta_1}\hat{\rho}_{\theta_2}\right)\hat{\sigma}_{\theta_1\theta_2} + \left(\hat{w}_{\theta_1\theta_1} + 2\hat{w}_{\theta_1}\hat{\rho}_{\theta_1}\right)\hat{\sigma}_{\theta_1\theta_1}\Big] + \frac{1}{2}\Big[\left(\hat{w}_{\theta_1}\hat{\sigma}_{\theta_2\theta_1} + \hat{w}_{\theta_2}\hat{\sigma}_{\theta_2\theta_2}\right)\left(\hat{\ell}_{\theta_2\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{\ell}_{\theta_1\theta_2\theta_2}\hat{\sigma}_{\theta_1\theta_2} + \hat{\ell}_{\theta_2\theta_1\theta_2}\hat{\sigma}_{\theta_2\theta_1} + \hat{\ell}_{\theta_2\theta_2\theta_2}\hat{\sigma}_{\theta_2\theta_2}\right) + \left(\hat{w}_{\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{w}_{\theta_2}\hat{\sigma}_{\theta_1\theta_2}\right)\left(\hat{\ell}_{\theta_1\theta_1\theta_1}\hat{\sigma}_{\theta_1\theta_1} + \hat{\ell}_{\theta_1\theta_2\theta_1}\hat{\sigma}_{\theta_1\theta_2} + \hat{\ell}_{\theta_2\theta_1\theta_1}\hat{\sigma}_{\theta_2\theta_1} + \hat{\ell}_{\theta_2\theta_2\theta_1}\hat{\sigma}_{\theta_2\theta_2}\right)\Big] \quad \ldots(23)$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the MLE's of θ_1 and θ_2 respectively, σ_{ij} is the $(i,j)^{th}$ elements of matrix $\left[(-)\frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_1\partial\theta_2}\right]^{-1}$; $i,j = 1, 2$. Subscripts (i,j) refer to θ_1,θ_2 respectively, and:

$$\hat{w}_{\theta_1} = \frac{\partial w}{\partial\theta_1}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} ; \quad \hat{w}_{\theta_1\theta_2} = \frac{\partial^2 w}{\partial\theta_1\partial\theta_2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} ;$$

$$\hat{w}_{\theta_1\theta_1} = \frac{\partial^2 w}{\partial{\theta_1}^2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} ; \quad \hat{w}_{\theta_2} = \frac{\partial w}{\partial\theta_2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} ;$$

$$\hat{w}_{\theta_2\theta_1} = \frac{\partial^2 w}{\partial\theta_2\partial\theta_1}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} ; \quad \hat{w}_{\theta_2\theta_2} = \frac{\partial^2 w}{\partial{\theta_2}^2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}}$$

$$\hat{\rho}_{\theta_1} = \frac{\partial \ln\pi(\theta_1,\theta_2)}{\partial\theta_1}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} = \frac{a-1}{\theta_1} - b;$$

$$\hat{\rho}_{\theta_2} = \frac{\partial \ln\pi(\theta_1,\theta_2)}{\partial\theta_2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} = \frac{c-1}{\theta_2} - d;$$

Now:

$$\hat{\ell}_{\theta_1\theta_1} = \frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_1^2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} \text{ as in (12)}$$

$$\hat{\ell}_{\theta_1\theta_2} = \frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_1\partial\theta_2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} = \hat{\ell}_{\theta_2\theta_1} = \frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_2\partial\theta_1}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} \text{ as in (13)}$$

$$\hat{\ell}_{\theta_2\theta_2} = \frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_2^2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}} \text{ as in (14)}$$

$$\hat{\ell}_{\theta_1\theta_1\theta_1} = \frac{\partial^3\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_1^3}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}}$$

$$= \sum_{i=1}^{n}\frac{\int\left[\frac{3p_1}{x^4}e^{-\theta_1/x} - \frac{\theta_1 p_1}{x^5}e^{-\theta_1/x}\right]\mu_{\widetilde{x_i}}(x)dx}{\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx}$$

$$(-3)\sum_{i=1}^{n}\frac{\left(\int\left[\frac{-2p_1}{x^3}e^{-\theta_1/x} + \frac{p_1\theta_1}{x^4}e^{-\theta_1/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^2} \times \left(\int\left[\frac{p_1}{x^2}e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3}e^{-\theta_1/x}\right]\mu_{\tilde{x}_i}(x)dx\right)(+2)$$

$$\sum_{i=1}^{n}\left(\frac{\left(\int\left[\frac{p_1}{x^2}e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3}e^{-\theta_1/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}\right)^3 \quad \ldots(24)$$

$$\hat{\ell}_{\theta_2\theta_2\theta_2} = \frac{\partial^3\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_2^3}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}}$$

$$= \sum_{i=1}^{n}\frac{\int\left[\frac{3(1-p_1)}{x^4}e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^5}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx}{\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx} -$$

$$3\sum_{i=1}^{n}\frac{\left(\int\left[\frac{-2(1-p_1)}{x^3}e^{-\theta_2/x} + \frac{(1-p_1)\theta_2}{x^4}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^2} \times \left(\int\left[\frac{(1-p_1)}{x^2}e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right)$$

$$+2\sum_{i=1}^{n}\left(\frac{\left(\int\left[\frac{(1-p_1)}{x^2}e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}\right)^3 \quad \ldots(25)$$

$$\hat{\ell}_{\theta_2\theta_1\theta_1} = \frac{\partial^3\ell((\theta_1,\theta_2)|\underline{\tilde{x}})}{\partial\theta_2\partial\theta_1\partial\theta_1}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}}$$

$$= -\sum_{i=1}^{n}\frac{\left(\int\left[\frac{(1-p_1)}{x^2}e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^2} \times \left(\int\left[\frac{p_1\theta_1}{x^4}e^{-\theta_1/x} - \frac{2p_1}{x^3}e^{-\theta_1/x}\right]\mu_{\tilde{x}_i}(x)dx\right) +$$

$$2\sum_{i=1}^{n}\frac{\left(\int\left[\frac{(1-p_1)}{x^2}e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^3} \times \left(\int\left[\frac{p_1}{x^2}e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3}e^{-\theta_1/x}\right]\mu_{\tilde{x}_i}(x)dx\right)^2 \quad \ldots(26)$$

$$\hat{\ell}_{\theta_2\theta_1\theta_2} = \frac{\partial^2\ell(\theta_1,\theta_2|\underline{\tilde{x}})}{\partial\theta_2\partial\theta_1\partial\theta_2}\Big|_{\substack{\theta_1=\hat{\theta}_1\\ \theta_2=\hat{\theta}_2}}$$

$$= (-)\sum_{i=1}^{n}\frac{\left(\int\left[\frac{p_1}{x^2}e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3}e^{-\theta_1/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^2} \times \left(\int\left[\frac{-2(1-p_1)}{x^3}e^{-\theta_2/x} + \frac{(1-p_1)\theta_2}{x^4}e^{-\theta_2/x}\right]\mu_{\tilde{x}_i}(x)dx\right)$$

$$+2\sum_{i=1}^{n}\frac{\left(\int\left[\frac{(1-p_1)}{x^2}e^{-\theta_2/x} - \frac{(1-p_1)\theta_2}{x^3}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^2}{\left(\int\left[\frac{p_1\theta_1}{x^2}e^{-\theta_1/x} + \frac{(1-p_1)\theta_2}{x^2}e^{-\theta_2/x}\right]\mu_{\widetilde{x_i}}(x)dx\right)^3} \times \left(\int\left[\frac{p_1}{x^2}e^{-\theta_1/x} - \frac{p_1\theta_1}{x^3}e^{-\theta_1/x}\right]\mu_{\tilde{x}_i}(x)dx\right) \quad \ldots(27)$$

3.1 Bayes estimation based on square error loss function:

The two parameters θ_1 and θ_2 and reliability function can be estimate by using

Lindely approximation from eq. (23), as follows:

Approximate Bayes Estimate of θ_1 based on SELF:

Assume that w in (23) equal to $w\,(\theta_1,\theta_2)=\theta_1$ and then:
$w_{\theta_1}=1, w_{\theta_1\theta_1}=w_{\theta_2}=w_{\theta_2\theta_2}=w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$

Now, Bayes estimate of θ_1 based on SELF according to Lindley's approximation referred to as $\hat{\theta}_{1BS}^{L}$ can be obtained by the following expression,

$$\hat{\theta}_{1BS}^{L}=E(\theta_1|\underline{\tilde{x}}) \qquad \ldots(28)$$

Approximate Bayes Estimate of θ_2 based on SELF:

Assume that w in (23) equal to $w\,(\theta_1,\theta_2)=\theta_2$, and then:
$w_{\theta_2}=1, w_{\theta_2\theta_2}=w_{\theta_1}=w_{\theta_1\theta_1}=w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$

Now, Bayes estimate of θ_2 based on SELF according to Lindley's approximation referred to as $\hat{\theta}_{2BS}^{L}$, can be obtained by the following expression:

$$\hat{\theta}_{2BS}^{L}=E(\theta_2|\underline{\tilde{x}}) \qquad \ldots(29)$$

Approximate Bayes Estimate of $R(t)$ based on SELF:

Assume that w in (23) equal to:

$$w\,(\theta_1,\theta_2)=\mathrm{R(t)}=p_1\left(1-e^{-\theta_1/t}\right)+(1-p_1)\left(1-e^{-\theta_2/t}\right)$$

and then:

$w_{\theta_1}=\frac{p_1}{t}e^{-\theta_1/t}, w_{\theta_1\theta_1}=\frac{-p_1}{t^2}e^{-\theta_1/t}$
$w_{\theta_2}=\frac{(1-p_1)}{t}e^{-\theta_2/t}, w_{\theta_2\theta_2}=\frac{-(1-p_1)}{t^2}e^{-\theta_2/t}$,
$w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$

Now, Bayes estimate of R(t)based on SELF according to Lindley's approximation, referred to as $\hat{R}_{BS}^{L}(t)$, can be obtained by the following expression:

$$\hat{R}_{BS}^{L}(t)=E\left(R(t)|\underline{\tilde{x}}\right) \qquad \ldots(30)$$

3.2 Bayes estimation based on quadratic loss function:

De Groot (1970) [4] discussed different types of loss function and obtained the Bayes estimates based on quadratic loss function (QLF) which is defined as:

$$L(\theta,\hat{\theta})=\left(1-\frac{\hat{\theta}}{\theta}\right) \qquad \ldots(31)$$

According to eq. (31), Bayes estimator of θ based on QLF is obtained by [3]:

$$\hat{\theta}_{BQ}=\frac{E\left(\frac{1}{\theta}|\underline{\tilde{x}}\right)}{E\left(\frac{1}{\theta^2}|\underline{\tilde{x}}\right)} \qquad \ldots(32)$$

Bayes estimate of scale parameters θ_1,θ_2 and reliability function R(t)based on QLF can be obtained as follows:

Approximate Bayes Estimate of θ_1 based on QLF:

For the estimation of the scale parameter θ_1 of MTIED based on quadratic loss function according to Lindley's approximation referred to as θ_{1BQ}^{L} can be obtained by the following expression,

$$\hat{\theta}_{1BQ}^{L}=\frac{E\left(\frac{1}{\theta_1}|\underline{\tilde{x}}\right)}{E\left(\frac{1}{\theta_1^{2}}|\underline{\tilde{x}}\right)} \qquad \ldots(33)$$

Put w in (23) equal to:

1) $w\,(\theta_1,\theta_2)=\frac{1}{\theta_1}$, and then $w_{\theta_1}=\frac{-1}{\theta_1^{2}}$, $w_{\theta_1\theta_1}=\frac{2}{\theta_1^{3}}$, $w_{\theta_2}=w_{\theta_2\theta_2}=w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$

2) $w\,(\theta_1,\theta_2)=\frac{1}{\theta_1^{2}}$,and then $w_{\theta_1}=\frac{-2}{\theta_1^{3}}$, $w_{\theta_1\theta_1}=\frac{6}{\theta_1^{4}}$, $w_{\theta_2}=w_{\theta_2\theta_2}=w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$

Approximate Bayes Estimate of θ_2 based on QLF:

For the estimation of the scale parameter θ_2 of MTIED based on quadratic loss function according to Lindley's approximation referred to as θ_{2BQ}^{L} can be obtained by the following expression:

$$\hat{\theta}_{2BQ}^{L}=\frac{E\left(\frac{1}{\theta_2}|\underline{\tilde{x}}\right)}{E\left(\frac{1}{\theta_2^{2}}|\underline{\tilde{x}}\right)} \qquad \ldots(34)$$

Assume that w in (23) equal to:

1) $w\,(\theta_1,\theta_2)=\frac{1}{\theta_2}$, and then $w_{\theta_2}=\frac{-1}{\theta_2^{2}}$, $w_{\theta_2\theta_2}=\frac{2}{\theta_2^{3}}$, $w_{\theta_1}=w_{\theta_1\theta_1}=w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$

2) $w(\theta_1,\theta_2)=\frac{1}{{\theta_2}^2}$,and then $w_{\theta_2}=\frac{-2}{{\theta_2}^3}$, $w_{\theta_2\theta_{21}}=\frac{6}{{\theta_2}^4}$, $w_{\theta_1}=w_{\theta_1\theta_1}=w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0$.

Approximate Bayes Estimate of $R(t)$ based on QLF:

For the estimation of R(t) of MTIED based on quadratic loss function according to Lindley's approximation referred to as $\mathrm{R(t)}_{BQ}^{L}$ can be obtained by the following expression,

$$\hat{R}(t)_{BQ}^{L}=\frac{E\left(\frac{1}{\mathrm{R(t)}}\middle|\tilde{x}\right)}{E\left(\frac{1}{(\mathrm{R(t)})^2}\middle|\tilde{x}\right)} \quad \ldots(35)$$

Assume that w in (23) equal to:

1) $w(\theta_1,\theta_2)=\frac{1}{\mathrm{R(t)}}$, and then

$$w_{\theta_1}=\frac{\frac{-p_1}{t}e^{-\theta_1/t}}{(\mathrm{R(t)})^2},$$

$$w_{\theta_1\theta_1}=\frac{\frac{p_1}{t^2}e^{-\theta_1/t}}{(\mathrm{R(t)})^2}\left[1+\frac{2p_1\,e^{-\theta_1/t}}{\mathrm{R(t)}}\right],$$

$$w_{\theta_1\theta_2}=\frac{\frac{2p_1(1-p_1)}{t^2}e^{-(\theta_1+\theta_2)/t}}{(\mathrm{R(t)})^3},$$

$$w_{\theta_2}=\frac{\frac{-(1-p_1)}{t}e^{-\theta_2/t}}{(\mathrm{R(t)})^2},$$

$$w_{\theta_2\theta_2}=\frac{\frac{(1-p_1)}{t^2}e^{-\theta_2/t}}{(\mathrm{R(t)})^2}\left[1+\frac{2(1-p_1)e^{-\theta_2/t}}{\mathrm{R(t)}}\right],$$

$$w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$$

2) $w(\theta_1,\theta_2)=\frac{1}{(\mathrm{R(t)})^2}$, and then:

$$w_{\theta_1}=\frac{\frac{-2p_1}{t}e^{-\theta_1/t}}{(\mathrm{R(t)})^3},$$

$$w_{\theta_1\theta_1}=\frac{\frac{2p_1}{t^2}e^{-\theta_1/t}}{(\mathrm{R(t)})^3}\left[1+\frac{3p_1\,e^{-\theta_1/t}}{\mathrm{R(t)}}\right],$$

$$w_{\theta_2}=\frac{\frac{-2(1-p_1)}{t}e^{-\theta_2/t}}{(\mathrm{R(t)})^3},$$

$$w_{\theta_2\theta_2}=\frac{\frac{2(1-p_1)}{t^2}e^{-\theta_2/t}}{(\mathrm{R(t)})^3}\left[1+\frac{3(1-p_1)e^{-\theta_2/t}}{\mathrm{R(t)}}\right],$$

$$w_{\theta_1\theta_2}=\frac{\frac{6p_1(1-p_1)}{t^2}e^{-(\theta_1+\theta_2)/t}}{(\mathrm{R(t)})^4},$$

$$w_{\theta_1\theta_2}=w_{\theta_2\theta_1}=0.$$

4. Simulation Study and Results

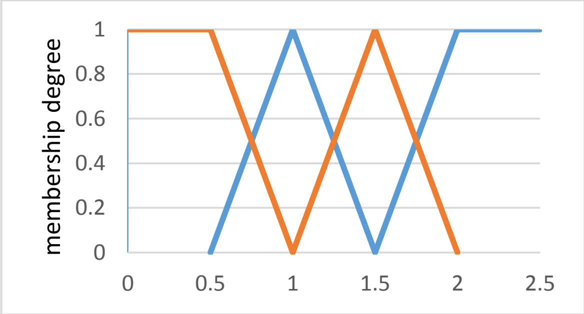
We obtained, in the above section, MLEs and Bayesian estimates of two parameters θ_1, θ_2 and reliability function $\mathrm{R(t)}$ for mixture of MTIED. We can obtain Bayes estimation based on square error loss function (LS-NR) and quadratic loss functions (LQ-NR).The MLEs are obtained as well via the Newton-Raphson (NR) algorithm. The following algorithm will be used to generate the samples, each observation of the generated samples was made to be fuzzy observation and then calculate the estimators:

1. We have generate (100) i.i.d. random samples from the (MTIED) with different sample sizes $n_1=15,30,75$ for first subpopulation and $n_2=15,30,75$ for second subpopulation where $n=n_1+n_2$, i.e., $(n=30,60,150)$ represent small, median and large sample sizes respectively, through the adoption of inverse transformation method with scale parameters $\theta_1=0.5,1$ for first subpopulation and $\theta_2=0.5,0.6$ for second subpopulation and $p=\frac{n_1}{n}$.

Table (1)
Parameter choices for mixture two inverse exponential distributions

Case	$\boldsymbol{\theta_1}$	$\boldsymbol{\theta_2}$
1	0.5	0.5
2	0.5	0.6
3	1	0.5
4	1	0.6

2. Then, by employing fuzzy information system $\{\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4\}$ corresponding to the following membership functions that shown in figure (1), each observation of the generated samples was made to be fuzzy observation.



FIS used to encode the Simulated Data.

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1\,, x \le 0.5 \\ \frac{1-x}{0.5}\,, 0.5 \le x \le 1 \\ 0\,, o.w \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.5}{0.5}\,, 0.5 \le x \le 1 \\ \frac{1.5-x}{0.5}\,, 1 \le x \le 1.5 \\ 0\,, o.w \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-1}{0.5}\,, 1 \le x \le 1.5 \\ \frac{1-x}{0.5}\,, 1.5 \le x \le 2 \\ 0\,, o.w \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-1.5}{0.5}\,, 1.5 \le x \le 2 \\ 1\,, x \ge 2 \\ 0\,, o.w \end{cases}$$

3. In order to deal with non-informative gamma priors, the hyper parameters are chosen to be respectively, $a = b = c = d = 0.0000001$.
4. The initial values which required for proceeding algorithm are used to be symmetrical rank regression estimators. The iterative process stops when the absolute difference between two successive iteration becomes less than $\epsilon = 0.0001$.
5. The obtained Bayes estimates of the parameters θ_1 and θ_2 were compared based on average values from Mean Square Error (MSE) whereas the obtained Bayes estimates of the reliability function were compared based on average values from Integrated Mean Square Error (IMSE), where:

$$MSE(\hat{\theta}_1) = \frac{\sum_{j=1}^{L}(\hat{\theta}_1-\theta_1)^2}{L}$$

$$MSE(\hat{\theta}_2) = \frac{\sum_{j=1}^{L}(\hat{\theta}_2-\theta_2)^2}{L}$$

$$IMSE(\hat{R}(t)) = \frac{1}{L}\sum_{j=1}^{L}\left(\frac{1}{n_t}\sum_{i=1}^{n_t}\hat{R}_j(t_i) - R(t_i)\right)$$

6. The simulation program has been written by using MATLAB (R2010b) program and the computational results have been summarized in the tables (2)…(4).

5. Conclusions and Recommendations

- Results in tables (2,3,4) appears that the MSE and IMSE values are decreasing as the sample sizes increasing.
- Tables (2,3) indicate that the MLE based on NR algorithm introduced the best perform "smallest MSE values "comparing with the Bayes estimates under square error loss function(LS-NR) and the Bayes estimates under quadratic loss functions (LQ-NR) for (MTIED) with all sample sizes and four different cases.
- Table (2) increase the value of the scale parameter for first subpopulation **from** $\boldsymbol{\theta_1 = 0.5}$ to $\boldsymbol{\theta_1 = 1}$, increasing the values MSE.
- Table (3) increase the value of the scale parameter for second subpopulation, from $\boldsymbol{\theta_2 = 0.5}$ to $\boldsymbol{\theta_2 = 0.6}$, increasing the values MSE associated with MLE based on NR algorithm and LS-NR.
- Table(4) indicate that the IMLE based on NR algorithm introduced the best perform comparing with LS-NR and LQ-NR for small sample size As well as for moderate sample size for all cases expect $\boldsymbol{\theta_{1=}1}, \boldsymbol{\theta_2 = 0.5}$

Based on conclusions stated above, for estimating the parameters of (MTIED), we recommend choosing the MLE based on NR algorithm for estimating the parameters whereas for estimating the reliability function choosing the LS-NR for large sample sizes.

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Table (2)
MSE values of the estimates of θ_1 for different samplesizes.

Case	*Estimate / Sample sizes*	$\hat{\theta}_{1\,ML}^{NR}$	$\hat{\theta}_{1_{BS}}^{L}$	$\hat{\theta}_{1_{BQ}}^{L}$
1	30	0.0233932	0.2005009	0.0991962
	60	0.0183723	0.0806426	0.0578716
	150	0.0163274	0.0294155	0.0404439
2	30	0.0158582	0.0459260	0.1170193
	60	0.0121490	0.0240735	0.0818985
	150	0.0106996	0.0184793	0.0289910
3	30	0.2466995	0.6436168	0.6857820
	60	0.2360822	0.4862397	0.6218401
	150	0.2213613	0.4511455	0.5171946
4	30	0.2102429	0.9497045	0.5782250
	60	0.2082938	0.5096119	0.5412154
	150	0.2043110	0.3965683	0.4589043

Table (3)
MSE values of the estimates of θ_2 for different sample sizes.

Case	Estimate / Sample sizes	$\hat{\theta}_{2\,ML}^{NR}$	$\hat{\theta}_{2BS}^{\;L}$	$\hat{\theta}_{2BQ}^{\;L}$
1	30	0.0244574	0.1348539	0.0908540
	60	0.0183518	0.0808005	0.0598297
	150	0.0163358	0.0295862	0.0403106
2	30	0.0457090	0.0881518	0.1482921
	60	0.0421992	0.0756958	0.1177494
	150	0.0393457	0.0730716	0.0938337
3	30	0.0178490	0.2143132	0.1053265
	60	0.0083644	0.0280595	0.0602245
	150	0.0030963	0.0068288	0.0078554
4	30	0.0231579	1.1713622	0.0860626
	60	0.0091552	0.0918873	0.0583447
	150	0.0052717	0.0097797	0.0194922

Table (4)
IMSE values of the estimates of $R(t)$ for different sample sizes.

Case	Estimate / Sample sizes	$\hat{R}(t)_{ML}^{NR}$	$\hat{R}(t)_{BS}^{L}$	$\hat{R}(t)_{BQ}^{L}$
1	30	0.0044231	0.1891220	0.0600726
	60	0.0033442	0.1375865	0.0599771
	150	0.0029349	0.0019619	0.0598943
2	30	0.0048832	0.0044912	0.0698393
	60	0.0041619	0.0028614	0.0697503
	150	0.0037947	0.0025382	0.0696669
3	30	0.0085685	0.3069143	0.1078105
	60	0.0071472	0.0074717	0.1077407
	150	0.0070861	0.0060068	0.1075071
4	30	0.0084504	122.271391	0.1207336
	60	0.0080880	0.0787380	0.1206586
	150	0.0073630	0.0063894	0.1203874