Some Results of Principal Fuzzy Metric Spaces and Their Completeness

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Abstract

The main objective of this paper is to introduce the concept of principal fuzzy metric spaces based on the definition of the distance function between fuzzy points. Also, among the obtained results in this paper is the relationship between p-convergent and convergent sequences in principal fuzzy metric spaces, which may be used later to prove the main result of this paper that is the completeness of principal fuzzy metric spaces. [DOI: 10.22401/ANJS.00.1.16]

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1- Introduction

Among the aims of fuzzy set theory is to develop the methodology of the formulations and solutions of problems that are ill defined or too complicated to be acceptable to analysis by conventional techniques.

Historically, the general accepted birth date of the theory of fuzzy sets is 1965, when the first article entitled "Fuzzy sets" written by L.A. Zadeh appeared in the Journal of Information and Control. Also, the phrase, fuzzy, was introduced and coined by Zadeh in this article, [15].

Zadeh's original definition of fuzzy sets is to consider a class objects characterized by a characteristic or a membership function that assigns to each object from the set a grade of membership that ranging between 0 and 1, [10].

The characteristic function is a weighting function that reflects the ambiguity of the element in the set. As the membership value approaches unity, the grade of membership of the event in the fuzzy set $\tilde{A}$ with membership function $M_{\tilde{A}}$ higher, for example $M_{\tilde{A}}(x) = 1$ indicates that the event x is strictly contained in the set $\tilde{A}$, while on the other hand, $M_{\tilde{A}}(x) = 0$ indicate that x is strictly does not belong to $\tilde{A}$ and any intermediate value would reflects the degree on which x could be a member of $\tilde{A}$, [11].

2- Fundamental Concepts

In this section, some preliminary and fundamental basic concepts, as well as, results previously obtained in fuzzy metric spaces are given, which are necessary for the present work of this paper. More elementary concepts related to fuzzy sets and real analysis will not be given and for more details one can see the standard texts,[16,1,15,12].

Definition (2-1), [15]:

Let X be any non-empty set of points. A Fuzzy set, denoted by $\tilde{A}$ in X is characterized by a membership function $M_{\tilde{A}}: X \to [0,1]$. This fuzzy set may be written as:

$$\tilde{A} = \{(x, M_{\tilde{A}}(x)): x \in X, 0 \leq M_{\tilde{A}}(x) \leq 1\}$$

Some of the well-known relations enclosing fuzzy sets may be given in the next remark.

Remark (2-2), [10,13]:

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subset of X with membership function $M_{\tilde{A}}$ *and* $M_{\tilde{B}}$ respectively then:

1) $\tilde{A} \subseteq \tilde{B}$ iff $M_{\tilde{A}}(x) \leq M_{\tilde{B}}(x), \quad \forall\, x \in X$.
2) $\tilde{A} = \tilde{B}$ iff $M_{\tilde{A}}(x) = M_{\tilde{B}}(x), \quad \forall\, x \in X$.
3) $\tilde{A}$ is the complement of $\tilde{A}$ with membership function
   $M_{\tilde{A}}(x) = 1 - M_{\tilde{A}}(x), \quad \forall\, x \in X$.
4) The empty fuzzy set $\tilde{\phi}$ and the universal set X are defined for all $x \in X$, with membership function $M_{\tilde{A}}(x) = 0$ and $M_{\tilde{A}}(x) = 1$, respectively.

The extension principle of fuzzy set theory may be used to generalize crisp mathematical concepts to fuzzy mathematical concepts, which is used also to define fuzzy function, i.e., crisp or non-fuzzy concepts may be extended or generalized to fuzzy sets depending on the so called the extension principle as it is defined below:

Definition (2-3), [8]:

Let X be the Cartesian product of universes $X_1, X_2, \dots . X_n$ and $\widetilde{A_1}, \widetilde{A_2}, \dots . \widetilde{A_n}$ be n-fuzzy subsets of $X_1, X_2, \dots . X_n$, respectively, f is a mapping for X into a universal set $Y(y = f(x_1, x_2, \dots, x_n)$, then the fuzzy set $\tilde{B}$ in Y is define by:

$$\tilde{B} = f(\tilde{A})$$
$$= \{(y, \mu_{\tilde{B}}(y))\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\dots,x_s)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), \dots, \mu_{\tilde{A}_n}(x_n)\}, & \text{if } f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

where f^{-1} is the inverse image of f.

Now, many different approaches may be used to define fuzzy metric spaces, but we will use that one which is based on the distance between points of a fuzzy set, as in the next definition:

Definition (2-4), [4]:

Let X^* be the set of all fuzzy points in X,,i.e,

$$X^* = \{\tilde{q}_x^\lambda : x \in X,\ \lambda \in (0,1]\}$$

and let $\tilde{q}_x^\lambda, \tilde{q}_{x'}^{\lambda'}, \tilde{q}_{x_i}^{\lambda_i},\ i = 1,2,3$, be fuzzy points in X, where $\lambda, \lambda', \lambda_i \in (0,1], x, x', x_i \in X,\ \forall\, i = 1,2,3$

A function $d^* = X^* \times X^* \to [0, \infty)$ is called fuzzy distance function if d^* satisfies the following conditions:

1) $d^*\left(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}\right) = 0$ if and only if $\lambda_1 = \lambda_2$ and $x_1 = x_2$
2) $d^*\left(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}\right) = d^*\left(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_1}^{\lambda_1}\right)$
3) $d^*\left(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}\right) \leq d^*\left(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}\right) + \mathrm{d}^*\left(\tilde{\mathrm{q}}_{\mathrm{x}_2}^{\lambda_2}, \tilde{\mathrm{q}}_{\mathrm{x}_3}^{\lambda_3}\right)$
4) If $d^*\left(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}\right) < r$, where $r > 0$, then there exists $\lambda' > \lambda_1 > \lambda_2$, such that $d^*\left(\tilde{q}_{x_2}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}\right) < r$

(X^*, d^*) is called fuzzy metric space.

Based on definition (2.4), the convergence of the sequence of fuzzy points and Cauchy sequence are also defined in the next two definitions:

Definition (2-5) [4]:

A sequence $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}, n \in \mathbb{N}$ of fuzzy points in $\tilde{A}$ is said to be converge to a fuzzy point $\tilde{q}_x^\lambda$ (termed as $\tilde{q}_{x_n}^{\lambda_n} \to \tilde{q}_x^\lambda$) if for all $\varepsilon > 0$, thier exist $N \in \mathbb{N}$ such that $d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^\lambda\right) < \varepsilon$, for all $n \geq \mathbb{N}$
where $x, x_n \in X,\ \lambda, \lambda_n \in (0,1],\ \forall\, n \in \mathbb{N}$.

Definition (2-6), [4]:

A sequence $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}, n \in \mathbb{N}$ of fuzzy points in $\tilde{A}$ is said to be fuzzy Cauchy sequence, if for all $\varepsilon > 0$, thier exist $N \in \mathbb{N}$, such that $d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}\right) < \varepsilon$, for all $n, m \geq N$, where $x_n, x_m \in X,\ \lambda_n, \lambda_m \in (0,1], \forall\, n, m \in \mathbb{N}$.

The next theorem is of great importance that will be used later in the statement of the isometric mapping between two fuzzy metric spaces.

Theorem (2-7), [5]:

If $f: X \to Y$ is a mapping, then $f^*: X^* \to Y^*$ is a fuzzy mapping which is defined by:

$$f^*\left(\tilde{q}_{x_0}^{\lambda_0}\right) = \tilde{q}_{x_0}^{\lambda_0} = \begin{cases} \lambda_0, if\ y_0 = f(x_0) \\ 0, if\ y_0 \neq f(x_0) \end{cases}$$

Definition (2-8), [14]:

Let (X^*, d^*) and (Y^*, D^*) be two fuzzy metric spaces, a mapping $f: X^* \to Y^*$ is said to be an isometry if :

$$d^*\left(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}\right) = D^*\left(f\left(\tilde{q}_{x_1}^{\lambda_1}\right), f\left(\tilde{q}_{x_2}^{\lambda_2}\right)\right)$$

for all $\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2} \in X^*$. In this case X^*, Y^* are called isometric.

Definition (2-9):

A fuzzy metric completion of (X^*, d^*) is complete fuzzy metric space, such that (X^*, d^*) is isometric to a dense subspace of X^*, X^* is called completeable if it admits a fuzzy metric completion.

3- The Main Results

In this section the main results of this paper are presented. The obtained results are based on introducing the definition of the fuzzy distance function given in definition (2.5).

First, we give an alternative definition for the convergent and Cauchy fuzzy sequences.

Definition (3-1), [14]:

Let (X^*, d^*) be a fuzzy metric space a sequence $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}$ of fuzzy points of fuzzy set $\tilde{A}$ of X^* is said to be p-convergent if for each $\varepsilon \in (0,1)$, there is $N \in \mathbb{N}$, such that $d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_0}^{\lambda_0}\right) > 1-\varepsilon,\ \forall\ n \in \mathbb{N}$.

Definition (3-2), [14]:

Let (X^*, d^*) be a fuzzy metric space a sequence $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}$ of fuzzy points of fuzzy set $\tilde{A}$ ofX^* is said to be p-Cauchy if for each $\varepsilon \in (0,1)$ there is $N \in \mathbb{N}$, such that:

$d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}\right) = 1, \forall\, n, m \geq N$

The relationship between p-convergent and p-Cauchy sequences of fuzzy points maybe introduced in the next theorem:

Theorem (3-3):

Every p-convergent sequence of fuzzy points $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}$, of a fuzzy subset $\tilde{A}$ of a fuzzy metric space (X^*, d^*) is a p-Cauchy sequence.

Proof:

Let $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}$ be a Cauchy sequence of a fuzzy subset $\tilde{A}$of(X^*, d^*), then by using [theorem 3.7, [15]. There exists a sequence $\{x_n\} \subset X$ of monotonic sequence of supports and a sequence of images $\{\lambda_n\} \subset (0,1], n \in \mathbb{N}$ such that $x_n \to x, \lambda_n \to \lambda, x \in X, \lambda \in (0,1]$.

Since $\{x_n\}$ is p-convergent non-fuzzy sequence hence it is a p-Cauchy sequence in X.

Also, $\{\lambda_n\}$ is p-convergent of real numbers in (0,1].

Therefore, by using properties (3.8) [14], the sequence of fuzzy points $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}$ is a p-Cauchy sequence. ■

Definition (3-4), [14]:

The fuzzy metric space (X^*, d^*) is said to be p-complete if every p-Cauchy sequence in X^* or is p-convergent.

Definition (3-5), [12]:

A fuzzy metric space (X^*, d^*) is said to be principal, if the set of balls $\{B\left(\tilde{q}_{x_1}^{\lambda_1}, r\right): r \in (0,1)\}$ is a local base at $\tilde{q}_x^{\lambda} \in X$, for each $\tilde{q}_{x_1}^{\lambda_1} \in X^*$.

The relationship between principal fuzzy metric spaces and p-convergent may be stated and proved in the next theorem:

Theorem (3.6):

The fuzzy metric space (X^*, d^*) is principal if and only if all p-convergent sequence an convergent.

***Proof*:**

Suppose that X^* is principal and that $\left\{\tilde{q}_{x_n}^{\lambda_n}\right\}$ is a sequence which is p-convergent to $\tilde{q}_{x_0}^{\lambda_0}$

Let $\varepsilon \in (0,1)$.

Since X^* is principal, then $\{B\left(\tilde{q}_{x_0}^{\lambda_0}, \frac{1}{n}\right): n \in N\}$ is local base of fuzzy points at $\tilde{q}_{x_0}^{\lambda_0}$ Hence, their exist $m \in N$ such that

$B\left(\tilde{q}_{x_0}^{\lambda_0}, \frac{1}{n}\right) \subset B\left(\tilde{q}_{x_0}^{\lambda_0}, \varepsilon\right)$

Since$\lim_{n\to\infty} d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_0}^{\lambda_0}\right) = 1$

Hence, there exist $\delta \in (0,1)$, with $\delta < \frac{1}{m}$, and $n \in N$, such that

$\tilde{q}_{x_n}^{\lambda_n} \in B\left(\tilde{q}_{x_0}^{\lambda_0}, \delta\right)$, for all $n \geq n_1$ and thus $\tilde{q}_{x_n}^{\lambda_n} \in B\left(\tilde{q}_{x_0}^{\lambda_0}, \varepsilon\right)$ for all $n \geq n_1$

Hence$d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_0}^{\lambda_0}\right) > 1-\varepsilon$, for all $n \geq n_1$

and so $\lim_{n\to\infty} d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_0}^{\lambda_0}\right) = 1$ and this is true for all $n > 0$.

For the converse assume that X^* is not principal then it may find a fuzzy point $\tilde{q}_{x_0}^{\lambda_0} \in X^*$, such that $\{B\left(\tilde{q}_{x_0}^{\lambda_0}, \frac{1}{n}\right): n \in N\}$ is not local base at $\tilde{q}_{x_0}^{\lambda_0}$

Hence, there exists $r \in (0,1)$, such that $B\left(\tilde{q}_{x_0}^{\lambda_0}, \frac{1}{n}\right) \nsubseteq B\left(\tilde{q}_{x_0}^{\lambda_0}, r\right)$, for all $n \in N$

Now, by induction a sequence $\{\tilde{q}_{x_n}^{\lambda_n}\}$ may be formed as follows:

For each $n \in \mathbb{N}$, take:

$\tilde{q}_{x_n}^{\lambda_n} \in B\left(\tilde{q}_{x_0}^{\lambda_0}, \frac{1}{n}\right) | B\left(\tilde{q}_{x_0}^{\lambda_0}, r\right)$
and chosen $n_1 \in \mathbb{N}$ for a given $\varepsilon \in (0,1)$ with $\frac{1}{n_1} < \varepsilon$ hence for $m \geq n_1$
Therefore $d^*\left(\tilde{q}_{x_0}^{\lambda_n}, \tilde{q}_{x_0}^{\lambda_0}\right) > 1 - \frac{1}{m} > 1 - \frac{1}{n_1} > 1 - \varepsilon$ and since ε is arbitrary,which implies that $\lim_{n\to\infty} d^*\left(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_0}^{\lambda_0}\right) = 1$.
So, $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is a p-convergent sequence of fuzzy point to $\tilde{q}_{x_0}^{\lambda_0}$andon the other hand, by construction $\tilde{q}_{x_n}^{\lambda_n} \in X^* | B\left(\tilde{q}_{x_0}^{\lambda_0}, r\right)$, for all $n \in \mathbb{N}$and also $\{\tilde{q}_{x_n}^{\lambda_n}\}$ dose not converge to $\tilde{q}_{x_0}^{\lambda_0}$ and by corollary (6), [16] then $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is not hence $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is principal.

The completeness of fuzzy metric spaces may be obtained as a result of the p-completeness of X^*, as it is stated and proved in the next theorem.

Theorem (3.7):
Let (X^*, d^*) be a principal fuzzy metric space, if X^* is p-complete, then X^* is complete.
Proof:
Let $\{\tilde{q}_{x_n}^{\lambda_n}\}$ be a Cauchy sequence in X^* then $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is p-Cauchy and so $\{\tilde{q}_{x_n}^{\lambda_n}\}$is p-convergent to some points $\tilde{q}_{x_0}^{\lambda_0} \in X^*$, and since X^* is principal then $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is convergent to $\tilde{q}_{x_0}^{\lambda_0}$.
Hence X^* is complete.

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