Special Issus: 1st Scientific International Conference, College of Science, Al-Nahrain University, 21-22/11/2017, Part I, pp.123-136

The Continuous Classical Boundary Optimal Control of a Couple Nonlinear Parabolic Partial Differential Equations

Jamil A. Ali Al-Hawasy and Ahmed Abdul Hasan Naeif
Department of Math., College of Science, Al-Mustansiriyah University, Baghdad-Iraq.
Corresponding author: Jhawassy17@uomustansiriyah.edu.iq, hawasy20@yahoo.com

Abstract

In this paper the continuous classical boundary optimal control problem of a couple nonlinear partial differential equations of parabolic type is studied. The Galerkin method is used to prove the existence and uniqueness theorem of the state vector solution of a couple nonlinear parabolic partial differential equations for given (fixed) continuous classical boundary control vector. The theorem of the existence of a continuous classical optimal boundary control vector associated with the couple of nonlinear parabolic partial differential equations is proved. The existence of a unique vector solution of the adjoint equations is studied. The Fréchet derivative is derived; Finally The Kuhn-Tucker-Lagrange multipliers theorems is developed and then is used to prove the necessary conditions theorem and the sufficient conditions theorem of optimality of a couple of nonlinear parabolic equations with equality and inequality constraints. [DOI: 10.22401/ANJS.00.1.17]

Keywords: boundary optimal control, couple nonlinear parabolic partial differential equations.

1. Introduction

The objective of optimal control theory is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion, [8]. Control theory is an application-oriented mathematics that deals with the basic principles underlying the analysis and design of (control) system. Systems can be engineering (air conditioner, air craft, and CD player etc.), economic, and biological, [12]. In general, there are many optimal control problems are governed either by ODEs as Orpel in 2009[11] or by different types of PDEs and are subject to control and state constraints, as El-Borari and et al in 2013 [9], and Wang, Y. and et al in 2015 [15], which are studied an optimal control of parabolic partial differential equations, Farag, M. H. in 2014[10] studied classical optimal control of hyperbolic partial differential equations, Diaz and et al in 2012 [7] studied a optimal control of elliptic partial differential equations, Al-Rawdanee, E. in 2014 [3] studied an a classical optimal control of a coupled of nonlinear elliptic partial differential equations and M. K. Ghufran in 2016 [4] studied a classical optimal control of a coupled of nonlinear parabolic partial differential equations while, Al-Hawasy, J. in 2016 [2] studied a classical optimal control of a coupled of nonlinear hyperbolic partial differential equations.

This paper deals with, the proof of the existence and uniqueness theorem of the state vector solution of a couple nonlinear parabolic partial differential equations where the continuous classical boundary control vector is given, the existence theorem of a continuous classical boundary optimal control vector associated with a couple nonlinear partial differential equations of parabolic type is proved, also the derivation of the Fréchet derivative is done, the study of the existence and uniqueness of the vector solution of the adjoint equations which corresponds to the state vector. Finally, the Kuhn-Tucker-Lagrange multipliers theorem is developed and is used to prove the necessary conditions theorem and the sufficient conditions theorem of optimality of a couple of nonlinear parabolic equations with equality and inequality constraints.

2. Description of the Problem

Let $I = (0,T)$, $T < \infty$, $\Omega \subset \mathbb{R}^2$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial\Omega$, $Q = \Omega \times I$, $\Sigma = \Gamma \times I$. Consider the following continuous boundary optimal control problem:

The state equation is given by the following nonlinear parabolic equation:

$y_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y_1}{\partial x_j}) + b_1(x,t)y_1 - b(x,t)y_2 = f_1(x,t,y_1)$, in Q .... (1)

$y_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial y_2}{\partial x_j}) + b_2(x,t)y_2 + b(x,t)y_1 = f_2(x,t,y_2)$, in Q .... (2)

$\sum_{i,j=1}^{n} a_{ij}\frac{\partial y_1}{\partial n} = u_1(x,t)$, on Σ .... (3)

$y_1(x,0) = y_1^0(x)$, on Ω .... (4)

$\sum_{i,j=1}^{n} b_{ij}\frac{\partial y_2}{\partial n} = u_2(x,t)$, on Σ .... (5)

$y_2(x,0) = y_2^0(x)$, on Ω .... (6)

where for all $x = (x_1, x_2)$, $(y_1, y_2) \in (H^1(\mathrm{Q}))^2$ is the state vector, $(u_1, u_2) \in (L^2(\Sigma))^2$ is the classical boundary control vector, $(f_1, f_2) \in (L^2(\mathrm{Q}))^2$ is a vector of a given function defined $(Q \times \mathbb{R}) \times (\mathrm{Q} \times \mathbb{R})$, and $a_{ij}(x,t)$, $b_{ij}(x,t)$, $b(x,t)$ and $b_i(x,t) \in C^\infty(Q)$.

$\vec{W}_A = \{\vec{w} \in L^2(\Sigma) \times L^2(\Sigma) | \vec{w} \in \vec{U} \text{ a.e. in } \Sigma, G_1(\vec{w}) = 0, G_2(\vec{w}) \leq 0\}$

$\vec{U} \subset \mathbb{R}^2$.

The cost function is

$G_0(\vec{u}) = \int_Q [g_{01}(x,t,y_1) + g_{02}(x,t,y_2)]dxdt + \int_\Sigma [h_{01}(x,t,u_1) + h_{02}(x,t,u_2)]d\sigma$ .... (7)

The constraints on the state and the control vectors are

$G_1(\vec{u}) = \int_Q [g_{11}(x,t,y_1) + g_{12}(x,t,y_2)]dxdt + \int_\Sigma [h_{11}(x,t,u_1) + h_{12}(x,t,u_2)]d\sigma = 0$ .... (8)

$G_2(\vec{u}) = \int_Q [g_{21}(x,t,y_1) + g_{22}(x,t,y_2)]dxdt + \int_\Sigma [h_{21}(x,t,u_1) + h_{22}(x,t,u_2)]d\sigma \leq 0$ .... (9)

where $(y_1, y_2) = (y_{u_1}, y_{u_2})$ is the solution of (1-6) corresponding to the boundary control vector (u_1, u_2).

Let $\vec{V} = V \times V = \{\vec{v} : \vec{v} \in (H^1(\Omega))^2\}$, $\vec{v} = (v_1, v_2)$.

We denote by $(v,v)_\Omega$ and $\|v\|_0$ (by $(v,v)_\Gamma$ and $\|v\|_\Gamma$) the inner product and the norm in $L^2(\Omega)$ (in $L^2(\Gamma)$), by $(v,v)_1$ and $\|v\|_1$ the inner product and the norm in $H^1(\Omega)$**,** by $(\vec{v},\vec{v})_\Omega$ and $\|\vec{v}\|_0$ (by $(\vec{v},\vec{v})_\Gamma$ and $\|\vec{v}\|_\Gamma$) the inner product and the norm in $L^2(\Omega) \times L^2(\Omega)$(in $L^2(\Gamma) \times L^2(\Gamma)$) by $(\vec{v},\vec{v})_1 = (v_1,v_1)_1 + (v_2,v_2)_1$ and $\|\vec{v}\|_1^2 = \|v_1\|_1^2 + \|v_2\|_1^2$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$.

3. Weak Formulation of the State Equations

The weak forms of the problem (1-6) when $\vec{y} \in (H_0^1(\mathrm{Q}))^2$ are given almost everywhere on I $(\forall v_1, v_2 \in V, y_1(.,t), y_2(.,t) \in V)$ by

$\langle y_{1t}, v_1\rangle + a_1(t,y_1,v_1) + (b_1(t)y_1, v_1)_\Omega - (b(t)y_2, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma$, ...(10a)

$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega$ .... (10b)

and

$\langle y_{2t}, v_2\rangle + a_2(t,y_2,v_2) + (b_2(t)y_2, v_2)_\Omega + (b(t)y_1, v_2)_\Omega = (f_2, v_2)_\Omega + (u_2, v_2)_\Gamma$, ...(11a)

$(y_2^0, v_2)_\Omega = (y_2(0), v_2)_\Omega$ .... (11b)

Where $a_1(t,y_1,v_1) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j}dx$,

$a_2(t,y_2,v_2) = \int_\Omega \sum_{i,j=1}^{n} b_{ij}\frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j}dx$.

To study the existence of unique solution of the weak form (10-11), we consider the following assumption.

***Assumptions (A)*:**

(i) f_i is of a Carathéodory type on $Q \times \mathbb{R}$, satisfies the following sub linearity condition for y_i,i.e. $|f_i(x,t,y_i)| \leq \eta_i(x,t) + c_i|y_i|$
Where $(x,t) \in Q$, $y_i \in \mathbb{R}$, $c_i > 0$ and $\eta_i \in L^2(Q,\mathbb{R})$, $\forall i = 1,2$

(ii) f_i is Lipschitz w.r.t. y_i, i.e. $|f_i(x,t,y_i) - f_i(x,t,\hat{y}_i)| \leq L_i|y_i - \hat{y}_i|$

(iii)Where $(x,t) \in Q$, $y_i, \hat{y}_i \in \mathbb{R}$ and $L_i > 0$, $\forall i = 1,2$

(iv)$c(t,\vec{y},\vec{y}) = a_1(t,y_1,y_1) + (b_1(t)y_1,y_1)_\Omega + a_2(t,y_2,y_2) + (b_2(t)y_2,y_2)_\Omega$, and $|c(t,\vec{y},\vec{y})| \leq \alpha\|\vec{y}\|_1\|\vec{v}\|_1$,$c(t,\vec{y},\vec{y}) \geq \bar{\alpha}\|\vec{y}\|_1^2$, where $\alpha, \bar{\alpha}$ are real positive constants

Proposition (3.1), [6]:

Let $f: \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is of a Carathéodory type, let F be a functional, such that

$F(y) = \int_\Omega f(x, y(x))dx$,

where Ω is measurable subset of $\mathbb{R}^d (d = 2,3)$, and suppose that

$\|f(x,y)\| \leq \zeta(x) + \eta(x)\|y\|^\alpha$, $\forall (x,y) \in \Omega \times \mathbb{R}^n$, $y \in L^p(\Omega \times \mathbb{R}^n)$

Where $\zeta(x) \in L^1(\Omega \times \mathbb{R})$, $\eta \in L^{\frac{p}{p-\alpha}}(\Omega \times \mathbb{R})$, and $\alpha \in [0,p]$, if $p \in [1,\infty)$, and $\eta \equiv 0$, if $p = \infty$. Then F is continuous on $L^p(\Omega \times \mathbb{R}^n)$.

Proposition (3.2), [6]:

Let $f \& f_y: \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ are Carathéodory type, let $F: L^p(\Omega) \to \mathbb{R}$ be a functional, such that, $F(y) = \int_\Omega f_y(x, y(x))dx$ where Ω is a measurable subset of $\mathbb{R}^d$, and

$\|f_y(x,y)\| \leq \zeta(x) + \eta(x)\|y\|^{\frac{\beta}{q}}$, $\quad \forall (x,y) \in \Omega \times \mathbb{R}^n$,

Where $\zeta \in L^q(\Omega \times \mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in L^{\frac{pq}{p-\beta}}(\Omega \times \mathbb{R})$, $\beta \in [0, p]$ if $p \neq \infty$, and $\eta \equiv 0$, if $p = \infty$

Then the Fréchet derivative of F exists for each $y \in L^p(\Omega \times \mathbb{R}^n)$ and is given by:

$\acute{G}(y)h = \int_\Omega f_y(x, y(x))h(x)dx.$

Theorem (3.1), [14]:

Let D be a measurable subset of $\mathbb{R}^d$, $\emptyset: D \to \mathbb{R}$ and $\emptyset \in L^1(D, \mathbb{R})$.

If the following inequality is satisfied

$\int_S \emptyset(v)dv \geq 0$ (or $\leq 0, = 0$), for each measurable set $S \subset D$, then

$\emptyset(v) \geq 0$ (or $\leq 0, = 0$), a.e. in D.

Theorem (3.2) (Existence and Uniqueness of Solution of the State Equations):

With assumptions (A), for each fixed boundary control $\vec{u} \in (L_2(\Sigma))^2$, the weak form of the state equations (10-11) has a unique solution $\vec{y} = (y_1, y_2)$, s.t. $\vec{y} \in \left(L^2(I,V)\right)^2$ and $\vec{y}_t = (y_{1t}, y_{2t}) \in \left(L^2(I,V^*)\right)^2$

Proof:

Let $\vec{V}_n \subset \vec{V}$ be the set of continuous and piecewise affine functions in Ω, let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be basis of $\vec{V}_n$ where $n = 2N$ (where N is the dimension of each V), then the approximate solution $\vec{y}$ of (10-11) is approximated by $\vec{y}_n = (y_{1n}, y_{2n})$, such that, for each n

$\vec{y}_n = \sum_{j=1}^n C_j(t)\vec{v}_j(x)$ ....(12)

where

$\vec{v}_j = ((2-\ell)v_{1k}, (\ell-1)v_{2k})$, and $C_j = c_{lj}$

For $j = k + n(\ell - 1)$, $\quad k = 1, \dots N, \ell = 1,2$, and $c_{lj}(t)$ is unknown function of t.

The weak forms of the state equations (10) and (11) can be approximated w.r.t. the space variable, using the Galerkin's method to get,

$\langle y_{1nt}, v_1\rangle + a_1(t, y_{1n}, v_1) + (b_1(t)y_{1n}, v_1)_\Omega - (b(t)y_{2n}, v_1)_\Omega = (f_1(y_{1n}), v_1)_\Omega + (u_1, v_1)_\Gamma$ ....(13a)

$(y_{1n}^0, v_1)_\Omega = (y_1^0, v_1)_\Omega, \forall v_1 \in V_n$ .... (13b)

and

$\langle y_{2nt}, v_2\rangle + a_2(t, y_{2n}, v_2) + (b_2(t)y_{2n}, v_2)_\Omega + (b(t)y_{1n}, v_2)_\Omega = (f_2(y_{2n}), v_2)_\Omega + (u_2, v_2)_\Gamma$ ....(14a)

$(y_{2n}^0, v_2)_\Omega = (y_2^0, v_2)_\Omega$, ... (14b)

where $y_{in}^0 = y_{in}(x, 0) \in V_n \subset V \subset L^2(\Omega)$ is the projection of y_i^0 for the norm $\|.\|_0$,i.e.

$(y_{in}^0, v_i)_\Omega = (y_i^0, v_i)_\Omega \; \forall v_i \in V_n \Leftrightarrow \|y_{in}^0 - y_i^0\|_0 \leq \|y_i^0 - v_i\|_0$, $\forall v_i \in V_n, \forall i = 1,2$

By substituting (12) in (13 a& b) and in (14 a & b), one obtains

$A_1 C_1'(t) + D_1 C_1(t) - E_1 C_2(t) = b_1(\bar{V}_1^T(x)C_1(t))$ .... (12´a)

$A_1 C_1(0) = b_1^0$ ....(12´b)

$A_2 C_2'(t) + D_2 C_2(t) + E_2 C_1(t) = b_2(\bar{V}_2^T(x)C_2(t))$ .... (13´a)

$BC_2(0) = b_2^0$ ....(13´b)

where:

$A_1 = (a_{ij})_{n\times n}$, $a_{ij} = (v_{1j}, v_{1i})_\Omega$, $D_1 = (d_{ij})_{n\times n}$, $d_{ij} = [a_1(t, v_{1j}, v_{1i}) + (b_1(t)v_{1j}, v_{1i})_\Omega]$, $E_1 = (e_{ij})_{n\times n}$, $e_{ij} = (b(t)v_{2j}, v_{1i})_\Omega$, $C_\ell(t) = (c_{\ell j}(t))_{n\times 1}$, $C_\ell'(t) = (c_{\ell j}'(t))_{n\times 1}$, $C_\ell(0) = (c_{\ell j}(0))_{n\times 1}$, $b_\ell = (b_{li})_{n\times 1}$, $b_{\ell i} = (f_\ell(\bar{v}_\ell^T C_\ell(t)), v_{\ell i})_\Omega + (u_\ell, v_{\ell i})_\Gamma$, $\bar{V}_\ell = (v_\ell)_{n\times 1}$, $b_\ell^0 = (b_{\ell i}^0)$, $b_{\ell i}^0 = (y_\ell^0, v_{\ell i})_\Omega$, and $A_2 = (b_{ij})_{n\times n}$, $b_{ij} = (v_{2j}, v_{2i})_\Omega$, $D_2 = (f_{ij})_{n\times n}$, $f_{ij} = [a_2(t, v_{2j}, v_{2i}) + (b_2(t)v_{2j}, v_{2i})_\Omega]$, $E_2 = (h_{ij})_{n\times n}$, $h_{ij} = (b(t)v_{1i}, v_{2i})_\Omega$, $\ell = 1,2$.

From assumptions (A), easily once can get that the matrices $A_1 \& A_2$ are positive definite, therefore the system (12´-13´) of 1^{st} order differential equation has unique solution [5].

Now, to show the norm $\|\vec{y}_n^0\|_0$ is bounded:

Since $\vec{y}^0 \in (L^2(\Omega))^2$, then there exists $\{\vec{v}_n^0\}$, with $\vec{v}_n^0 \in \vec{V}_n$ such that $\vec{v}_n^0 \longrightarrow \vec{y}^0$ strongly in $(L^2(\Omega))^2$ and then from the projection theorem and (13b & 14b) one obtains that $\vec{y}_n^0 \longrightarrow \vec{y}^0$ Strongly in $(L^2(\Omega))^2$ with $\|\vec{y}_n^0\|_{\mathbf{0}} \leq b_1$.

The norm $\|\vec{y}_n(t)\|_{L^\infty(I, L^2(\Omega))}$ and $\|\vec{y}_n(t)\|_Q$ are bounded:

Setting $v_1 = y_{1n}$ in (13a) and $v_2 = y_{2n}$ in (14a), integrating both sides of each obtaining equation for t from 0 to T, and adding them finally with Assumption (A-iii), one has

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n \rangle dt + \int_0^T c(t, \vec{y}_n, \vec{y}_n) dt =$
$\int_0^T (f_1(y_{1n}), y_{1n})_\Omega dt + \int_0^T (f_2(y_{2n}), y_{2n})_\Omega dt +$
$\int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (u_2, y_{2n})_\Gamma dt$, ... (15)

Since $\vec{y}_{nt} \in (L^2(I, V^*))^2 = (L^2(I, V))^2$ and $\vec{y}_n \in (L^2(I, V))^2$ in the 1^{st} term of the L.H.S. of (15), hence for this term we can use Lemma 1.2 in [13] and since the 2^{nd} term is positive, taking $T = t \in [0, T]$, finally using, and Assumption (A-i) for the 1^{st} two terms in the right hand side (for briefly will use R.H.S. from here and next) of (15), one has

$\int_0^t \frac{d}{dt} \|\vec{y}_n(t)\|_0^2 \, dt \le \|\eta_1\|_Q^2 + \|\eta_2\|_Q^2 +$
$\|u_1\|_\Sigma^2 + \|u_2\|_\Sigma^2 + c_5 \int_0^t \|\vec{y}_n\|_0^2 \, dt,$
$\Rightarrow \|\vec{y}_n(t)\|_0^2 - \|\vec{y}_n(0)\|_0^2 \le m_1 + m_2 + c_1 +$
$c_2 + c_5 \int_0^t \|\vec{y}_n\|_0^2 \, dt$
$\Rightarrow \|\vec{y}_n(t)\|_0^2 \le m^* + c_5 \int_0^t \|\vec{y}_n\|_0^2 \, dt$

By using the Belman- Gronwall inequality, one gets

$\Rightarrow \|\vec{y}_n(t)\|_{L^\infty(I, L^2(\Omega))} \le h_9$

hence

$\|\vec{y}_n(t)\|_Q^2 = \int_0^T \|\vec{y}_n\|_0^2 dt$
$\le T \max_{t \in [0,T]} \|\vec{y}_n(t)\|_0^2$
$\le T h_8 = h_{10}^2 = h_{10}$

The norm $\|\vec{y}_n(t)\|_{L^2(I,V)}$ is bounded:

Again by using Lemma 1.2 in [13] for the 1^{st} term in the L.H.S. of (15), then using same results which are obtained from the R.H.S., finally setting $t = T$, and $\|\vec{y}_n(T)\|_0^2 \ge 0$, equation (15) becomes

$\|\vec{y}_n(T)\|_0^2 + 2\bar{\alpha} \int_0^T \|\vec{y}_n\|_1^2 dt \le \|\eta_1\|_Q^2 +$
$\|\eta_2\|_Q^2 + \|u_1\|_\Sigma^2 + \|u_2\|_\Sigma^2 + c_5 \|\vec{y}_n\|_Q^2 +$
$\|\vec{y}_n(0)\|_0^2$
$\Rightarrow \|\vec{y}_n\|_{L^2(I,V)} \le h_{11}$

***The convergence of the solution*:**

Let $\{\vec{V}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\vec{V}$, such that $\forall \vec{v} = (v_1, v_2) \in \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}) \in \vec{V}_n$, $\forall n$, and $\vec{v}_n \longrightarrow \vec{v}$ strongly in $\vec{V} \Rightarrow \vec{v}_n \longrightarrow \vec{v}$ strongly in $(L^2(\Omega))^2$.

Since for each, with $\vec{V}_n \subset \vec{V}$, problems (13a&b) and (14a&b) have a unique solutions y_{1n}, y_{2n} respectively, hence corresponding to the sequence subspaces $\{\vec{V}_n\}_{n=1}^\infty$, one obtain a sequence of approximation problems (13 a&b) and (14 a&b), by substituting $\vec{v} = \vec{v}_n = (v_{1n}, v_{2n})$ for $n = 1,2, \ldots$, in these approximation problem, one gets

$\langle y_{1nt}, v_{1n} \rangle + a_1(t, y_{1n}, v_{1n}) +$
$(b_1(t) y_{1n}, v_{1n})_\Omega - (b(t) y_{2n}, v_{1n})_\Omega =$
$(f_1(y_{1n}), v_{1n})_\Omega + (u_1, v_{1n})_\Gamma, \quad \forall y_{1n}, y_{2n} \in$
$L^2(I, V_n)$a.e in I ....(16a)
$(y_{1n}^0, v_{1n})_\Omega = (y_1^0, v_{1n})_\Omega, \forall\, v_{1n} \in V_n, \forall n$
.... (16b)

and

$\langle y_{2nt}, v_{2n} \rangle +$
$a_2(t, y_{2n}, v_{2n}) + (b_2(t) y_{2n}, v_{2n})_\Omega +$
$(b(t) y_{1n}, v_{2n})_\Omega =$
$(f_2(y_{2n}), v_{2n})_\Omega + (u_2, v_{2n})_\Gamma, \quad \forall y_{1n}, y_{2n} \in$
$L^2(I, V_n)$ a.e.in I ....(17a)
$(y_{2n}^0, v_{2n})_\Omega = (y_2^0, v_{2n})_\Omega, \forall v_{2n} \in V_n, \forall n$
.... (17b)

which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^\infty$, where nm$\vec{y}_n = (y_{1n}, y_{2n})$. Since the norms $\|\vec{y}_n\|_{L^2(Q)}$ and $\|\vec{y}_n\|_{L^2(I,V)}$ are bounded, then by Alaoglu's theorem, there exists a subsequence of $\{\vec{y}_n\}_{n \in N}$, say again $\{\vec{y}_n\}_{n \in N}$ such that $\vec{y}_n \longrightarrow \vec{y}$ weakly in $(L^2(Q))^2$ and in $(L^2(I, V))^2$.

Then through the First Compactness Theorem, Assumption (A-i), and the bounded norms results, once get $\vec{y}_n \longrightarrow \vec{y}$ strongly in $(L^2(Q))^2$.

Now, consider the weak state equations (16a&b), (17a&b) and take any arbitrary, $v_1, v_2 \in V$,then there exists a sequence $\{v_{1n}\}$, $\{v_{2n}\}$ respectively, $v_{in} \in V_n$, $\forall n$, such that $v_{in} \longrightarrow v_i$ strongly in V (which gives $v_{in} \longrightarrow v_i$ strongly in $L^2(\Omega)$), $\forall i = 1,2$.

Multiplying both sides of (16a) and (17a) by $\varphi_i(t) \in C^1[0, T]$ respectively, with $\varphi_i(T) = 0, \forall i = 1,2$, integrating with respect to t from 0 to T, and then integrating by parts the 1^{st} term in the L.H.S. of each obtained equation, one gets that

$-\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt +$
$\int_0^T [a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b(t)y_{2n}, v_{1n})_\Omega]\varphi_1(t)dt =$
$\int_0^T (f_1(y_{1n}), v_{1n})_\Omega\varphi_1(t)dt +$
$\int_0^T (u_1, v_1)_\Omega\, \varphi_1(t)dt + (y_{1n}^0, v_{1n})_\Omega\varphi_1(0)$
.... (18)

and

$-\int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt +$
$\int_0^T [a_2\,(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b(t)y_{1n}, v_{2n})_\Omega]\varphi_2(t)dt =$
$\int_0^T (f_2(y_{2n}), v_{2n})_\Omega\varphi_2(t)dt +$
$\int_0^T (u_2, v_2)_\Gamma\varphi_2(t)dt + (y_{2n}^0, v_{2n})_\Omega\varphi_2(0)$
.... (19)

Since $\forall i = 1,2,\ y_{in} \longrightarrow y_i$ weakly in $L^2(Q)$, $y_{in}^0 \longrightarrow y_i^0$ strongly in $L^2(\Omega)$, and

$\left.\begin{array}{l} v_{in} \longrightarrow v_i \text{ strongly in } L^2(\Omega) \\ v_{in} \longrightarrow v_i \text{ strongly in } V \end{array}\right\} \Rightarrow$

$\begin{cases} v_{in}\varphi_i' \longrightarrow v_i\varphi_i' \text{strongly in } L^2(Q) \\ v_{in}\varphi_i \longrightarrow v_i\varphi_i \text{ strongly in } L^2(I, V) \end{cases}$

Then the following convergences hold

$\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt + \int_0^T [a_1\,(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b(t)y_{2n}, v_{1n})_\Omega]\varphi_1(t)dt \longrightarrow$
$\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b(t)y_2, v_1)_\Omega]\varphi_1(t)dt,$
.... (20)

$(y_{1n}^0, v_{1n})_\Omega\varphi_1(0) \longrightarrow (y_1^0, v_1)_\Omega\varphi_1(0)$ ...... (21)

and

$\int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt + \int_0^T [a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b(t)y_{1n}, v_{2n})_\Omega]\varphi_2(t)dt \longrightarrow$
$\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b(t)y_1, v_2)_\Omega]\varphi_2(t)dt$
.... (22)

$(y_{2n}^0, v_{2n})_\Omega\varphi_2(0) \longrightarrow (y_2^0, v_2)_\Omega\varphi_2(0)$ ..... (23)

On the other hand, let $w_{in} = v_{in}\varphi_i$ and $w_i = v_i\varphi_i$ then $\forall i = 1,2,\ w_{in} \longrightarrow w_i$ strongly in $L^2(Q)$ and then w_{in} is measurable w.r.t. (x, t), using assumption (A-i), then applying Proposition (3.1), the integral $\int_Q f_i(x, t, y_{in})w_{in}dxdt$ is continuous w.r.t. (y_{in}, w_{in}), but $y_{in} \longrightarrow y_i$strongly in $L^2(Q)$, then

$\int_0^T (f_i(y_{in}), v_{in})_\Omega\varphi_i(t)dt \longrightarrow$
$\int_0^T (f_i(y_i), v_i)_\Omega\varphi_i(t)dt$, $\forall i = 1,2$

From (20-23) and the above converges, (18) and (19) become

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b(t)y_2, v_1)_\Omega]\varphi_1(t)dt =$
$\int_0^T (f_1(y_1), v_1)_\Omega\varphi_1(t)dt +$
$\int_0^T (u_1, v_1)_\Gamma\, \varphi_1(t)dt + (y_1^0, v_1)_\Omega\varphi_1(0)$
.... (24)

and

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b(t)y_1, v_2)_\Omega]\varphi_2(t)dt =$
$\int_0^T (f_2(y_2), v_2)_\Omega\varphi_2(t)dt +$
$\int_0^T (u_2, v_2)_\Gamma\, \varphi_2(t)dt + (y_2^0, v_2)_\Omega\varphi_2(0),$
.... (25)

Now we have following two cases:

Case 1:

Choose $\varphi_i \in D[0, T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall i = 1,2$ substituting these values for φ_i in (24) and (25), finally using integration by parts for the 1^{st} terms in the L.H.S. of each one of the obtained equations, yield

$\int_0^T (y_{1t}, v_1)\varphi_1(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b(t)y_2, v_1)_\Omega\varphi_1(t)]dt =$
$\int_0^T (f_1(y_1), v_1)_\Omega\varphi_1(t)dt +$
$\int_0^T (u_1, v_1)_\Gamma\, \varphi_1(t)dt$ .... (26)

and

$\int_0^T (y_{2t}, v_2)\varphi_2(t)dt + \int_0^T [a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b(t)y_1, v_2)_\Omega\varphi_2(t)]dt =$
$\int_0^T (f_2(y_2), v_2)_\Omega\varphi_2(t)dt +$
$\int_0^T (u_2, v_2)_\Omega\, \varphi_2(t)dt,$ .... (27)

i.e., y_1&$y_2$are solutions of the state equations (10a) & (11a) respectively.

Case 2:

Choose$\varphi_i \in C^1[0, T]$, $\forall i = 1,2$, such that $\varphi_i(T) = 0$ &$\varphi_i(0) \neq 0$,

Using integration by parts for the 1^{st} term in the L.H.S. of (26) & (27)and subtracting two each obtained equations of (24),(25) respectively one gets

$(y_1^0, v_1)_\Omega\varphi_1(0) = (y_1(0), v_1)_\Omega\varphi_1(0), \Rightarrow$
$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega$

i.e., the initial condition (10b) holds. Easily one can see that the initial condition (11b) holds.

The strong convergence for $\vec{y}_n$ in $L^2(I, V)$:

By substituting $v_1 = y_1$ and $v_1 = y_{1n}$ in (10a) and (13a) respectively and also substituting $v_2 = y_2$ and $v_2 = y_{2n}$ in (11a) and (14a) respectively, integrating these four equations from $t = 0$ to $t = T$ finally adding the equations which is obtained from (10a) with that obtained from (13a) to gather and the same thing happened for (11a), (14a), to get:

$$\int_0^T \langle \vec{y}_t, \vec{y} \rangle \, dt + \int_0^T c(t, \vec{y}, \vec{y}) dt = \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega] dt + \int_0^T (u_1, y_1)_\Gamma + \int_0^T (u_2, y_2)_\Gamma \quad \text{.......(28a)}$$

and

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n \rangle \, dt + \int_0^T c(t, \vec{y}_n, \vec{y}_n) dt = \int_0^T [(f_1(y_{1n}), y_{1n}) + (f_2(y_{2n}), y_{2n})] dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (u_2, y_{2n})_\Gamma dt \quad \text{.......(28b)}$$

Using Lemma 1.2 in [13] for the 1^{st} terms in the L.H.S. of (28a&b), once get,

$$\tfrac{1}{2}\|\vec{y}(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T c(t, \vec{y}, \vec{y}) dt = \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega] \, dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (u_2, y_2)_\Gamma dt \quad \text{.......(29a)}$$

and

$$\tfrac{1}{2}\|\vec{y}_n(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T c(t, \vec{y}_n, \vec{y}_n) dt = \int_0^T [(f_1(y_{1n}), y_{1n})_\Omega + (f_2(y_{2n}), y_{2n})_\Omega] \, dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (u_2, y_{2n})_\Gamma dt, \quad \text{.......(29b)}$$

Now, consider the following equality:

$$\tfrac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 + \int_0^T c(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y}) \, dt = A_1 - A_2 - A_3 \quad \text{.......(30)}$$

where:

$$(A_1) = \tfrac{1}{2}\|\vec{y}_n(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T c\big(t, \vec{y}_n(T), \vec{y}_n(T)\big) dt$$

$$(A_2) = \tfrac{1}{2}\big(\vec{y}_n(T), \vec{y}(T)\big) - \tfrac{1}{2}\big(\vec{y}_n(0), \vec{y}(0)\big) + \int_0^T c\big(t, \vec{y}_n(T), \vec{y}(T)\big) \, dt$$

and

$$(A_3) = \tfrac{1}{2}\big(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) - \tfrac{1}{2}\big(\vec{y}(0), \vec{y}_n(0) - y(0)\big) + \int_0^T c\big(t, \vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) dt$$

Since (form the projection theorem, see prove theorem 3.2)

$$\vec{y}_n^0 = \vec{y}_n(0) \longrightarrow \vec{y}^0 = \vec{y}(0) \text{ strongly in } \big(L^2(\Omega)\big)^2 \quad \text{.......(31a)}$$

$$\vec{y}_n(T) \longrightarrow \vec{y}(T) \text{ strongly in } \big(L^2(\Omega)\big)^2 \quad \text{..... (31b)}$$

Then

$$\big(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)\big) \longrightarrow 0 \;\&\; \big(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) \longrightarrow 0 \quad \text{.......(31c)}$$

$$\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 \longrightarrow 0 \;\&\; \|\vec{y}_n(T) - \vec{y}(T)\|_0^2 \longrightarrow 0 \quad \text{.......(31d)}$$

and since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\big(L^2(I, V)\big)^2$, then

$$\int_0^T c\big(t, \vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) dt \longrightarrow 0 \quad \text{.......(31e)}$$

From proposition (3.1), the integral $\int_0^T (f_i(y_{in}), y_{in}) \, dt$ is continuous w.r.t. y_i, then

$$\int_0^T [(f_1(y_{1n}), y_{1n}) + (f_2(y_{2n}), y_{2n})] \, dt \longrightarrow \int_0^T [(f_1(y_1), y_1) + (f_2(y_2), y_2)] dt \quad \text{.......(31f)}$$

since $y_{in} \longrightarrow y_i$ strongly in $L^2(Q)$, $\forall i = 1,2$.

Now, when $n \to \infty$ in both sides of (30), one has the following results:

1. The first two terms in the L.H.S. of (30) are tending to zero (from 31d)

2. $$\text{Eq.}(A_1) \underset{(29b)}{\overset{\text{from}}{=}} \int_0^T [(f_1(y_{1n}), y_{1n}) + (f_2(y_{2n}), y_{2n})] \, dt + \int_0^T [(u_1, y_{1n})_\Gamma + (u_2, y_{2n})_\Gamma] dt \underset{(31f)}{\overset{\text{from}}{\longrightarrow}} \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega] \, dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (u_2, y_2)_\Gamma dt$$

3. $$\text{Eq.}(A_2) \to \text{L.H.S.of(29a)} = \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega] \, dt + \int_0^T [(u_1, y_{1n})_\Gamma + (u_2, y_{2n})_\Gamma] dt$$

4. The three terms in (A_3) are tending to zero from(31c) and c(31e).

From the above steps, (30) gives that:

$$\int_0^T c(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y}) dt \longrightarrow 0$$

which means that:

$$\bar{\alpha} \int_0^T \|\vec{y}_n - \vec{y}\|_1^2 \, dt \longrightarrow 0 \Rightarrow \vec{y}_n \longrightarrow \vec{y} \text{ strongly in } \big(L^2(I, V)\big)^2.$$

***Uniqueness of the solution*:**

Let $\vec{y} = (y_1, y_2)$, $\hat{\vec{y}} = (\hat{y}_1, \hat{y}_2)$ be two solutions of the state equations (10a)-(11a), i.e. $\forall v_1, v_2 \in V$, i.e., first from (10a), one has

$$\langle y_{1t}, v_1 \rangle + a_1(t, y_1, v_1) + (b_1(t) y_1, v_1)_\Omega - (b(t) y_2, v_1)_\Omega = (f_1(y_1), v_1)_\Omega + (u_1, v_1)_\Gamma,$$

$$\langle \hat{y}_{1t}, v_1 \rangle + a_1(t, \hat{y}_1, v_1) + (b_1(t) \hat{y}_1, v_1)_\Omega - (b(t) \hat{y}_2, v_1)_\Omega = (f_1(\hat{y}_1), v_1)_\Omega + (u_1, v_1)_\Gamma,$$

By subtracting the 2^{nd} equation from the 1^{st} one, then substituting $v_1 = y_1 - \hat{y}_1$, to get

$\langle (y_1-\hat{y}_1)_t, y_1-\hat{y}_1\rangle + a_1(t, y_1-\hat{y}_1, y_1-\hat{y}_1) + (b_1(t)(y_1-\hat{y}_1), y_1-\hat{y}_1)_\Omega - (b(t)(y_2-\hat{y}_2), v_1)_\Omega = (f_1(y_1)-f_1(\hat{y}_1), y_1-\hat{y}_1)_\Omega$, ...(32)

Second form (11a), and by the same above way but for $y_2, \hat{y}_2$, the following equality is also obtained

$\langle (y_2-\hat{y}_2)_t, y_2-\hat{y}_2\rangle + a_2(t, y_2-\hat{y}_2, y_2-\hat{y}_2) + (b_2(t)(y_2-\hat{y}_2), y_2-\hat{y}_2)_\Omega + (b(t)(y_1-\hat{y}_1), y_1-\hat{y}_1)_\Omega = (f_2(y_1)-f_2(\hat{y}_1), y_2-\hat{y}_2)_\Omega$, ...(33)

Adding (32) and (33), applying Lemma 1.2 in [13] for the 1^{st} term of L.H.S of above equality, using assumption A-iii, yields

$\frac{1}{2}\frac{d}{dt}\|\vec{y}-\vec{\hat{y}}\|_0^2 + \bar{\alpha}\|\vec{y}-\vec{\hat{y}}\|_1^2 \le |(f_1(y_1)-f_1(\hat{y}_1), y_1-\hat{y}_1)_\Omega + (f_2(y_2)-f_2(\hat{y}_2), y_2-\hat{y}_2)_\Omega|$ ....(34)

Keep in mind the second term in the L.H.S. of (34) is positive, integration both sides of (34) with respect to t from 0 to t, then using assumptions (A-ii) of the R.H.S, finally using Belman - Gronwall inequality, one gets $\|\vec{y}(t)-\vec{\hat{y}}(t)\|_0^2 = 0,\ \forall t \Rightarrow \|\vec{y}-\vec{\hat{y}}\|_{L^2(I,V)} = 0 \Rightarrow \vec{y} = \vec{\hat{y}}$.

4. Existence of a classical Boundary Optimal Control

The following theorem and lemma are important to study the existence of a classical boundary optimal control vector.

Theorem (4.1):

(a) In addition to assumptions (A), if $\vec{y}$ and $\vec{y}+\overrightarrow{\Delta y}$ are the states vectors corresponding to the controls vectors $\vec{u}$ and $\vec{u}+\overrightarrow{\Delta u}$, if $\vec{u}$ and $\overrightarrow{\Delta u}$ are bounded in $(L^2(\Sigma))^2$, then

$\|\overrightarrow{\Delta y}\|_{L^\infty(I,L^2(\Omega))} \le K\|\overrightarrow{\Delta u}\|_\Sigma$, $\|\overrightarrow{\Delta y}\|_{L^2(Q)} \le K\|\overrightarrow{\Delta u}\|_\Sigma$ and $\|\overrightarrow{\Delta y}\|_{L^2(I,V)} \le K\|\overrightarrow{\Delta u}\|_\Sigma$

(b) With assumptions (A), the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ from$(L^2(\Sigma))^2$in to $(L^\infty(I, L^2(\Omega)))^2$, or in to $(L^2(I,V))^2$, or in to $(L^2(Q))^2$ is continuous.

Proof:

(a) Let $\vec{u} = (u_1, u_2) \in (L^2(\Sigma))^2$ then by theorem (3.1) there exists $\vec{y} = (y_1 = y_{u_1}, y_2 = y_{u_2})$ which satisfies the weak forms (10a&b) - (11a&b),$\forall v_1, v_2 \in V$, also let $\vec{\hat{u}} = (\hat{u}_1, \hat{u}_2) \in (L^2(\Sigma))^2$, then $\vec{\hat{y}} = (\hat{y}_1, \hat{y}_2)$ is the corresponding solution of the following weak forms

$\langle \hat{y}_{1t}, v_1\rangle + a_1(t, \hat{y}_1, v_1) + (b_1(t)\hat{y}_1, v_1)_\Omega - (b(t)\hat{y}_2, v_1)_\Omega = (f_1(\hat{y}_1), v_1)_\Omega + (\hat{u}_1, v_1)_\Gamma$ ....(35a)

$(\hat{y}_1(0), v_1)_\Omega = (y_1^0, v_1)_\Omega$ ....(35b)

and

$\langle \hat{y}_{2t}, v_2\rangle + a_2(t, \hat{y}_2, v_2) + (b_2(t)\hat{y}_2, v_2)_\Omega + (b(t)\hat{y}_1, v_2)_\Omega = (f_1(\hat{y}_2), v_2)_\Omega + (\hat{u}_1, v_2)_\Gamma$ ....(36a)

$(\hat{y}_2(0), v_2)_\Omega = (y_2^0, v_2)_\Omega$ ....(36b)

By subtracting (10a&b) and (11a&b) from (35a&b), (36a&b) respectively, setting $\Delta y_1 = \hat{y}_1 - y_1$, $\Delta y_2 = \hat{y}_2 - y_2$, $\Delta u_1 = \hat{u}_1 - u_1$ and $\Delta u_2 = \hat{u}_2 - u_2$ in each one of the two obtained equations, i.e.

$\langle \Delta y_{1t}, v_1\rangle + a_1(t, \Delta y_1, v_1) + (b_1(t)\Delta y_1, v_1)_\Omega - (b(t)\Delta y_2, v_1)_\Omega = (f_1(y_1+\Delta y_1), v_1)_\Omega - (f_1(y_1), v_1)_\Omega + (\Delta u_1, v_1)_\Gamma$ ....(37a)

$(\Delta y_1(0), v_1)_\Omega = 0$ ....(37b)

and

$\langle \Delta y_{2t}, v_2\rangle + a_2(t, \Delta y_2, v_2) + (b_2(t)\Delta y_2, v_2)_\Omega + (b(t)\Delta y_1, v_2)_\Omega = (f_2(y_2+\Delta y_2), v_2)_\Omega - (f_2(y_2), v_2)_\Omega + (\Delta u_2, v_2)_\Gamma$, ...(38a)

$(\Delta y_2(0), v_2)_\Omega = 0$ ....(38b)

By substituting $v_1 = \Delta y_1$ in (37a) and $v_2 = \Delta y_2$ in (38a), adding the obtained equations, using Lemma 1.2 in [13] for the 1^{st} term and Assumption (A-iii) in the L.H.S. of the obtained equation, one gets

$\frac{1}{2}\frac{d}{dt}\|\overrightarrow{\Delta y}\|_0^2 + \bar{\alpha}\|\overrightarrow{\Delta y}\|_1^2 \le |(f_1(y_1+\Delta y_1) - f_1(y_1), \Delta y_1)| + |(f_2(y_2+\Delta y_2) - f_2(y_2), \Delta y_2)|$
$|(\Delta u_1, \Delta y_1)| + |(\Delta u_2, \Delta y_2)|$ ....(39)

Since the 2^{nd} term of L.H.S. of (39) is positive, integrating both sides w.r.t. t from 0 to t, then using assumptions (A-ii), and then using the Cauchy-Schwarz inequality for the R.H.S., finally using the Trace operator, to get

$\|\overrightarrow{\Delta y}(t)\|_0^2 \le 4\|\overrightarrow{\Delta u}\|_\Sigma^2 + L_3\int_0^t \|\overrightarrow{\Delta y}\|_0^2\, dt$,

where L_3 refers to a summation for constants

Applying the Belman-Gronwall inequality gives

$\|\overrightarrow{\Delta y}(t)\|_0 \le K\|\overrightarrow{\Delta u}\|_Q, t \in [0,T]$

$\Rightarrow \|\overrightarrow{\Delta y}\|_{L^\infty(I,L^2(\Omega))} \le K\|\overrightarrow{\Delta u}\|_\Sigma, t \in [0,T]$

From this result, easily once can get that

$\|\overrightarrow{\Delta y}\|_{L^2(Q)} \leq K\|\overrightarrow{\Delta u}\|_{\Sigma}$, and $\|\overrightarrow{\Delta y}\|_{L^2(I,V)} \leq K\|\overrightarrow{\Delta u}\|_{\Sigma}$

(b) Let $\overrightarrow{\Delta u} = \vec{u}_1 - \vec{u}_2$ and $\overrightarrow{\Delta y} = \vec{y}_1 - \vec{y}_2$ where $\vec{y}_1$and $\vec{y}_2$ are the correspond states to the boundary controls $\vec{u}_1$and$\vec{u}_2$, then frompart (a) of this theorem,once get that the operator $\vec{u} \longmapsto \vec{y}$ is Lipschitz continuous from $(L^2(\Sigma))^2$ in to$(L^\infty(I, L^2(\Omega)))^2$. The other result is obtained easily.

Assumptions (B):

Consider g_{li} and h_{li} (for each $l = 0,1,2$ and $i = 1,2$) is of Carathéodory type on $(Q \times \mathbb{R})$ and on$(\Sigma \times \mathbb{R})$ respectly and satisfies the following sub quadratic condition w.r.t. y_i and u_i

$|g_{\ell i}(x,t,y_i)| \leq \gamma_{li}(x,t) + c_{li}(y_i)^2, |h_{li}(x,t,u_i)| \leq \delta_{li}(x,t) + d_{li}(u_i)^2$

where $y_i, u_i \in \mathbb{R}$ with $\gamma_{li} \in L^1(Q), \delta_{li} \in L^1(\Sigma)$

Lemma (4.2):

With assumptions (B), the functional $G_l(\vec{u})$, is continuous on$(L^2(\Sigma))^2$ for each $l = 0,1,2$.

Proof:

From assumptions (B), with Using proposition (3.1), the integrals $\int_Q g_{li}(x,t,y_i)\,dxdt$ and $\int_\Sigma h_{li}(x,t,u_i)\,d\sigma$ are continuous on $L^2(Q)$ and $L^2(\Sigma)$ respectively for each $i = 1,2$, $l = 0,1,2 \Rightarrow G_l(\vec{u})$ is continuous on $(L^2(\Sigma))^2$, $\forall l = 0,1,2$.

Theorem (4.3):

In addition to assumptions (A), and (B), If the set of controls is of the form $\vec{W} = \vec{W}_{\vec{U}}$ with $\vec{U}$ is convex and compact,$\vec{W}_A \neq \emptyset$, if for each $i = 1,2$, $G_1(\vec{u}\ )$ is independent of u_i, $G_0(\vec{u})$ and$G_2(\vec{u})$ are convex w.r.t. u_i for fixed (x,t,y_i) for each $i = 1,2$. Then there exists a boundary optimal control vector.

Proof:

Since the set W_i is convex, closed and bounded for each $i = 1,2$, then $W_1 \times W_2$ is convex, closed and bounded, which gives $W_1 \times W_2$ is weakly compact. Since $\vec{W}_A \neq \emptyset$, then there exists $\vec{\vec{u}} \in \vec{W}_A$ and there exists a minimum sequence $\{\vec{u}_k\}$ with $\vec{u}_k \in \vec{W}_A$, $\forall k$ such that, $\lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}}\in\vec{W}_A} G_0(\vec{\vec{u}})$.

But $\vec{W}$ is weakly compact, there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ which converges weakly to some point $\vec{u}$ in $\vec{W}$, i.e. $\vec{u}_k \longrightarrow \vec{u}$ weakly in $(L^2(\Sigma))^2$, and $\|\vec{u}_k\|_\Sigma \leq c$, $\forall k$.

From theorem (3.2) for each boundary control $\vec{u}_k$, the state equation has a unique solution $\vec{y}_k = \vec{y}_{\vec{u}_k}$ and the norms $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and $\|\vec{y}_k\|_{L^2(I,V)}$ are bounded, then by Alaoglu's theorem there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ which converges weakly to some point $\vec{y}$ w.r.t the above norms, i.e. $\vec{y}_k \longrightarrow \vec{y}$ weakly in $(L^\infty(I, L^2(\Omega)))^2$, in $(L^2(Q))^2$, and in $(L^2(I,V))^2$.

Also, from theorem (3.2), the norm $\|\vec{y}_k\|_{L^2(I,V^*)}$ is bounded and since

$(L^2(I,V))^2 \subset (L^2(Q))^2 \cong ((L^2(Q))^*)^2 \subset (L^2(I,V^*))^2$

Then by using the First Compactness Theorem [13], there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ such that $\vec{y}_k \longrightarrow \vec{y}$ strongly in $(L^2(Q))^2$.

Since for each k, y_{1k} and y_{2k} are corresponding solutions to the controls u_{1k} and u_{2k}, i.e.,

$\langle y_{1kt}, v_1\rangle + a_1(t, y_{1k}v_1) + (b_1(t)y_{1k}, v_1)_\Omega - (b(t)y_{2k}, v_1)_\Omega = (f_1(x,t,y_{1k}), v_1)_\Omega + (u_{1k}, v_1)_\Gamma$ ....(40)

and

$\langle y_{2kt}, v_2\rangle + a_2(t, y_{2k}, v_2) + (b_2(t)y_{2k}, v_2)_\Omega + (b(t)y_{1k}, v_2)_\Omega = (f_2(x,t,y_{2k}), v_2)_\Omega + (u_{2k}, v_2)_\Gamma$ ....(41)

Let $\varphi_i \in C^1[I]$, with $\varphi_i(T) = 0$, $\forall i = 1,2$. Multiplying both sides of (40) and (41) by $\varphi_1(t)$ and $\varphi_2(t)$ respectively, and then integrating both sides w.r.t. t from 0 to T, finally using integration by parts for the 1^{st} terms in the L.H.S. of the two obtain equations, to get

$-\int_0^T (y_{1k}, v_1)\acute{\varphi}_1(t)dt + \int_0^T [a_1(t, y_{1k}v_1) + (b_1(t)y_{1k}, v_1)_\Omega - (b(t)y_{2k}, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(x,t,y_{1k}), v_1)_\Omega\varphi_1(t)dt + \int_0^T (u_{1k}, v_1)_\Gamma\varphi_1(t)\,dt + (y_{1k}(0), v_1)_\Omega\varphi_1(0)$

..............................(42)

and

$-\int_0^T (y_{2k}, v_2)\acute{\varphi}_2(t)dt + \int_0^T [a_2(t, y_{2k}v_2) + (b_2(t)y_{2k}, v_2)_\Omega + (b(t)y_{1k}, v_1)_\Omega]\varphi_2(t)dt =$

$\int_0^T (f_2(x,t,y_{2k}),v_2)_\Omega \varphi_2(t)dt +$
$\int_0^T (u_{2k},v_2)_\Gamma \varphi_2(t)\,dt + (y_{2k}(0),v_2)_\Omega \varphi_2(0),$
.... (43)

Since $\vec{y}_k \longrightarrow \vec{y}$ weakly in $(L^2(Q))^2$ and in $(L^2(I,V))^2$, then

$-\int_0^T (y_{1k},v_1)\acute{\varphi}_1(t)dt + \int_0^T [a_1(t,y_{1k},v_1) + (b_1(t)y_{1k},v_1)_\Omega - (\mathrm{b}(t)y_{2k},v_1)_\Omega]\varphi_1(t)dt \longrightarrow$
$-\int_0^T (y_1,v_1)\acute{\varphi}_1(t)dt + \int_0^T [a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b(t)y_2,v_1)_\Omega]\varphi_1(t)dt$
..............................(44a)

and

$-\int_0^T (y_{2k},v_2)\acute{\varphi}_2(t)dt + \int_0^T [a_2(t,y_{2k},v_2) + (b_2(t)y_{2k},v_2)_\Omega + (\mathrm{b}(t)y_{1k},v_2)_\Omega]\varphi_1(t)dt \rightarrow$
$-\int_0^T (y_2,v_2)\acute{\varphi}_2(t)dt + \int_0^T [a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b(t)y_1,v_2)_\Omega]\varphi_2(t)dt$
..............................(45a)

Since $y_{ik}(0), y_{2k}(0)$ are bounded in $L^2(\Omega)$ and from Projection theorem, one gets that:

$(y_{ik}(0),v_i)_\Omega \varphi_1(0) \longrightarrow (y_i^0,v_i)_\Omega \varphi_i(0), i = 1,2$
.... (45b)

Let $\forall i = 1,2$, $w_i = v_i\varphi_1(t)$, then $w_i(x,t)$ is fixed for fixed $(x,t) \in Q$, and then $w_i \in L^\infty(I,V) \subset L^2(I,V) \subset L^2(Q)$. Let $v_i \in C[\overline{\Omega}]$, then $w_i \in C[\overline{\mathrm{Q}}]$ is measurable w.r.t. (x,t) and let $\bar{f}_i(y_{1k}) = f_i(y_{ik})w_l$, then $\bar{f}_i: Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous with respect to y_{ik} for fixed $(x,t) \in Q$, then

$\|\bar{f}_i(x,t,y_{ik}(X)\| \le \eta_i|w_i| + c_1|\,y_{ik}||w_i| = \bar{\eta}_i^2 + \bar{c}_2\|y_{ik}\|^2$, where $\bar{\eta}_i^2 = \frac{1}{2}(\eta_i^2 + \bar{c}_1|w_i|^2)$

By applying proposition (3.1), the integral $\int_Q f_i(\,y_{ik})w_i dxdt$ is continuous w.r.t. y_{1k} but $y_{ik} \longrightarrow y_i$ strongly in $L^2(Q)$, then

$\int_Q f_i(\,y_{ik})w_i dxdt \longrightarrow \int_Q f_i(\,y_i)w_i dxdt,$
$\forall w_i \in C[\bar{Q}]$ ....(44c)

on the other hand, since $u_{ik} \longrightarrow u_i$ weakly in $L^2(\Sigma)$, $\forall i = 1,2$, then

$\int_\Gamma (u_{ik},v_i)\varphi_i(t)\,d\gamma dt \longrightarrow \int_\Gamma (\,u_i,v_i)\varphi_i(t)d\gamma dt$
.... (44d)

Finally, using (44a, b, c& d) and (45b) in (42-43), once get that

$-\int_0^T (y_1,v_1)\acute{\varphi}_1(t)dt + \int_0^T [a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b(t)y_2,v_1)_\Omega]\varphi_1(t)dt =$
$\int_0^T (f_1(x,t,y_1),v_1)_\Omega \varphi_1(t)dt +$
$\int_0^T (u_1,v_1)_\Gamma\,\varphi_1(t)dt + (y_1^0,v_1)_\Omega \varphi_1(0)$ ....(46)

$-\int_0^T (y_2,v_2)\acute{\varphi}_2(t)dt + \int_0^T [a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b(t)y_1,v_2)_\Omega]\varphi_2(t)dt =$
$\int_0^T (f_2(x,t,y_2),v_2)_\Omega \varphi_2(t)dt +$
$\int_0^T (u_2,v_2)_\Gamma \varphi_2(t))dt + (y_2^0,v_2)_\Omega \varphi_2(0)$ ....(47)

Equations (46) and (47) are also hold for each $v_i \in V$, $\forall i = 1,2$ (Since $C[\overline{\Omega}]$ is dense in V).

Now, using the same steps which are used in Case 1 and Case2 in the proof of theorem 3.1, once get that y_1 and y_2 are solutions of the weak form of the state equations.

Now, since g_{1i} is independent of u_i, for each $i = 1,2$. i.e.,

$G_1(\vec{u}_k) = \int_Q g_{11}(x,t,y_{1k})\,dxdt + \int_Q g_{12}(x,t,y_{2k})\,dxdt$

From the continuity of $g_{1i}(x,t,y_{ik})$ w.r.t. y_k and the proof of Lemma (4.2), the integral $\int_Q g_{1i}(x,t,y_{ik})\,dxdt$ is continuous w.r.t. y_{ik} and since $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$, hence from proposition 3.1, one gets that

$G_1(\vec{u}) = \lim_{k\to\infty} G_1(\vec{u}_k) = 0.$

Again since for each $i = 1,2$ and $l = 0,2$, $g_{li}(x,t,y_{ik})$ is continuous w.r.t y_{ik}, then from the proof of Lemma (4.2), one has

$\int_Q g_{li}(x,t,y_{ik})\,dxdt \longrightarrow \int_Q g_{li}(x,t,y_i)\,dxdt$
..............................(48)

From the hypotheses on h_{li}, $h_{li}(x,t,u_i)$ is weakly lower semi continuous w.r.t. u_i, for each $i = 1,2$ and $l = 0,2$, then from (48), one has

$\int_Q g_{li}(x,t,y_i)\,dxdt + \int_\Sigma h_{li}(x,t,u_i)\,d\sigma \le$
$\lim_{k\to\infty} \inf \int_\Sigma h_{li}(x,t,u_{ik})d\sigma + \int_Q g_{li}(x,t,y_i)\,dxdt$
$= \lim_{k\to\infty} \inf \int_\Sigma (h_{li}(x,t,u_{ik})d\sigma +$
$\lim_{k\to\infty} \int_Q (g_{li}(x,t,y_i) - g_{li}(x,t,y_{ik}))dxdt +$
$\lim_{k\to\infty} \int_Q g_{li}(x,t,y_{ik})\,dxdt$
$= \lim_{k\to\infty} \inf \int_\Sigma h_{li}(x,t,u_{ik})\,d\sigma +$
$\lim_{k\to\infty} \inf \int_Q g_{li}(x,t,y_{ik})\,dxdt$
$\Rightarrow G_l(\vec{u}) \le \lim_{k\to\infty} \inf G_l(\vec{u}_k)$, (for each $l = 0,2$)

But $G_2(\vec{u}_k) \le 0, \forall k$, then $G_2(\,\vec{u}) \le 0$, and one gets that $\vec{u} \in \overrightarrow{W}_A$ and

$G_0(\vec{u}) \le \lim_{k\to\infty} \inf G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}}\in\overrightarrow{W}_A} G_0\left(\vec{\vec{u}}_k\right)$

$\Rightarrow G_0(\vec{u}) = \min_{\vec{\vec{u}}\in\overrightarrow{W}_A} G_0\left(\vec{\vec{u}}_k\right) \Rightarrow \vec{u}$ is a classical boundary optimal control.

5. The Necessary Conditions for Optimality

This section concerns with the derivation of the Fréchet derivative under some suitable assumptions, it concerns also with the proof of theorem of necessary conditions for optimality so as the theorem of sufficient conditions under some additional assumptions. Hence, the following assumption is very useful.

Assumptions (C):

If $f_{iy_i}, g_{l_iy_i}$and $h_{l_iu_i}$, $(l = 0,1,2$ & $i = 1,2)$ are of Carathéodory type on $Q \times (\mathbb{R})$, $Q \times (\mathbb{R})$ and on $\Sigma \times (\mathbb{R})$ respectively, such that
$(\mathbb{R})$, $Q \times (\mathbb{R})$and on $\Sigma \times (\mathbb{R})$ respectively, such that

$|f_{iy_i}(x,t,y_i)| \leq \acute{L}_i$

$|g_{l_iy_i}(x,t,y_i)| \leq \zeta_{li}(x,t) + e_{li}|y_i|$

and

$|h_{l_iu_i}(x,t,u_i)| \leq \eta_{li}(x,t) + f_{li}|u_i|$

where$(x,t) \in Q$, $y_i, u_i \in \mathbb{R}$, $\zeta_{li}(x,t) \in L^2(Q)$, $\eta_{li}(x,t) \in L^2(\Sigma)$, and $e_{li}, f_{li} > 0$.

Theorem (5.1):

Dropping index l, the Hamiltonian H which is defined by

$H(x,t,\vec{y},\vec{z},\vec{u}) = \sum_{i=1}^{2}(z_i f_i(x,t,y_i) + g_i(x,t,y_i) + h_i(x,t,u_i))$

and the adjoint state $z_i = z_{iu}$(where $y_i = y_{ui}$) equation satisfies

$-z_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial z_1}{\partial x_i}) + b_1(x,t)z_1 + b(x,t)z_2 = z_1 f_{y_1}(x,t,y_1) + g_{y_1}(x,t,y_1)$, inQ

$-z_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial z_2}{\partial x_i}) + b_2(x,t)z_2 - b(x,t)z_1 = z_2 f_{y_2}(x,t,y_2) + g_{y_2}(x,t,y_2)$, in$Q$

$z_1(x,T) = 0$, on Ω

$z_2(x,T) = 0$, on Ω

$\frac{\partial z_1}{\partial n} = 0$, on Σ

$\frac{\partial z_2}{\partial n} = 0$, on Σ

Then the Fréchet derivative of G is given by,

$\acute{G}(\vec{u})\overrightarrow{\Delta u} = \int_\Sigma \begin{pmatrix} z_1 + h_{u_1} \\ z_2 + h_{u_2} \end{pmatrix} \cdot \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} d\sigma$

$= \int_\Sigma H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) \cdot \overrightarrow{\Delta u} d\sigma$

where:

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) = \sum_{i=1}^{2}(z_i + h_{iu_i}(x,t,u_i))$

Proof:

The weak forms of the adjoint equations are given by

$-\langle z_{1t}, v_1\rangle + a_1(t,z_1,v_1) + (b_1(t)z_1,v_1)_\Omega + (b(t)z_2,v_1)_\Omega = (z_1 f_{1y_1}, v_1)_\Omega + (g_{y_1}, v_1)_\Omega$ .... (49)

$-\langle z_{2t}, v_2\rangle + a_2(t,z_2,v_2) + (b_2(t)z_2,v_2)_\Omega - (b(t)z_1,v_2)_\Omega = (z_2 f_{2y_2}, v_2)_\Omega + (g_{y_2}, v_2)_\Omega$ .... (50)

These weak forms have a unique solution and this can be proved by the same way which is used in the proof of theorem 3.2.

Now, substituting $v_1 = z_1$ in (37) and $v_2 = z_2$ in (38), integrating both sides with respect to t from 0 to T, then adding two obtained equations to get

$\int_0^T \langle\overrightarrow{\Delta y_t}, \vec{z}\rangle dt + \int_0^T [a_1(t,\Delta y_1,z_1) + (b_1(t)\Delta y_1, z_1)_\Omega - (b(t)\Delta y_2, z_1)_\Omega + a_2(t,\Delta y_2,v_2) + (b_2(t)\Delta y_2, v_2)_\Omega + (b(t)\Delta y_1, v_2)_\Omega]dt = \int_0^T (f_1(y_1 + \Delta y_1), z_1)_\Omega dt - \int_0^T (f_1(y_1), z_1)_\Omega dt + \int_0^T (\Delta u_1, z_1)_\Gamma dt + \int_0^T (f_2(y_2 + \Delta y_2), z_2)_\Omega dt - \int_0^T (f_2(y_2), z_2)_\Omega dt + \int_0^T (\Delta u_2, z_2)_\Gamma dt$ .... (51)

From assumption (A-ii), and proposition (3.2), the Fréchet derivative of f_i exists for each $y_i \in L^2$, which gives after using the result of Theorem (4.1)

$\int_0^T (f_i(x,t,y_i + \Delta y_i) - f_i(x,t,y_i), z_i)_\Omega dt = \int_0^T (f_{iy_i}\Delta y_i, z_i)dt + \varepsilon_{i1}(\Delta y_i)\|\Delta y_i\|_Q$
$\varepsilon_1(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_\Sigma$, ... (52)

where, $\varepsilon_1(\overrightarrow{\Delta u}) \longrightarrow 0$as $\|\overrightarrow{\Delta u}\|_\Sigma \longrightarrow 0$

By substituting (52) in the R.H.S. of (51), one has that

$\int_0^T \langle\overrightarrow{\Delta y_t}, \vec{z}\rangle dt + \int_0^T [a_1(t,\Delta y_1,z_1) + (b_1(t)\Delta y_1, z_1)_\Omega - (b(t)\Delta y_2, z_1)_\Omega + a_2(t,\Delta y_2,v_2) + (b_2(t)\Delta y_2, v_2)_\Omega + (b(t)\Delta y_1, v_2)_\Omega]dt = \int_0^T (f_{1y_1}\Delta y_1, z_1)_\Omega dt + \int_0^T (f_{2y_2}\Delta y_2, z_2)_\Omega dt + \int_0^T (\Delta u_1, z_1)_\Gamma dt + \int_0^T (\Delta u_2, z_2)_\Gamma dt + \varepsilon_1(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_\Sigma$ .... (53)

Now, substituting $v_1 = \Delta y_1$ and $v_2 = \Delta y_2$ in the adjoint equations (49) and (50) respectively, integrating both sides with respect to t from 0 to T, using integrating by part for the 1^{st} term of each obtained equation, finally adding these two equations, to get:

$\int_0^T \langle\overrightarrow{\Delta y_t}, \vec{z}\rangle dt + \int_0^T [a(t,z_1,\Delta y_1) + (b_1(t)z_1, \Delta y_1)_\Omega + (b(t)z_2, \Delta y_1)_\Omega + a_2(t,z_2,\Delta y_2) + (b_2(t)z_2, \Delta y_2)_\Omega - (b(t)z_1, \Delta y_2)_\Omega]dt = \int_0^T (z_1 f_{1y_1}, \Delta y_1)_\Omega dt + \int_0^T (g_{1y_1}, \Delta y_1)_\Omega dt +$

$\int_0^T (z_2 f_{2y_2}, \Delta y_2)_\Omega dt + \int_0^T (g_{2y_2}, \Delta y_2)_\Omega dt$ .... (54)

By subtracting (54) from (53), one gets

$\int_0^T \left(g_{1y_1}, \Delta y_1\right)_\Omega dt + \int_0^T \left(g_{2y_2}, \Delta y_2\right)_\Omega dt = \int_0^T (\Delta u_1, z_1)_\Gamma dt + \int_0^T (\Delta u_2, z_2)_\Gamma dt + \varepsilon_1(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma$ .... (55)

Now, let $G_A(\vec{u}) = \int_Q k_1(x,t,y_1,y_2)dxdt$, $G_B(\vec{u}) = \int_\Sigma k_2(x,t,u_1,u_2)d\sigma$

where:

$k_1(x,t,y_1,y_2) = g_1(x,t,y_1) + g_2(x,t,y_2)$,
$k_2(x,t,u_1,u_2) = h_1(x,t,u_1) + h_2(x,t,u_2)$,

From the Fréchet derivative and the result of theorem (4.1), one has

$G_A\left(\vec{u} + \overrightarrow{\Delta u}\right) - G_A(\vec{u}) = \int_Q (k_{1y_1} \Delta y_1 + k_{1y_2} \Delta y_2)dxdt + \varepsilon_2(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma$ .... (56)

$G_B\left(\vec{u} + \overrightarrow{\Delta u}\right) - G_B(\vec{u}) = \int_\Sigma (k_{2u_1} \Delta u_1 + k_{2u_2} \Delta u_2)d\sigma + \varepsilon_3(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma$ .... (57)

Adding (56) and (57), to get

$G(\vec{u} + \overrightarrow{\Delta u}) - G(\vec{u}) = \int_Q (g_{1y_1} \Delta y_1 + g_{2y_2} \Delta y_2)\, dxdt + \int_\Sigma (h_{1u_1} \Delta u_1 + h_{2u_2} \Delta u_2)d\sigma + \varepsilon_4(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma$ .... (58)

By substituting (55) in (58), gives

$G(\vec{u} + \overrightarrow{\Delta u}) - G(\vec{u}) = \int_\Sigma (\Delta u_1, z_1)\, d\sigma + \int_\Sigma (\Delta u_2, z_2)\, d\sigma + \int_\Sigma (h_{1u_1} \Delta u_1 + h_{2u_2} \Delta u_2)d\sigma + \varepsilon_5(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma$

where $\varepsilon_5(\overrightarrow{\Delta u}) \to 0$ as $\left\|\overrightarrow{\Delta u}\right\|_\Sigma \to 0$

Using proposition (3.2), the Fréchet derivative of G is

$$\left(\acute{G}(\vec{u}), \overrightarrow{\Delta u}\right) = \int_\Sigma \begin{pmatrix} z_1 + h_{1u_1} \\ z_2 + h_{2u_2} \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} d\sigma.$$

***Theorem (5.2) Necessary Conditions for Optimality (Multipliers Theorem)*:**

If $\vec{u} \in \overrightarrow{W}_A$ is an optimal control, i.e., there exists multipliers $\lambda_l \in R, l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^2 |\lambda_l| = 1$ such that

$\sum_{l=0}^2 \lambda_l \, \acute{G}_l(\vec{u}\,)(\vec{\acute{u}} - \vec{u}) \geq 0, \forall \vec{\acute{u}} \in \overrightarrow{W}$ .... (59)

and

$\lambda_2 G_2(\vec{u}) = 0$ (Transversality condition) ... (60)

The above relation is equivalent (59) to the following (weak) point wise minimum principle.

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{\acute{u}} \in U} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{\acute{u}}$ a.e on Σ .... (61)

Where

$$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) = \sum_{i=1}^{2} (z_i + h_{iu_i}(x,t,u_i))$$

with $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$ and $h_i = \sum_{l=0}^{2} \lambda_l h_{li}$, (for $i = 1,2$).

Proof:

With assumptions (A),(B) and (C), the functional $G_l(\vec{u})$ and $\acute{G}_l(\vec{u})$ (for l=0,1,2) are continuous and liner w.r.t. $(\vec{\acute{u}} - \vec{u})$, then $G_l(\vec{u})$ is ρ −differentiable at each $\vec{u} \in \overrightarrow{W}$, $\forall \rho$, Then by using the Kuhn-Tucker-Lagrange multipliers theorems [14], there exists multipliers $\lambda_l \in \mathbb{R}, l = 0,1,2,$ with $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, $\sum_{l=0}^2 |\lambda_l| = 1$ such that (60) and (61) are hold, i.e.

$\left(\lambda_0 \acute{G}_0(\vec{u}) + \lambda_1 \acute{G}_1(\vec{u}) + \lambda_2 \acute{G}_2(\vec{u})\right) \cdot \left(\vec{\acute{u}} - \vec{u}\right) \geq 0$, $\forall \vec{\acute{u}} \in \overrightarrow{W}$

Applying Theorem (5.1), setting $\overrightarrow{\Delta u} = \vec{\acute{u}} - \vec{u}$ and substituting the Fréchet derivative of G_l, for $l = 0,1,2$ in (58), one has that

$$\sum_{l=0}^2 \sum_{i=1}^2 \int_\Sigma \left(\lambda_l (z_{li} + h_{li_{u_i}})\right) (\acute{u}_i - u_i) d\sigma \geq 0$$

Let $z_i = \sum_{l=0}^2 \lambda_l \, z_{li}$, $h_i = \sum_{l=0}^2 \lambda_l h_{li}$, for each $i = 1,2$

$\Rightarrow \int_\Sigma H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) \cdot \overrightarrow{\Delta u} d\sigma \geq 0$ .... (62)

To prove that (62) is equivalent to (61)

Let

$\overrightarrow{W}_{\vec{U}} = \left\{\vec{u} \in (L^2(\Sigma,\mathbb{R}))^2 \middle| \vec{u}(x,t) \in \vec{U} \text{a.e.in}\Sigma\right\}$, with $\vec{U} \subset \mathbb{R}^2$, let $\left\{\vec{u}\right\}$ be a dense sequence in $\overrightarrow{W}_{\vec{U}}$, μ is Lebesgue measure on Σ and let $S \subset \Sigma$ be a measurable set such that

$$\vec{\acute{u}}(x,t) = \begin{cases} \vec{u}_k(x,t), \text{if } (x,t) \in S \\ \vec{u}(x,t), \text{if } (x,t) \notin S \end{cases}$$

Therefore (66) becomes

$\int_S H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) \cdot (\vec{u}_k - \vec{u})dS \geq 0, \forall S$

Using theorem (3.1), to get

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}).(\vec{u}_k - \vec{u}) \geq 0$, a.e. in Σ, which means this inequality is satisfied on the boundary Σ of the region Q except in a subset Σ_k such that $\mu(\Sigma_k) = 0$, $\forall k$, where μ is a Lebesgue measure, i.e. the it satisfies on the boundary Σ except in the union of $\bigcup_k \Sigma_k$ with $\mu(\bigcup_k \Sigma_k) = 0$, but $\{\vec{\vec{u}}_k\}$ is a dense sequence in the control set $\overrightarrow{W}$, then there exists $\vec{\vec{u}} \in \overrightarrow{W}$ such that

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\overrightarrow{w} \in \vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{\acute{u}}$, a.e. in Σ, $\forall \vec{\vec{u}} \in \overrightarrow{W}$. The proof of the converse is obtained directly.

6. Sufficient Conditions for Optimality

Theorem (6.1) (Sufficient Conditions for Optimality):

In addition to the assumptions (A), (B) and (C), suppose that $\vec{W} = \vec{W}_{\vec{U}}$ is convex, with $\vec{U}$ convex, f_i and $g_{1i}$are affine w.r.t. $y_i$for each(x,t) and $i = 1,2$, g_{0i} and $g_{2i}$are convex w.r.t. y_i, h_{0i} and $h_{2i}$are convex with respect to u_i for each (x,t) and, $\forall i = 1,2$. Then the necessary conditions in Theorem (5.2) with $\lambda_0 > 0$, are sufficient.

Proof:

Suppose$\vec{u}$ is satisfied the K.T.L. condition, the Transversality condition and $\vec{u} \in \vec{W}_A$, i.e.

$\int_\Sigma \begin{pmatrix} z_1 + h_{1u1} \\ z_2 + h_{2u2} \end{pmatrix} \cdot \overrightarrow{\Delta u} d\sigma \geq 0$, $\quad \forall \vec{u} \in \vec{W}$ and $\lambda_2 G_2(\vec{u}) = 0$,

where $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$ and $h_i = \sum_{l=0}^{2} \lambda_l h_{li}$ (for $i = 1,2$).

Let $G(\vec{u}) = \sum_{l=0}^{2} \lambda_l G_l(\vec{u})$, then

$\acute{G}(\vec{u}) \cdot \overrightarrow{\Delta u} = \sum_{l=0}^{2} \lambda_l \acute{G}_l(\vec{u}) \cdot \overrightarrow{\Delta u} =$
$\lambda_0 \int_\Sigma \sum_{i=1}^{2} \left(z_{0i} + h_{0iu_i}\right) \Delta u_i \, d\sigma +$
$\lambda_1 \int_\Sigma \sum_{i=1}^{2} \left(z_{1i} + h_{1iu_i}\right) \Delta u_i \, d\sigma$
$\lambda_2 \int_\Sigma \sum_{i=1}^{2} \left(z_{2i} + h_{2iu_i}\right) \Delta u_i \, d\sigma \geq 0$

since the functions $f_1 \& f_2$ in the R.H.S. of the state equation (1) and (2) are affine with respect to y_1, and y_2 for each$(x,t) \in Q$ respectively, i.e.,

$f_1(x,t,y_1) = f_{11}(x,t)y_1 + f_{12}(x,t)$ &
$f_2(x,t,y_2) = f_{21}(x,t)y_2 + f_{22}(x,t)$

Let $\vec{u} = (u_1, u_2)$ & $\vec{\bar{u}} = (\bar{u}_1, \bar{u}_2)$ are two given controls and then (by Theorem (3.2)), $\vec{y} = \left(y_{u_1}, y_{u_2}\right) = (y_1, y_2)$ & $\vec{\bar{y}} = \left(\bar{y}_{\bar{u}_1}, \bar{y}_{\bar{u}_2}\right) = (\bar{y}_1, \bar{y}_2)$ are their corresponding solutions, i.e. and for the 1st state equations and their corresponding initial condition

$y_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y_1}{\partial x_j}) + b_1(x,t)y_1 - b(x,t)y_2 = f_{11}(x,t)y_1 + f_{12}(x,t)$
$\sum_{i,j=1}^{n} a_{ij} \frac{\partial y_1}{\partial n} = u_1(x,t),$
$y_1(x,0) = y_1^0(x)$

and

$\bar{y}_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial \bar{y}_1}{\partial x_j}) + b_2(x,t)\bar{y}_1 - b(x,t)\bar{y}_2 = f_{11}(x,t)\bar{y}_1 + f_{12}(x,t)$
$\sum_{i,j=1}^{n} b_{ij} \frac{\partial y_1}{\partial n} = \bar{u}_1\,(x,t)$
$\bar{y}_1(x,0) = y_1^0(x)$

By multiplying the 1^{st} above equation and its initial condition by θ, $\theta \in [0,1]$, and the 2^{nd} equation and its initial condition by $(1-\theta)$, and adding the obtained equations and their obtained initial conditions, one has

$(\theta y_1 + (1-\theta)\bar{y}_1)_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial x_j}) + b_1(x,t)(\theta y_1 + (1-\theta)\bar{y}_1) - b(x,t)(\theta y_2 + (1-\theta)\bar{y}_2) = f_{11}(x,t)(\theta y_1 + (1-\theta)\bar{y}_1) + f_{12}(x,t)$ ....(63a)
$\theta y_1(x,0) + (1-\theta)\bar{y}_1(x,0) = y_1^0(x)$ .... (63b)
$\sum_{i,j=1}^{n} a_{ij} \frac{\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial n} = (\theta u_1 + (1-\theta\bar{u}_1)$ on Σ ....(63c)

By the same way and for the second differential equations and their initial conditions, one gets

$(\theta y_2 + (1-\theta)\bar{y}_2)_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial(\theta y_2 + (1-\theta)\bar{y}_2)}{\partial x_j}) + b_2(x,t)(\theta y_{21} + (1-\theta)\bar{y}_2) + b(x,t)(\theta y_1 + (1-\theta)\bar{y}_1) = f_{21}(t)(\theta y_2 + (1-\theta)\bar{y}_2) + f_{22}(x,t)$ ....(64a)
$\theta y_2(x,0) + (1-\theta)\bar{y}_2(x,0) = y_2^0(x)$ ... (64b)
$\sum_{i,j=1}^{n} b_{ij} \frac{\partial(\theta y_2 + (1-\theta)\bar{y}_2)}{\partial n} = (\theta u_2 + (1-\theta\bar{u}_2)$,on Σ ....(64c)

Equations (63) and (64), tell us that the control $\vec{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2)$, with $\vec{\tilde{u}} = \theta\vec{u} + (1-\theta)\vec{\bar{u}}$ has the corresponding solutions, $\vec{\tilde{y}} = (\tilde{y}_1, \tilde{y}_2)$, with $\vec{\tilde{y}} = \theta\vec{y} + (1-\theta)\,\vec{\bar{y}}$, which means the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is convex – linear with respect to $(\vec{y}, \vec{u})$ for each(x,t).

Now, since for each $i = 1,2$, $g_{1i}(x,t,y_i)$ is affine w.r.t y_i, $\forall (x,t) \in Q$ and $h_{1i}(x,t,u_i)$ is affine w.r.t. u_i, $\forall (x,t) \in \Sigma$, i.e.,

$g_{1i}(x,t,y_i) = I_{1i}(x,t)y_i + I_{2i}(x,t),$
$h_{1i}(x,t,y_i) = I_{1i}(x,t)u_i + I_{3i}(x,t).$

Let $\vec{u} \& \vec{\bar{u}}$ are two controls, and $\vec{y} = \vec{y}_{\vec{u}}$ & $\vec{\bar{y}} = \vec{\bar{y}}_{\vec{\bar{u}}}$ are their corresponding solutions, then

$G_1\left(\theta\vec{u} + (1-\theta)\,\vec{\bar{u}}\right) =$
$\sum_{i=1}^{2} \int_Q g_{1i}\left(x,t,y_{i(\theta u_i + (1-\theta)\bar{u}_i)}\right) dxdt + \sum_{i=1}^{2} \int_\Sigma [\, h_{1i}(x,t,\theta u_i + \ (1-\theta)\bar{u}_i)] d\sigma$
$= \sum_{i=1}^{2} \left[\int_Q I_{1i}\left(x,t,y_{i(\theta u_i + (1-\theta)\bar{u}_i)}\right) + I_{2i}(x,t)\right] dxdt + \sum_{i=1}^{2} \left[\int_\Sigma I_{1i}(x,t,\theta u_i + (1-\theta)\bar{u}_i) + \ I_{3i}(x,t)\right] d\sigma$

Since the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is convex – linear, then

$G_1(\theta\vec{u} + (1-\theta)\,\vec{\bar{u}}) =$
$\sum_{i=1}^{2}\left\{\int_Q [\, I_{1i}(x,t)(\theta y_i + (1-\theta)\bar{y}_i) + I_{2i}(x,t)]\, dxdt + \right.$
$\sum_{i=1}^{2}\{\int_\Sigma [\, I_{1i}(x,t)(\theta u_i + (1-\theta)\bar{u}_i) + I_{3i}(x,t)]\, d\sigma$

$G_1(\theta\vec{u} + (1-\theta)\,\vec{\bar{u}}) = \theta G_1(\vec{u}) + (1-\theta)G_1(\vec{\bar{u}})$

$\Rightarrow G_1(\vec{u})$ is convex – linear w.r.t. $(\vec{y},\vec{u})$, $\forall(x,t)\in Q$.

From the Assumptions on $g_{0i}\&g_{2i}$ $(h_{0i}\&h_{2i})$ $\forall i = 1,2$, the integrals $\sum_{i=1}^{2}\int_Q g_{0i}\,dxdt$ & $\sum_{i=1}^{2}\int_Q g_{2i}\,dxdt$ $(\sum_{i=1}^{2}\int_\Sigma h_{0i}\,d\sigma \& \sum_{i=1}^{2}\int_\Sigma h_{2i}\,d\sigma)$ are convex w.r.t. y_i $\forall(x,t)\in$ Q and (are convex w.r.t. u_i $\forall(x,t)\in\Sigma$), then $G_0(\vec{u})\&G_2(\vec{u})$ are convex w.r.t. $(\vec{y},\vec{u})$, $(\forall(x,t)\in Q, \forall(x,t)\in\Sigma)$, i.e. $G(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u})$, $(\forall(x,t)\in Q, \forall(x,t)\in\Sigma)$. On the other hand, since $\vec{W} = \vec{W}_{\vec{U}}$ is convex, and the Fréchet derivative of $G_l(\vec{u})$, $(l = 0,1,2)$ exists for each $\vec{u}\in\vec{W}$ and it is continuous (by Theorem (5.1) and assumptions (A),(B)and (C)), then it satisfies $\acute{G}(\vec{u})\overrightarrow{\Delta u} \geq 0$, which means $G(\vec{u})$ has a minimum at $\vec{u}$, i.e.

$\lambda_0 G_0(\vec{u}) + \lambda_1 G_1(\vec{u}) + \lambda_2 G_2(\vec{u}) \leq \lambda_0 G_0(\vec{w}) + \lambda_1 G_1(\vec{w}) + \lambda_2 G_2(\vec{w})$ .... (65)

Let $\vec{w}\in\vec{W}_A$, with $\lambda_2 \geq 0$, then Transversality condition (64), gives

$\Rightarrow G_0(\vec{u}) \leq G_0(\vec{w}), \forall\vec{w}\in\vec{W}$, since $(\lambda_0 > 0)$

Hence $\vec{u}$ is a continuous classical boundary optimal control for the problem.

Conclusions

In this paper, the existence and uniqueness theorem of a continuous classical boundary optimal control vector governing by the considered couple of nonlinear partial differential equation of parabolic type with equality and inequality constraints is proved using the Galerkin method, the existence of a classical boundary optimal control is proved under a suitable conditions, while the existence and uniqueness solution of the couple of adjoint vector equations associated with the considered couple equations of the state equations is proved and the derivation of the Fréchet derivative of the Hamiltonian is derived. Finally the theorem of necessary conditions and the theorem of sufficient conditions of optimality problem with equality and inequality constrained are proved.

References

[1] Al-Hawasy, J., "*Optimaization and Approximation of Parabolic Boundary Value Problem*", Athens-Greece, ph. D. Thesis, School of Applied Mathematics and Physics Science, Department of Mathematics, NTUA, 2004.

[2] Al-Hawasy, J., "*The Continuous Classical Optimal Control of a Coupled of nonlinear hyperbolic Equations"* Vol. 57, No.2C, BaghdadIraqi Journal of science, 2016, pp: 1528-1538.

[3] Al-Hawasy, J., Al- Rawdhanee, E., "*The Continuous Classical Optimal Control of a Coupled of a Nonlinear Elliptic Equations*", Mathematical Theory and Modeling, Vol.4, No.14, 2014.

[4] Al-Hawasy, J., and Kadhem Gh., "*The Continuous Classical Optimal Control of a Coupled of a Nonlinear* parabolic *Equations*" vol. 19(1). Jornal of AL.Nahrainuniversity., March,2016, pp. 173 -186.

[5] Brauer, F., Nohel, J., "*OrdinaryDifferential Equations*", Philippines, W.A. Benjamin, Inc., 1973.

[6] Chryssoverghi, I. and Bacopoulos, A., "*Approximation of Relaxed Nonlinear Parabolic Optimal Control Problems*", Journal of Optimization Theory and Applications, Vol.77, No.1, 1993.

[7] Diaz, J.I., Mingazzini, T., and Ramos, A.M., "*on the Optimal Control for a Semi linear Equation with Cost Depending on the Free Boundary*", Vol.7 American Institute of Mathematical Sciences, No.4, 2012.

[8] Donald E.Kirk,"Optimal Control Theory An introduction" Copyright, 1998.

[9] El-Borari, M. M., El-Banna, A. H. and Ahmed, W. H., "*Optimal Control of a Class of Parabolic Partial Differential Equations*", International Journal of Advanced Computing, Vol.36, Issue.1, 2013.

[10] Farag, M.H., "*on an Optimization Problem with Hyperbolic partial Differential Equations*", Vol. 04, Issue 02, IOSR Journal of Engineering, pp. 27-33, 2014.

[11] Orpel, A., "Optimal Control Problems with Higher Order Constraints. Folia Mathematical", 16 (1), pp:31-44. 2009
[12] Sasane, A., "*Variational Calculus and Optimal Control*", LSE, 2004.
[13] Temam, R. "*Navier –Stokes Equations*", New York, North-Holland Publishing Company, 1977.
[14] Warga, J., "*Optimal Control of Differential and Functional Equations*", Academic Press, New York and London, 1972.
[15] Wang, Y., Luo, X. and Li, S., "*Optimal Control Method of Partial Differential Equations and Its Application to Heat transfer Model in Continuous Cast Secondary Cooling Zone*", Copyright, 2015.