Special Issus: 1st Scientific International Conference, College of Science, Al-Nahrain University, 21-22/11/2017, Part I, pp.186-195

Small duo and Fully small stable modules

Mehdi Sadiq Abbas and Hiba Ali Salman
Department of Mathematics, College of Science, Mustansiriya University, Baghdad-Iraq.

Abstract

New types of modules, namely fully small stable and small duo modules over a ring are introduced and investigated. These concepts lead to study the relation between these types and other classes of modules; such as, uniserial, some classes of multiplication and quasi-injective modules. It is shown that a projective module is small duo if and only if it is a small multiplication. Also, uniserial Artinian module is fully small stable. In addition, if R is a commutative ring then a fully small stable module is equivalent to small duo and small principally quasi injective. Also, we discuss full small stability of direct sum of modules. [DOI: 10.22401/ANJS.00.1.25]

1.Introduction

Throughout this paper, R represent an associative ring with non-zero identity, unless otherwise stated, all modules are unitary left R-modules. Let M be an R-module, a submodule N of M is called ***stable (fully invariant)***, if N contains $\alpha(N)$ for each R-homomorphism $\alpha: N \longrightarrow M$ ($\alpha: M \longrightarrow M$). an R-module M is called ***fully stable (duo)*** if each submodule of M is stable (fully invariant), [1], [2]. It is clear that every fully stable module is duo but the converse may not be true.

A good source of duo modules is provided by multiplication modules. If R is a commutative ring, then an R-module M is called ***multiplication*** if each submodule of M has the form IM for some left ideal I of R [3].

Many well-known concepts are generalized relative to small submodules, for example, recall that an R-module M is injective if and only if for each left ideal I of R and each R-homomorphism from I into M can be extended to all R. A submodule N of M is called ***small*** if N+K=M implies that K=M for each proper submodule K of M [4]. In [5], an R-module M is called ***small injective*** if and only if every R-homomorphism from small left ideal I of R into M can be extended to an R-homomorphism from R into M.

Let M be an R-module. We denote the Jacobson radical of M by $J(M)$ which is defined as the intersection of all maximal submodules of M and $J(M) = M$ in case M has no maximal submodule. Equivalently, $J(M)$ is the sum of all small submodules of M. A cyclic submodule Rx of M is small if and only if $x \in J(M)$ [6]. For a subset X of an R-module M, the left annihilator of X in R is denoted by:

$$l_R(X) = \{r \in R | rx = 0 \; for \; all \; x \; in \; X\}$$

the right annihilator of a subset Y of R in M is denoted by:

$$r_M(Y) = \{m \in M | ym = 0 \; for \; all \; y \; in \; Y\}$$

In this paper, we introduce and study the concepts of fully small stable modules, small duo and small multiplication modules by restricting the conditions of the above concepts to small submodules. Some properties and characterizations of these new concepts are obtained. We prove results that provide a good source of these concepts. Finally, we discuss full small stability and small duo on internal direct sum.

2. Fully Small Stable Modules

Definition (2.1): A left R-module M is called ***fully small stable*** if every small submodule N of M is stable.

Next, we shall state some of the results that appears in [6] about small submodules, for the sake of completion and the reader can find the proof in [6].

Lemma (2.2): Let M be an R-module. Then:

a. If $A \subseteq B \subseteq M \subseteq N \; and \; B \ll M$, then $A \ll N$.

b. If $A_i \ll M$ for each $i = 1, \dots, n$, then $\sum_{i=1}^{n} A_i \ll M$.

c. If $\alpha: M \longrightarrow N$ is an R-homomorphism and $A \ll M$, then $\alpha(A) \ll N$.

The following proposition shows that the property of a module being fully small stable can depend on a very restricted kind of small submodules.

Proposition (2.3): Let M be an R-module. Then M is fully small stable if and only if every small cyclic submodule of M is stable.

Proof:

Let N be a small submodule of M and $\alpha: N \longrightarrow M$ an R-homomorphism. Then for each x in N we have $Rx \subseteq N$ and $Rx \ll M$ by lemma (2.2(a)), thus $\alpha(Rx) \subseteq Rx$ and hence $\alpha(N) \subseteq N$. ■

It is clear that every fully stable module is a fully small stable one, but the converse is not true generally. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is fully small stable since the zero submodule (0) is the only small submodule of it which is stable. But $\mathbb{Z}$ as $\mathbb{Z}$-module is not fully stable, since there exists a homomorphism $f: 2\mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(2x)=3x$ I for each $x \in \mathbb{Z}$,clearly $f(2\mathbb{Z}) \nsubseteq 2\mathbb{Z}$.

Recall that, a ring R is called ***Quasi-Frobenius,*** satisfying that R as a left R-module is ***Noetherian*** [6] and for each submodule A of ${}_RR$, then $l_R(r_R(A)) = A$ [4,p.336].

A ***semisimple*** module is the module that every submodule of it is a direct summand [9, p.166].

Examples and remarks (2.4):

a. The two concepts of fully stable modules and fully small stable modules coincide if the module has no maximal submodules; that is, $J(M)=M$.
b. The $\mathbb{Z}$-module $\mathbb{Q}$ is not fully small stable. Since it is well-known that $\mathbb{Q}$ has no maximal submodules, and by using the above remark along with the fact that $\mathbb{Q}$ as $\mathbb{Z}$-module is not fully stable module [1], that if we define an R-homomorphism $\alpha: \mathbb{Z} \longrightarrow \mathbb{Q}$ by $(z) = \left(\frac{2}{3}\right) z$, then it is an easy matter to verify that α is a well-defined R-homomorphism. But $\alpha(1) = \frac{2}{3} \notin \mathbb{Z}$, and hence $\alpha(\mathbb{Z}) \nsubseteq \mathbb{Z}$.
c. In the same manner of the proof of example (b) we can show that $\mathbb{Q}/\mathbb{Z}$ as $\mathbb{Z}$-module is not fully small stable module. It remains only to state the proof of $J(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ for the sake of completion, define $\pi: \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ to be the natural epimorphism and using the correspondence theorem for modules, we get that:
   $\mathbb{Q}/\mathbb{Z} = \pi(\mathbb{Q}) = \pi(J(\mathbb{Q})) \subseteq J(\mathbb{Q}/\mathbb{Z})$
   which ends the proof.
d. The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is fully small stable. Since every submodule $\mathbb{Z}_{p^k}$ of $\mathbb{Z}_{p^\infty}$ is a small submodule [10,p.73], and for each R-homomorphism $\alpha: \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^\infty}$, $\alpha(\mathbb{Z}_{p^k})$ consists of those elements of order less than or equal to p^k. Thus $\alpha(\mathbb{Z}_{p^k}) \subseteq \mathbb{Z}_{p^k}$.
e. Every semisimple module is fully small stable one trivially, but there exist semisimple modules which are not fully stable. For example, the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
f. It is mentioned in [1] that every Quasi-Frobenius ring is fully stable, and hence fully small stable.

[1] has mentioned that the sum of stable submodules is stable, and the finite sum of small submodules is small (proposition (2.2(b))) implies that the finite sum of small stable submodules is a small stable submodule.

The following proposition is a characterization of fully small stable modules.

Proposition (2.5): Let M be an R-module. Then the following statements are equivalent:

1. M is fully small stable module.
2. Every submodule N of M is fully small stable.
3. If N and K are two submodules of M such that K $\ll$ M and N is an epimorphic image of K, then N $\subseteq K$.

Proof:

(1)$\Leftrightarrow$ (2) Obvious.

(1) $\Rightarrow$ (3) Let N$\subseteq M$, K$\ll M$ and $\alpha: K \longrightarrow N$ be any R-epimorphism. Let $x \in N$ then there exists y in K such that $\alpha(y) = x$. Now, since K $\ll$ M and considering $i: N \longrightarrow M$ to be the inclusion homomorphism then $(i \circ \alpha)(K) \subseteq$ K, using M being fully small stable module. Then $\alpha(K) \subseteq$ K, but $\alpha(K) = N$ and hence N $\subseteq K$.

(3) $\Rightarrow$ (1) Let C be a small submodule of M and $\alpha: C \longrightarrow M$ be an R-homomorphism. Then $\alpha: C \longrightarrow \alpha(C)$ is an epimorphism and putting $\alpha(C) = N$ in the assumption we get that $\alpha(C) \subseteq C$. ■

Recall that an R-module M is called ***finitely supplemented*** if every finitely generated submodule of M has a supplement in M [9, p.349]; that is, if U and V are submodules of M, then we say that V is a supplement of U in M if U+V=M and $(U \cap V) \ll V$ [9].

Corollary (2.6): Let M be a finitely supplemented R-module. Then M is fully small stable if and only if every 2-generated small submodule B of M is fully small stable.

Proof:

$\Rightarrow$) Clear by the use of proposition (2.5).

$\Leftarrow$) Let N be a small submodule of M and $\alpha: N \longrightarrow M$ an R-homomorphism. Let $x\in N$ then $Rx \subseteq N \ll M$ implies that $Rx \ll M$ by proposition (1.1.2(a)) and $\alpha(Rx) \ll M$ by ( ). Let C=Rx + R$\alpha(x)$, then C$\ll M$ by proposition (2.2(b)) and generated by two elements x and $\alpha(x)$. Since C has a supplement in M by assumption, then R$x \ll$ C. Now, $\alpha|_{Rx}: Rx \longrightarrow C$ is an R-homomorphism and since C is fully small stable by assumption thus $\alpha(Rx) = \alpha|_{Rx}(Rx) \subseteq Rx \subseteq N$. Hence $\alpha(N) \subseteq N$.

The following corollary shows that there are modules that are not fully small stable.

Corollary (2.7): Suppose R is a ring and M a fully small stable R-module. If R as R-module is a small submodule of M, then ${}_RM = {}_RR$.

Proof:

Part (2) of proposition (2.5) we get that ${}_RR$ is fully small stable. Moreover, every cyclic R-module is a homomorphic image of ${}_RR$. Thus the result holds by using part (3) of proposition (2.5). ■

The following proposition gives a characterization of fully small stable modules with respect to the annihilators of its small cyclic submodules.

Proposition (2.8): Let M be an R-module. Consider the following statements:

1. M is a fully small stable module.
2. $r_M(l_R(Rx)) = Rx$ for each x in *J(M)*.
3. $l_R(x) \subseteq l_R(y)$ implies that $y \in Rx$ for each x in *J(M)* and y in *M*.
4. $r_M(Rb \cap l_R(x)) = r_M(b) + Rx$ for each x in *J(M)* and b in R.

Then (4) $\longrightarrow$ (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3). In additional, if R is commutative then (1) $\longrightarrow$ (4).

Proof:

(1) $\longrightarrow$ (2) Let x in *J(M)*. it is always true that $Rx \subseteq r_M(l_R(Rx))$. Let $y \in r_M(l_R(Rx))$. Define $\theta: Rx \longrightarrow M$ by $\theta(rx) = ry$ for all r in R. then θ is a well-defined clearly R-homomorphism. By (1) $\theta(Rx) \subseteq Rx$ and hence $y \in Rx$.

(2) $\longrightarrow$ (3) If $l_R(x) \subseteq l_R(y)$ then $Ry \subseteq r_M(l_R(Ry)) \subseteq r_M(l_R(Rx)) = Rx$ for each x in *J(M)* and y in *M*.

(3) $\longrightarrow$ (1) Let Rx be a small cyclic submodule of M and $\theta: Rx \longrightarrow M$ an R-homomorphism. Then $l_R(x) \subseteq l_R(\theta(x))$ and by (3) $\theta(x) \in Rx$ and hence $\theta(Rx) \subseteq Rx$.

(4) $\longrightarrow$ (2) By taking *b=1*.

(3) $\longrightarrow$ (4) Let $x \in r_M(Rb \cap l_R(a))$. Then $l_R(ab) \subseteq l_R(xb)$. By (3) we get $Rxb \subseteq Rab$, then $xb = tab$ for some t in R. Thus $(x - ta)b = 0$ and hence $(x - ta) \in r_M(b)$, so $x \in r_M(b) + Ra$. Thus $r_M(Rb \cap l_R(x)) \subseteq r_M(b) + Rx$. The other inclusion is always true. ■

Corollary (2.9): The following are equivalent for a ring R:

1. R is a left fully small stable ring.
2. $r_R(l_R(r)) = Rr$ for each r in *J(R)*.
3. $l_R(r) \subseteq l_R(s)$ implies that $Rs \subseteq Rr$ for all r in *J(R)* and $s \in R$.
4. $r_R(Rs \cap l_R(r)) = r_R(s) + Rr$ for all $r \in J(R)$ and $s \in R$.

A fully small stable R-module M can be characterized using the trace of a small submodule N of M. Where M is fully small stable if and only if every small submodule N of M is equal to its trace in M, where ***the trace*** of a submodule N of M in M is the set $tr(N, M) = \{\sum_{i=1}^{k} \varphi_i(n_i) \,|\varphi_i \in Hom(N, M), n_i \in N \text{ and } k \in \mathbb{N}\}$ [9, p.107].

Recall that an R-module M is called ***uniserial*** if $N \subseteq L \text{ or } L \subseteq N$ for all submodules N and L of M [9, p.539]. the following proposition gives a source of fully small stable modules.

Proposition (2.10): Let M be a uniserial R-module. If M has descending chain condition on small cyclic submodules, then M is fully small stable. In particular, every uniserial Artinian module is fully small stable.

Proof:

Let $x \in J(M)$ and $y \in M$ with $l_R(x) \subseteq l_R(y)$. If $y \notin Rx$ then $x \in Ry$. So there exists t in R such that $x = ty$. Consider the following descending chain

$Rx = Rty \supseteq Rt^2y \supseteq Rt^3y \supseteq \cdots$

of small cyclic submodules of M. Then there exists a positive integer n such that $Rt^ny = Rt^{n+1}y$ and so $t^ny = rt^{n+1}y$ for some r in R. Now, $t^nx = rt^{n-1}x$, and since $l_R(x) \subseteq l_R(y)$, then $t^ny = rt^{n-1}y$. Continue in this manner, we get $y = rty = rx \in Rx$ which is a contradiction! Therefore, $Ry \subseteq Rx$ and hence by proposition (2.8) M is a fully small stable module. ■

Corollary (2.11): Let R be a ring that satisfies the descending chain condition on principal left ideals. Then every uniserial R-module is fully small stable.

Proof:

Let M be a uniserial R-module. By proposition (2.10) it is enough to show that M has a descending chain condition on small cyclic submodules of M. Let $a, b \in J(M)$ and $Rb \subseteq Ra$. Then there is $r \in R$ such that $Rb = Rra$. Thus every descending chain of small cyclic submodules of M is of the form

$Ra \supseteq Rr_1a \supseteq Rr_1r_2a \supseteq \cdots$

Now, consider the descending chain

$R \supseteq Rr_1 \supseteq Rr_1r_2 \supseteq \cdots$

of principal left ideals of R. so there is a positive integer k such that $r_1r_2 \ldots r_kR = r_1r_2r_{k+1}R$. This shows that M has descending chain condition on small cyclic submodules of M. Therefore, M is fully small stable. ■

We need the following lemma which appear in [1].

Lemma (2.12): Let M be an R-module and A a left ideal of R. Then $r_M(A) \cong Hom_R(R/A, M)$.

Proposition (2.13): Let M be a fully small stable R-module. Then the following statements hold:

1. Distinct small submodules of M are not isomorphic.
2. $Rx \cong Hom_R(Rx, M)$ for each x in $J(M)$.

Proof:

1. Assume that M has distinct small submodules N_1, N_2 with $N_1 \cong N_2$. With no loss of generality, if we assume that $N_1 \nsubseteq N_2$, then there is an element x in N_1 not in N_2. Let $\theta: N_1 \longrightarrow N_2$ be an isomorphism and consider the following two R-homomorphism
   $i_{N_2} \circ \theta: N_1 \longrightarrow M$ and $i_{N_1} \circ \theta^{-1}: N_2 \longrightarrow M$
   where i_K is the inclusion mapping of a submodule K of M into M. Full small stability of M implies that $(i_{N_2} \circ \theta)(N_1) \subseteq N_1$ and $(i_{N_1} \circ \theta^{-1})(N_2) \subseteq N_2$. Now, $x = (i_{N_1} \circ \theta^{-1} \circ i_{N_2} \circ \theta)(x) \in N_2$ which is a contradiction.
2. Let $x \in J(M)$. then by lemma (2.12) and proposition (2.8), $Rx = r_M\big(l_R(Rx)\big) \cong Hom_R\left(\frac{R}{l_R(Rx)}, M\right) \cong Hom_R(Rx, M)$. ■

Recall that, a monomorphism $\alpha: M \longrightarrow N$ is called small monomorphism if $\alpha(M)$ is small in N.

Proposition (2.14): If M is fully small stable R-module, then every small monomorphism α in End_R is an epimorphism.

Proof:

Suppose $\alpha: M \longrightarrow M$ is a small monomorphism and define $\beta: \alpha(M) \longrightarrow M$ by $\beta\big(\alpha(m)\big) = m$ for all m in M. Then β is a well-defined R-homomorphism and clearly that $\beta\big(\alpha(M)\big) = M$. since M is fully small stable, then $M = \beta\big(\alpha(M)\big) \subseteq \alpha(M) \subseteq M$, so α is an epimorphism. ■

Recall that, a left ideal A of R is called an ***idempotent*** providing that $A^2 = A$. Moreover, a ***projective*** R-module M is characterized in many senses in [6,p.120].

Proposition (2.15): Every small and projective left ideal of a left fully small stable ring is idempotent.

Proof:

Let A be a small and projective left ideal of a lefty fully small stable ring R. Since A is projective, then A = *tr*(A,R)A, by [ ]. But *tr*(A,R) = A and hence $A^2 = A$. ■

Another characterization of fully small stable modules is applied next. First recall that an R-module M is said to satisfy ***Baer's criterion*** if for each submodule N of M and each R-homomorphism $\alpha: N \longrightarrow M$, there exists r in R such that $\alpha(n) = rn$ for each $n \in N$.

Proposition (2.16): Let M be an R-module. Then M is fully small stable if and only if M satisfies Baer criterion on its small cyclic submodules.

Proof:

$\Longrightarrow$) Let Rx be a small cyclic submodule of M and $\alpha: Rx \longrightarrow M$ be an R-homomorphism. Full small stability of M implies that $\alpha(Rx) \subseteq Rx$, and hence for each $x \in Rx$ there exists $r \in R$ such that $x = rx$.

$\Longleftarrow$) Let N be a small cyclic submodule of M and $\alpha: N \longrightarrow M$ an R-homomorphism. Then for each $x \in N$, $Rx \ll M$ and hence there exists $r \in R$ such that $\alpha(n) = rn$, thus $\alpha(Rx) \subseteq Rx$ for each $x \in N$. In particular, $\alpha(x) \in Rx \subseteq N$ and then $\alpha(N) \subseteq N$. Thus M is fully small stable. ■

Proposition (2.17): Let M be an R-module such that $l_R(N \cap K) = l_R(N) \cap l_R(K)$ for every finitely generated, small submodules N and K of M. then M is fully small stable if and only if M satisfies Baer's criterion on its finitely generated, small submodules.

Proof:

$\Longrightarrow$) Let N be a finitely generated small submodule of M and $\alpha: N \longrightarrow M$ an R-homomorphism. Then $N = Rx_1 + Rx_2 + \cdots + Rx_n$ for some $x_1, x_2, \ldots, x_n$ in N. we use induction on the number of generators of N.

If $n = 1$, then it is only proposition (2.16). Now, suppose Baer's criterion holds for all small submodules generated by m elements for $m \leq n-1$, full small stability of M implies that there exists two elements $r, s \in R$ such that $\alpha(x) = rx$ for each $x \in Rx_1 + Rx_2 + \cdots + Rx_{n-1}$ and $\alpha(x^\circ) = sx^\circ$ for each $x^\circ \in Rx_n$. Let $y \in ((Rx_1 + Rx_2 + \cdots + Rx_{n-1}) \cap Rx_n)$. Then $\alpha(y) = ry = sy$ implies that $r - s \in l_R\big((Rx_1 + Rx_2 + \cdots + Rx_{n-1}) \cap Rx_n\big)$, and by hypothesis there exists $u + v \in l_R(Rx_1 + Rx_2 + \cdots + Rx_{n-1}) + l_R(Rx_n)$ such that $r - s = u + v$, hence $r - u = s + v = t$. For each $z \in N, z = \sum_{i=1}^{n} r_i x_i$ for some $r_i \in R$ $i = 1, \ldots, n$. $\alpha(z) = \alpha(\sum_{i=1}^{n} r_i x_i) = \alpha(\sum_{i=1}^{n-1} r_i x_i) + \alpha(r_n x_n) = r(\sum_{i=1}^{n-1} r_i x_i) + s(r_n x_n) = r(\sum_{i=1}^{n-1} r_i x_i) - u(\sum_{i=1}^{n-1} r_i x_i) + s(r_n x_n) + v(r_n x_n) = (r - u)(\sum_{i=1}^{n-1} r_i x_i) + (s + v)(r_n x_n) = t(\sum_{i=1}^{n-1} r_i x_i) + t(r_n x_n) = t(\sum_{i=1}^{n} r_i x_i) = tz$.

$\Longleftarrow$) If Baer's criterion holds for finitely generated, small submodules of M, then it holds for small cyclic submodules of M and thus proposition (2.16) ends the proof. ■

Corollary (2.18): Let M be a Noetherian R-module and $l_R(N \cap K) = l_R(N) \cap l_R(K)$ for every small submodules N and K of M. Then M is fully small stable if and only if Baer's criterion holds for M.

In proposition (2.8) we showed that M is fully small stable if and only if every small cyclic submodule of M satisfies the double annihilator condition. So the next result is of some interest.

Proposition (2.19): Let M be a fully small stable R-module such that for each $x \in J(M)$ and left ideal A of R, each R-homomorphism from Ax into M can be extended to an R-homomorphism from Rx into M. If a submodule N of M satisfies the double annihilator condition, then so does N+Rx.

Proof:

Denote $l_R(N)$ by A and $l_R(Rx)$ by B. Then by the assumption and proposition (1.2.7) we get that $r_M(l_R(N)) = N$ and $r_M\big(l_R(Rx)\big) = Rx$. Now, since $l_R(N + Rx) = l_R(N) \cap l_R(Rx) = A \cap B$, then it is enough to show that $r_M(A \cap B) \subseteq N + Rx$. Now, let $y \in r_M(A \cap B)$ and define $\theta: Ax \longrightarrow M$ by $\theta(ax) = ay$ for each $a \in A$, if *ax=0* then $a \in l_R(x) = B$ hence $a \in (A \cap B)$ and since $y \in r_M(A \cap B)$ then *ay=0*. Therefore, θ is a well-defined clearly a homo. The use of our assumption implies that there exists an extension $\alpha: Rx \longrightarrow M$ of θ, and $\alpha(Rx) \subseteq Rx$ since M is a fully small stable module implies that $a\alpha(x) = \alpha(ax) = ay$ for each a in A. Then $a(\alpha(x) - y) = 0$ implies that $\alpha(x) - y \in r_M(A) = N$; that is, there exists $n \in N$ such that $\alpha(x) - y = n$ or $y = n + \alpha(x) \in N + Rx$. Thus:

$$N + Rx = r_M(l_R(N + Rx)). \quad ■$$

Proposition (2.20): Let M be a fully small stable R-module such that for each $x \in J(M)$ and left ideal A of R, each R-homomorphism from Ax into M can be extended to an R-homomorphism from Rx into M. If M is fully small stable, then each finitely generated submodule of *J(M)* satisfies the double annihilator condition.

Proof:

Let $N = \sum_{i=1}^{n} R\,x_i$ be a finitely generated submodule of *J(M)* for some $x_1, x_2, \dots, x_n \in J(M)$. We shall use induction on *n*. For *n=1*, there is nothing more than proposition (2.8). Suppose that the double annihilator condition holds for finitely generated submodules generated by m elements where $m \leq n+1$. Then by proposition (2.19) the double annihilator condition is satisfied for submodules generated by *m+1*. ■

Recall that, an R-module M is called ***quasi-injective*** providing that for any submodule X of M and any R-homomorphism ϕ: $X \rightarrow M$ there exists $\bar{\phi} \in End_R$(M) such that the restriction map $\bar{\phi}|_X$ coincides with ϕ.

Corollary (2.21): Let M be a quasi-injective R-module. Then the following are equivalent:

1. M is fully small stable.
2. $r_M\big(l_R(N)\big) = N$ for each finitely generated submodule N of *J(M).*
3. Small duo modules

Now, we shall introduce duo property of modules relative to small submodules.

Definition (3.1): An R-module M is called ***small duo*** if every small submodule of M is fully invariant. A ring R is called left small duo if ${}_R R$ is small duo; that is, every small left ideal of R is two sided.

Examples and remarks (3.2):

a. It is clear that duo modules are small duo, and the converse is true in case of ***hollow modules,*** where an R-module M is called hollow if every submodule of it is small [9, p.351].
b. It is an easy manner to see that an R-module M is small duo if and only if every cyclic small submodule of M is fully invariant.

The following proposition gives characterization of small duo modules.

Proposition (3.3): An R-module M is small duo if and only if for each R-endomorphism α of M and each element x in $J(M)$, there is an element r in R such that $\alpha(x) = rx$.

Proof:

Let α be an R-endomorphism of M and $x \in J(M)$, then Rx is a small submodule of M and hence the necessity follows from $\alpha(Rx) \subseteq Rx$. For sufficiency, note that the stated condition implies that $\alpha(N) \subseteq N$ for each small submodule N of M. Thus M is small duo. ■

The following theorem gives a good source of small duo modules.

Theorem (3.4): Let M be a uniserial R-module which is satisfying the ascending chain condition on small cyclic submodules of M. Then M is small duo. (In particular, every uniserial Noetherian module is small duo).

Proof:

Assume that M has ascending chain condition on small cyclic submodules of it, $m(\neq 0) \in J(M)$ and α an endomorphism of M with $\alpha(m) \notin Rm$. Then $m \in R\alpha(m)$ and there is r in R such that $m = r\alpha(m)$, note that $R\alpha(m)$ is a small cyclic submodule of M. it follows that $\alpha^n(m) = r\alpha^{n+1}(m)$ for each positive integer n. Consider the following ascending chain

$Rm \subseteq R\alpha(m) \subseteq R\alpha^2(m) \subseteq \cdots$

There is a positive integer k such that $R\alpha^k(m) = R\alpha^{k+1}(m)$ and hence there is s in R such that $\alpha^{k+1}(m) = s\alpha^k(m) = \alpha^k(sm)$. Then $\alpha(m) - sm \in \ker(\alpha^k)$. If $Rm \subseteq \ker(\alpha^k)$, then $\alpha^k(m) = 0$ and hence $m = r^k\alpha^k(m) = 0$ which is a contradiction, thus $\ker(\alpha^k) \subseteq Rm$ and hence $\alpha(m) - sm \in Rm$ implies that $\alpha(m) \in Rm$ a contradiction!. By proposition (3.3), M is small duo. ■

It was proved in proposition (2.10) the following: Let M be a uniserial R-module. If M has the descending chain condition on small cyclic submodules of M, then M is fully small stable and hence M is small duo.

Proposition (3.5): Every direct summand of small duo module is small duo.

Proof:

Let $M = M_1 \oplus M_2$ be a direct sum of submodules of M_1 *and* M_2. Let α be an endomorphism of M_1 and N a small submodule of M_1. Then $j_1 \circ \alpha \circ \rho_1$ is an endomorphism of M. Thus $(N) =$

$j_1(\alpha(\rho_1(N))) \subseteq N$. This shows that M_1 is small duo. ■

In general, small submodules of small duo modules need not be small duo modules. Recall that a module M is ***skew-injective*** if each endomorphism of each submodule of M can be extended to an endomorphism of M [11], and so the following proposition is interesting.

Proposition (3.6): Let M be a small duo module. If M is skew injective, then every submodule of M is small duo.

Proof:

Let $N \subseteq L$ be submodules of M with N small in L and α an endomorphism of L. Since M is skew-injective then α can be extended to an endomorphism $\bar{\alpha}$ of M. Then:

$\alpha(N) = \bar{\alpha}(N) \subseteq N$

This implies that L is small duo. ■

Recall that an epimorphism $\alpha: M \longrightarrow N$ is called ***small epimorphism*** if $\ker(\alpha)$ is small in M [9, p159]. An R-module M is called ***small quasi-projective*** if for each R-module A and small epimorphism $\alpha: M \longrightarrow A$, each R-homomorphism $\beta: M \longrightarrow A$ can be lifted to an endomorphism of M.

Proposition (3.7): Let M be a small duo R-module. If M is small quasi projective, then any small homomorphic image of M is small duo.

Proof:

Let K be a submodule of M and H a small submodule of M containing K. Let α be an endomorphism of M/K. Small quasi projectivity of M implies that there is an endomorphism $\bar{\alpha}$ of M such that $\pi \circ \bar{\alpha} = \alpha \circ \pi$ where π is the natural projection of M onto M/K. Thus $\bar{\alpha}(m) + K = \alpha(m + k)$ for each m in M. Since M is small duo, then $\bar{\alpha}(H) \subseteq H$ and hence $\alpha(M/K) \subseteq M/K$. Thus M/K is small duo. ■

We have noticed in corollary (2.6) that if M is a finitely supplemented R-module then M is fully small stable if and only if every 2-generated small submodule of M is fully small stable. The following proposition is in this direction, only that it is for small duo modules.

Proposition (3.8): Let M be a supplemented R-module in which every countably generated submodule is small duo. Then M is small duo.

Proof:

Let α be an endomorphism of M and $x \in J(M)$. Put $N = \sum_{n=0}^{\infty} R\alpha^n(x)$. Then N is countably generated submodule of M. Since M is supplemented, then $J(N) = J(M) \cap N$ and hence $x \in J(N)$. Furthermore, $\alpha(N) \subseteq N$. This shows that α is an endomorphism of N. By proposition (3.3), there is an element r in R such that $\alpha(x) = rx$. Again proposition (3.3) implies that M is small duo. ■

The following proposition shows that many modules are not small duo.

Proposition (3.9): Let T be a subring of a ring R with $T \subsetneq J(R)$. Then the left T-module R is not small duo.

Proof:

Let t be any element in $J(R)$ which is not in T. Define the mapping $\alpha: R \longrightarrow R$ by $\alpha(x) = tx$ for all x in R. Then α is an endomorphism of R, thus $t = \alpha(1)$ and this implies that T is not fully invariant submodule of the T-module R and hence is not a small duo T-module. ■

Let M be an R-module, then an R-module N is called ***small principally M-injective*** if every R-homomorphism from small and cyclic submodule of M into N can be extended to an R-homomorphism from M into N. An R-module M is called ***small principally quasi injective***, if M is small principally M-injective [7].

Theorem (3.10): Let M be an R-module. Consider the following conditions:

1. M is duo and small principally quasi-injective.
2. M is fully small stable.

   Then (1)⟶ (2) and (2) ⟶ (1) in case R is commutative.

Proof:

(1)⟶ (2) Let N be a small principal submodule of M and α be an R-homomorphism from N into M. Now, small principally quasi-injectivity of M implies that α can be extended to an endomorphism β of M. Then $\alpha(N) = \beta(N) \subseteq N$ since M is small duo. Hence m is fully small stable.

(2)⟶ (1) it is enough to show that every fully small stable module is small principally quasi-injective. Let Rx be a small principal submodule of M and $\alpha: Rx \longrightarrow M$ be an R-homomorphism. Then there exists an element r in R such that $\alpha(x) = rx$. Define $\beta: M \longrightarrow M$ by $\beta(m) = rm$ for all m in M. it is clear that β is an extension of α. ■

Corollary (3.11): Let R be a commutative ring. Then R is self, small principally quasi-injective ring if and only if R is fully small stable ring.

We shall call an R-module M ***small-multiplication***, if for each small submodule of M is of the form AM for some left ideal A of R, this concept is in fact a generalization of ***multiplication*** modules [3].

It is an easy matter to prove that small-multiplication modules are small duo ones as follows, Let N be a small submodule of small-multiplication R-module M, and α be an endomorphism of M. Then N=AM for some left ideal A of R, $\alpha(N) = A\alpha(M) \subseteq AM = N$. Thus M is small duo. For the converse we have the following theorem, but first recall that if an R-module M is ***projective,*** then it satisfies ***the dual basis lemma*** [6].

Theorem (3.12): Let M be a projective R-module. Then M is small duo if and only if M is small-multiplication. In particular, R is small duo if and only if R is small-multiplication.

Proof:

Assume that M is small duo, and B a submodule of M. By dual basis lemma there exists subsets $\{m_\alpha | \alpha \in \Lambda\}$ of M and $\{f_\alpha |\ \alpha \in \Lambda\}$ of $M^* = Hom(M, R)$ such that for every m in M, $m = \sum_{\alpha\in\Lambda} f_\alpha(m) m_\alpha$ where $f_\alpha(m)$ almost everywhere. Let A = $<f_\alpha(x) | x \in B>$, we claim that B=AM. It is clear that $B \subseteq AM$, now for $x \in B, a \in M$ and $\alpha \in \Lambda$ define $\mu_a: R \longrightarrow M$ by $\mu_a(r) = ra$ for all r in R and $\mu_a \circ f_a$ is an endomorphism of M. thus $f_\alpha(x)a = (\mu_a \circ f_a)(x) \in (\mu_a \circ f_a)(Rx) \subseteq Rx \subseteq B$ since M is small duo. Now, let $w \in AM$ then $w = \sum_{\alpha\in\Lambda} f_\alpha(x) m_\alpha$ where $x \in B$ and $m_\alpha \in M$. By the above we have $f_\alpha(x) m_\alpha \in B$ and hence $w \in B$, this shows that B=AM and hence M is a small-multiplication module. ■

A subclass of multiplication modules will be introduced which is contained in the class of fully small stable modules. An R-module M is called ***I-multiplication*** if each submodule of M is of the form AM for some idempotent left ideal A of R [3]. We call an R-module M ***small I-multiplication*** if for each small submodule N of M, there exists an idempotent Left ideal A of R such that N=AM. It is clear that every small I-multiplication module is small-multiplication and the converse is true in case R is regular.

Corollary (3.13): Let R be a regular ring and M a projective R-module. Then the following statements are equivalent:

1. M is small I-multiplication.
2. M is fully small stable.
3. M is small duo.
4. M is small multiplication.

Proof:

(1) ⟶ (2) Let N be a small submodule of m and $\alpha: N \longrightarrow M$ be an R-homomorphism. Then N=AM for some idempotent left ideal A of R. Hence, $\alpha(N) = \alpha(AM) = A\alpha(AM) \subseteq AM = N$.

(2) ⟶ (3) Clear.

(3) ⟶ (4) Follows from theorem (3.12).

(4) ⟶ (1) Clear.

A submodule of an R-module M is called ***small- essential***, if for each small submodule K of M, $N \cap K = (0)$ implies K = (0), [8].

Proposition (3.14): If M is a small duo R-module, then for every monomorphism α in $End_R(M)$, $\alpha(M)$ is small-essential.

Proof:

For each non-zero small submodule N of M, if $N \cap \alpha(M) = (0)$, then $\alpha(N) \subseteq N$ and so $\alpha(N) \subseteq N \cap \alpha(M)$. thus $\alpha(N) = (0)$ and hence N=(0). ■

Lemma (3.15): Let R be a commutative ring, M a small-multiplication R-module and N a small submodule of M. Then for each R-homomorphism $\theta: N \longrightarrow M$ we have the following:

1. $[\theta(x): M] \subseteq [l_R(M): l_R(x)]$.
2. $\theta(Rx) \subseteq [l_R(M): l_R(x)]M$ for each x in N.

Proof:

1. For each $r \in [\theta(Rx): M]$ and $t \in l_R(x)$, then $t \in l_R(\theta(x))$. For each $m \in M$, we have $rm \in \theta(Rx)$ and hence $trm = 0$.
2. $\theta(Rx)$ is a small submodule of M. Since M is small-multiplication, then $\theta(Rx) = [\theta(x): M]$ [13]. By (1) we have $\theta(Rx) \subseteq [l_R(M): l_R(x)]$. ■

Note that it is always true that $[Rx: M] \subseteq [l_R(M): l_R(x)]$, for each x in M.

Proposition (3.16): For a commutative ring R, if M is fully small stable then $[l_R(M): l_R(x)] \subseteq [Rx: M]$ for each x in $J(M)$.

Proof:

For each $w \in [l_R(M): l_R(x)]$ and $m \in M$. Define $\theta: Rx \longrightarrow M$ by $\theta(rx) = rwm$ for each r in R. Then θ is a well-defined R-homomorphism. Full small stability of M implies that $\theta(Rx) \subseteq Rx$, so $wm = \theta(x) \in Rx$ and hence $w \in [Rx: M]$. ■

Theorem (3.17): Let M be a small-multiplication over a commutative ring R. If $[l_R(M): l_R(x)] \subseteq [Rx: M]$ for each x in $J(M)$, then M is fully small stable.

Proof:

Let Rx be a small cyclic submodule of M. For each $m \in M$ and $e \in [l_R(M): l_R(x)]$, define $\alpha_{e,m}: Rx \longrightarrow M$ by $\alpha_{e,m}(rx) = rem$ for each r in R. Then $\alpha_{e,m}$ is a well-defined R-homomorphism. By the choice of the elements e, m and the condition above, we have $\alpha_{e,m}(Rx) \subseteq Rx$. Now, for each $\alpha: Rx \longrightarrow M$ and by lemma (3.12) we have:

$$\alpha(Rx) \subseteq [l_R(M): l_R(x)]M$$

Thus:

$$\alpha(rx) = r\alpha(x)$$
$$= r(\textstyle\sum_{i=1}^{n} s_i(re_i m_i))$$
$$= \textstyle\sum_{i=1}^{n} s_i(\alpha_{e_i,m_i}(rx))$$

Therefore, $\alpha = \sum_{i=1}^{n} s_i \alpha_{e_i,m_i}$ and hence by the above $\alpha(Rx) \subseteq Rx$. This proves that M is fully small stable. ■

Proposition (3.18):

Let $\{M_i | i \in \Lambda\}$ be a family of R-modules and $M = \prod_{i\in\Lambda} M_i, M^{\#} = \oplus_{i\in\Lambda}$. Then the following hold:

1. If $M(M^{\#})$ is small duo, then for each $i \in \Lambda$, M_i is small duo.
2. If $M(M^{\#})$ is fully small stable, then for each $i \in \Lambda$, M_i is fully small stable.

Proof:

1. For a fixed $j \in \Lambda$, let $f_j \in End_R(M_j)$ and $m_j \in J(M_j)$. Define $f: M \longrightarrow M$ by $f((m_i)_{i\in\Lambda}) = \begin{bmatrix} m_i & if\ i \neq j \\ f_j(m_j) & if\ i = j \end{bmatrix}$ for every $(m_i)_{i\in\Lambda}$ in M

   It is clear that $f \in End_R(M)$, since M is small duo then there is r in R such that $f((m_i)_{i\in\Lambda}) = r(m_i)_{i\in\Lambda} = (rm_i)_{i\in\Lambda}$ by proposition (3.3). Hence, $f_j(m_j) = rm_j$. Again by proposition (3.3) we have M_j is small duo. The proof for $M^{\#}$ is in the same manner.
2. For fixed $j \in \Lambda$. let $m_j \in J(M_j)$ and $n_j \in M_j$ with $l_R(m_j) \subseteq l_R(n_j)$. Then $l_R(\dots,0,0,m_j,0,0\dots) \subseteq l_R(\dots,0,0,n_j,0,0,\dots)$, siance M is fully small stable then $R(\dots,0,0,n_j,0,0,\dots) \subseteq R(\dots,0,0,m_j,0,0,\dots)$ and hence $Rn_j \subseteq Rm_j$, this shows that M_j is fully small stable. The proof for $M^{\#}$ goes in the same manner. ■

Proposition (3.19): Let $M = \oplus_{i\in\Lambda} N_i$ be an internal direct sum of submodules N_i of M. Then for every $i \in \Lambda$, N_i is fully invariant in M if and only if $Hom(N_i, N_k) = 0$ for all distinct i, k in Λ.

Proof:

Assume that N_i is fully invariant in M for every $i \in \Lambda$, and for $k \neq i \in \Lambda$, let $f \in Hom(N_i, N_k)$. Then $g = j_k \circ f \circ \rho_i: M \longrightarrow M$ where ρ_i is the projection of M into N_i and j_k is the injection of N_k into M. By the hypothesis, $g(N_i) \subseteq (N_i)$, on the other hand $g(N_i) = f(N_i) \subseteq (N_k)$ so $f(N_i) \subseteq N_i \cap N_k = (0)$, hence $f = 0$.

Conversely, assume $Hom(N_i, N_k) = 0$ for all distinct i, k in Λ. Let f be an endomorphism of M, then $\rho_k \circ f \circ j_i: N_i \longrightarrow N_k$. By hypothesis, we have $f(N_i) \subseteq \sum_{k\in\Lambda}(\rho_k \circ f \circ j_i)(N_i) = (\rho_i \circ f \circ j_i)(N_i) \subseteq N_i$. ■

Theorem (3.20): Let $M = \oplus_{i\in\Lambda} N_i$ be an internal direct sum of submodules N_i of M. Then the following statements are equivalent:

1. M is small duo.

2. For every $i \in \Lambda$, N_i is small duo and $Hom(N_i, N_k) = 0$ for all distinct i, k in Λ.

Proof:

(1) ⟶ (2) Follows from theorems (3.18) and (3.19).

(2) ⟶ (1) Let b be a small submodule of M and $f \in End_R(M)$. For every k in Λ, let $j_k(\rho_k)$ be the injection (projection) of N_k into M (M into N_k). Since N_k is a direct summand 0f then $N_k \cap B$ is small in M. But N_k is small duo, then $(\rho_k \circ f \circ j_k)(B \cap N_k) \subseteq B \cap N_k$ and $(\rho_k \circ f \circ j_i)(N_k \cap B)$=(0) for all distinct i, k in Λ. Since $B = \sum_{K\in\Lambda}(B \cap N_k)$, then $f(B) \subseteq \sum_{K\in\Lambda} f(B \cap N_k) \subseteq \sum_{K\in\Lambda}(\rho_k \circ f \circ j_k)(B \cap N_k) \subseteq \sum_{K\in\Lambda} (B \cap N_k) = B$. This proves that b is fully invariant in M and hence M is small duo. ■

Theorem (3.21): Let R be a commutative ring and $M = \oplus_{i\in\Lambda} N_i$ be an internal direct sum of submodules N_i of M. Then the following statements are equivalent:

1. M is fully small stable.
2. $N_i \oplus N_j$ is fully small stable for all distinct i, j in Λ.

Proof:

(1) ⟶ (2) Follows from corollary (3.6).

(2) ⟶ (1) Assume that for distinct i, j in Λ, $N_i \oplus N_j$ is fully small stable. Then $N_i \oplus N_j$ is small duo. So theorem (3.20) implies that $Hom(N_i, N_k) = 0$ for all distinct i, j in Λ, also for every i in Λ, N_i is small duo. We claim that M is small principally quasi-injective, suppose Rx is be a small principal submodule of M and $\alpha: Sx \longrightarrow M$ an R-homomorphism where S is a left ideal of R. Then we have two cases, either $x \in N_i$ and $\alpha(x) \in N_j$ for some distinct i, j in Λ or x and $\alpha(x) \in N_i$ for some i in Λ. If we have the first case, then $(x, \alpha(x)) \in N_i \oplus N_j$ since $l_R(x) \subseteq l_R(\alpha(x))$, by proposition (3.8), $\alpha(x) \in Sx \subseteq N_i$ and so $\alpha(x) \in N_j \cap N_i = (0)$. Then α can be extended trivially to $\bar{\alpha} \in End_R(M)$ and hence m is small principally quasi-injective. Otherwise, x and $\alpha(x) \in N_i$ for some i in Λ. Since N_i is fully small stable, N_i is small principally quasi-injective. Hence, there exists $\bar{\alpha}: N_i \longrightarrow N_i$ such that $\bar{\alpha}(n) = \alpha(n)$. Definer $\beta: M \longrightarrow M$ by $\beta(m) = \bar{\alpha}(m)$ if $m \in N_i$ and $\beta(m) = 0$, then it is an easy matter to see that β is a well-defined R-homomorphism and is an extension of α. It follows that M is small principally quasi-injective. Then theorem (3.10) completes the proof. ■

3.References

[1] M. S. Abbas: On fully stable modules, Ph.D. Thesis, Univ. of Baghdad, 1991.

[2] A. Ozcan, H. Harmanci and P. F. Smith: Duo modules, Glasgow Math. J., 48, p. 533-545, 2006.

[3] Z. A. El-Bast and P. P. Smith: Multiplication Modules, Comm. In Algebra, 16(1988),755-799.

[4] E. Mares: Semi-perfect modules, Math. Z. 82(1903), 347-360.

[5] L. Shen and J. chen: Small Injective Rings, arXiv, Math. RA/0505449, Vol. 21, 2005.

[6] F. Kasch: Modules and Rings, Academic press, London, 1982.

[7] S. Wonguai: Small Principally Quasi-Injective Modules, Int. J. Contemp. Math. Sciences, Vol. 6, 2011.

[8] Dx. Zhou, X. R. zhang: Small-Essential Submodules and Morita Duality, Southeast Asian Bulletin of Mathematics, Vol. 35, 2011.

[9] Robert Wisbauer: Foundations of Modules and Ring Theory, Gordon and Breach Science Publishers, Reading, 1991.

[10] Frank W. Anderson, Kent R. Fuller: Rings and Categories of Modules, Springer-Verlang 1992.

[11] A. A. Tuganbaev: Skew-Injective Modules, Journal of Mathematical Sciences, Vol.95, No.4, 1999.