On Conjugate –Gradient Algorithms

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Abstract

The aim of this paper is to recognize the attitude of the conjugate –Gradient Algorithms for solving linear systems $Ax=b$ under the existence of rounding errors. The effect of matrix condition number of $A$ on the relative error of the calculated series of approximations $\{x_k\}$ is analyzed. An especially appealing feature of the algorithm qualified is that error rating can be obtained very easily. Some examples are presented to support the theoretical results and to demonstrate the applicability and efficiency of the methods. The paper ends with some conclusions that sum up the finding of the study. The executed program for calculation is carried using “Matlb7”.

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1. Introduction

Let’s assume we want to resolve the following order of linear equation “$Ax=b$ for the vector $x$ where the known $nxn$ matrix $A$ is symmetrical (i.e., $A^T=A$), “positive definite (i.e., $x^TAx > 0$) for all non-zero vectors $x$ in $\mathbb{R}^n$), and $b$ is known as well. We indicate the unique resolution of this order by $x$. It is well determined that the cg. iterations impose “optimal complexity in a sense to be made precise.” In accurate arithmetic it produces a series of perpendicular residual vectors, and the solution is gained after mainly $n$ steps [1,2,3].Many of these theoretical properties do not hold in the presence of “rounding errors”. It is no longer correct that the calculated residual vectors are nearly perpendicular and that the $n^{th}$ calculated vector $x_n$ is a rational approach to $\alpha$.

A especially “attractive feature of the algorithm described” is that error estimates ability can be obtained very easily. The objective of this paper is to realize the action of several cg Algorithms in the presence of rounding errors. We are mainly concerned with discussing how the matrix “condition number” $K=||A|| ||A^{-1}||$, where $||A||$ donotes the spectral norm of $A$, influences the relative error of the calculated series $\{x_k\}$.

We notice that immediate algorithms of functional advantage as well as much “iterative algorithms” with iterative refinement are fully behaved, i.e, $(A+\delta A)y=b$, where $||\delta A||$ is of order $\mathcal{T}||A||$ and $\mathcal{T}$ is the relative computer precision. Equivalently, the “residual vector $r=ay-b$” has a norm of order $\mathcal{T}||A||||y||$. When it not possible to be determined that an algorithm is fully behaved, it is occasionally likely to be a weak property, namely an algorithm is numerically steady, i.e, the relative error of a calculate $y$ is of order $\mathcal{T}_k$ [4,5] for direct algorithms, and [7,10,11] “for iterative algorithms”.


We study “conjugate gradient algorithms” to work out uptight networks which are setup by a new modulate non monotonous planner study by Shi and Wany (2011). The use of non-monotonous planning can work out algorithm to cope with the status where the series of iterates runs into under most of a turned strict ravine, a common release in neural network practice procedure. Ours meant group of process guarantee enough descent, avoid there by the familiar inactive” restarts and it is globally convergent under” mid conditions. Our experimental results supply proof in order to suggest non monotonous conjugate gradient work out procedure are effective, outperforming classic procedure, provided extra settled, effective and credible study.

2.1 Conjugate-gradient algorithms [6]

The concept of what outcome can be expected form cg. Algorithms we record
numeral exam. We examined [13] conjugate – gradient algorithm for the numeral solution of special systems of linear equations “those whose matrix symmetric and positive-definite”. This method is often performed “as an iterative algorithm, usable to sparse systems that are “too large to be treated by direct application. We performed j(j ≥ n) iterative Steps finding the best possible approach x_k, k ≤ j, between all computed vectors. “Next we computed the relative error of x_k and its residual vector. Define the number s such that”

\[ \frac{\|x_k - \alpha\|}{\|x_k\|} = T k^k \] ................................ (2.1.1)

Let s=1, which suggests the numerical stabilization of the algorithm. But, for maximum cases, s was around \( \frac{3}{2} \) and “the residual vector had a spectral norm of order” \( T_k \|A\|\|x_k\| \). Thus, the algorithm is “neither well behaved nor numerically stable”. A natural problem is to know why this is so and to request a cg. Algorithm which is numerically steady and well behaved.

To know why s = \( \frac{3}{2} \), is called that the cg. algorithms decrease the inaccuracy in the A-norm, \( \|A^{1/2}(x_k - \alpha)\| \) thus it seems normal to size the inaccuracy by \( \|A^{1/2}(x_k - \alpha)\| \) instead of by conjugate-gradient algorithms. \( \|x_k - \alpha\| \). Assume there occurs a numerically steady cg. Algorithm in the A-norm, i.e.,

\[ \|A^{1/2}(x_k - \alpha)\| = O(\delta k\|A^{1/2}x_k\|) \] .... (2.1.2)

A reminder that the condition number of A in the A-norm coincides “with the condition number of A in the spectral norm. Since”

\[ \|x_k - \alpha\| \leq \|A^{1/2}\|\|A^{1/2}(x_k - \alpha)\| \] ... (2.1.3)

Produce

\[ \|(x_k - \alpha)\| = O(T^{2/3}_k\|x_k\|) \] .......................... (2.1.4)

This clarifies why s = \( \frac{3}{2} \) might be expected in (2.1.1).

We have not been successful in analyzing traditional cg. Algorithms, included that were suggested by [13]. In this paper we suggest a recent class of cg. Algorithms and demonstrate that for these algorithms these occur a calculate vector x_k such that

\[ \|A^{1/2}(x_k - \alpha)\| \leq C T_k \|A^{1/2}\|\|x_k\| \] .......................... (2.1.5)

Where C is a constant of order at most n., we shall indicate this class of algorithms by \( \phi \) where \( \phi \) is a set. Note that K places linearly (2.1.4). In general, we cannot say that (2.1.4) means numerical stabilization of the cg. Algorithms is its "own"norm, since we have \( \|A^{1/2}\|\|x_k\| \) instead of \( \|A^{1/2}x_k\| \). However, if \( \|A^{1/2}\|\|x_k\| \) is of \( O \|A^{1/2}x_k\| \), thereafter these cg. “Algorithms are numerically stable in the A-norm. For the residual vectors we are only able to prove that”

\[ \|r_k\| \leq C T_k \|A\|\|x_k\| \] .......................... (2.1.6)

We examined one algorithm \( \varphi \) form \( \phi \). For extreme cases \( \varphi \) matches a well-behaved algorithm, i.e., \( \|r_k\| \) was of rank \( T \|A\|\|x_k\| \). But, in few condition, \( \|r_k\| \) was of rank \( T \|A\|\|x_k\| \). This evidence that (2.5) is sharp and some cg. Algorithms from \( \phi \) are not well behaved.

Much iterative algorithms have this characteristic, i.e., “they are numerically stable but not well behaved. Examples”. Include the chebychev, SOR, Richardson and Jacobi iterative algorithms [10,11]. However, it was shown in [7] “that any algorithm (direct or indirect) which calculate an approach y such as “ \( \|y - \alpha\| \leq q \|\alpha\| \) with \( q \leq 1 \) followed by iterative refinement in single precision becomes numerically stable, and if \( T k^2 \) is of rank unity then it is also well behaved.

2.2 Gradient and Conjugate Gradient Iterations

We briefly conclude some basal properties of the gradient and conjugate – gradient iterations. We look it the solution of major linear system

\[ Ax = b \] ................................................. (2.2.1)

Where A=A*≠ 0 “is an n×n hermitian and positive definite matrix and b is a n×1 given vector. Assume that the datum “absolute the matrix A is given by a step which calculate y=Ax for a given x. For large system A is commonly scattered, who declarations the valuation of y in time and storage proportional to n.
We resolve (2.2.1) iteratively by structuring a series \( \{x_k\} \) approximate to the solution

\[ \alpha = A^{-1}b. \]

Let \( B = B^* > 0 \) be a matrix which replace which \( A : BA = AB. \) For case one can set \( B = A^p \) for a real \( p. \) Let \( \|x\|_B = \sqrt{(Bx,x)} = \|B^{1/2}x\|, \) where \( \|x\| = \sqrt{(x,x)} \) is the spectral norm.

We recall the introduction of the gradient iteration which build the sequence \( \{x_k\} \) as follows. Lets \( x_0 \) be a given initial approach and

\[ x_{k+1} = x_k - c_k r_k, r_k = A x_k - b, \quad (2.2.2) \]

Where \( c_k \) is election in such a way that the inaccuracy \( \xi_{k+1} = \|x_{k+1} - \alpha\|_B \) is reduced, i.e., \( \|x_{k+1} - \alpha\|_B = \inf \|x_k - cr_k\|_B \)

This produce

\[ c_k = \frac{(r_k A (x_k - \alpha))}{(B_k r_k)} \quad (2.2.3) \]

Retrieval that for \( A = B, \) the iteration (2.2.2),(2.2.3) is called the steepest – descent iteration. It has, in generic, very slow gathering and thus is not “recommended” in the numeral exercise. The conjugate –gradient iteration is so more effective. Next derivation of the cg. Iteration concentrates on its complexity optimality. Gaze a class of iterations for which the error formula secures the connection

\[ x_k - \alpha = w_k(A)(x_0 - \alpha) \quad (2.2.4) \]

Wherever \( w_k \) is a polynomial of degree at most \( k \) and \( w_k(0) = 1. \) A normal complication request is how to select the polynomials \( w_k. \) “Since we want to minimize the computational complexity (cost)”, we seek \( w_k \) such that inaccuracy \( e_k = \|x_k - \alpha\|_B \) is reduced. This wherewithal that the polynomials \( w_k \) are the solution of the following problem:

\[ \|w_k(A)(x_0 - \alpha)\|_B = \inf_{p \in w_k(0,1)} \|p(A)(x_0 - \alpha)\|_B \quad (2.2.5) \]

Wherever \( w_k(0,1) \) is the class of polynomials of degree at most \( K \) equalizes to unity in origin. The resolution of (2.2.5) is specific by the orthogonal polynomials defined as follows.

Let \( x_0 - \alpha = \sum_{j=1}^{m} c_j \xi_j \quad (2.2.6) \)

Where \( \xi_j \) is an eigenvector of \( A \) associated with the eigenvalue \( \lambda_j: A \xi_j = \lambda_j \xi_j \)

\[ \|\xi_j\| = 1, \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m, \text{ with } m \leq n \]

and \( cj \neq 0 \) for \( j = 1, 2, \ldots, m. \) memo that \( \xi_j \) is as well an eigenvector of \( B: \xi_j = B_j \xi_j \) for \( B_j > 0, j = 1, 2, \ldots, m. \) know the inner product

\[ (f,g) = \sum_{j=1}^{m} \xi_j^2 B_j \xi_j \lambda_j g(\lambda_j), \quad (2.2.7) \]

Wherever \( f \) and \( g \) are function realize on the interval \([\lambda_1, \lambda_m]. \) The polynomials \( w_k, w_k(0) = 1, \) “which minimize” (2.2.6) are the orthogonal polynomials with respect to inner production (2.2.7), i.e.,

\[ (w_k, w_i) = \sum_{j=1}^{m} \xi_j^2 B_j \lambda_j w_k(\lambda_j) w_i(\lambda_i) = 0 \quad (2.2.8) \]

till \( k \neq i, \) from the orthogonally of \( w_k \) it follow up that they satisfy a three –term repetition formula. We select a different shape of the three-term repetition formula than usual in order to confirm the relationship between the cg. iteration and the gradient one. This shape is realized as follows:

\[ w_0(\lambda) \equiv 1, \quad w_1(\lambda) \equiv 1 - c_0 \lambda \quad (2.2.9) \]

\[ w_{k+1}(\lambda) = \left\{ w_k(\lambda) - c_k \lambda w_k(\lambda) - u_k \{ w_{k-1}(\lambda) - w_k(\lambda) + c_k \lambda w_k(\lambda) \} \right\} k \geq 1 \quad (2.2.10) \]

Where

\[ c_k = \frac{(w_{k+1}w_k)}{(Aw_kw_k)} \quad \mu_0 = 0, \quad \mu_k = \frac{(w_k - c_k \lambda w_k \xi_{k+1} - c_k w_k) + c_k w_k}{(w_{k-1}w_k + c_k \lambda w_k \xi_{k+1} - c_k w_k)}, \quad k \geq 1 \]

\[ w_k(0,1) \]

For that we bring the three –term repetition formula for the series \( \{x_k\}, \)

\[ x_k - c_k r_k = z_k, \quad A x_k - b = r_k \quad (2.2.12) \]

\[ z_k - \mu_k y_k = x_{k+1}, \quad x_{k+1} - z_k = y_k \]

\[ u_0 = 0, u_k = \frac{(y_k, B y_k)}{(y_k, B y_k)}, \quad k \geq 1 \]

2.3 Round off Error Analysis of Gradient Algorithms

We will presentation the round off-error test of cg. Algorithms relationship to \( \phi \) ability
be firstly established on the round off error analysis of the gradient algorithms to be thoughtful in this division. “Therefore in this section we analyze gradient algorithms in the presence of rounding errors”. We focus our attention the steepest descent algorithm (A=B) and recall the identical outcome for the gradient algorithms with B = 1 or A^2 = B

We gaze a steepest descent algorithm in floating-point binary arithmetical (fp) with the relative computer precision T = 2^-t, “where t is the number of mantissa bits”. To extend further rating we shall use the relationship, which is known as follows. Let f and h be two scalar functions defined on [0,1]. By f(t) = h(t) = 1 + (t), where |t| for 0 < t. By f(t) = h(t).

We pass f(T) = h(T) or f(T) = h(T)

The relation ≤ enable as to eliminate the terms of rank T^2 in the existence of the term of order T. allow r_k and x_k denote the vectors calculate in fp by an algorithms. We suppose that

\[ f(A x_k - b) = r_k \] .................................. (2.3.1)

**Lemma 2.3.1:** [2] Suppose that \( \|r_k\| > \|A\|\|x_k\|C_1 \forall k \). Then series \{x_k\} is calculated by algorithm accept the following inaccuracy formula:

\[ \sqrt{e_k^2 - c_k^2} + \|A^{1/2}\|\|x_k\| + T \cdot c_k^2 \cdot (\|A^{3/2}\|\|x_k\|5C_1 + \|A\|e_k(C_1 + 2C_2 + 8)) \geq e_{k+1} \] .................................................. (2.3.2)

Where C_1, C_2 constants

Where \( r_k = (r_k^*; r_k^*)/(r_k^*; A r_k^*) = A x_k - b, c_k \)

**Lemma 2.3.2:** [2] Show how the inaccuracy \( e_{k+1} \) relies on the theoretical and “rounding errors”. It is good to notice that the bound-on the rounding error raise which \( c_k^* \), whilst the bounded on the theoretical error decreases with increasing \( c_k^* \).

We need to discovering the limiting possession of the series \{e_k\} which satisfies (2.3.2), to attain this we use the next lemma.

**Lemma 2.3.3:** [2] allow

\[ \sqrt{e_k^2 - c_k^2} + ak + c_k b_k + c_k d \geq e_{k+1} \]

Till specified nonnegative sequence \{a_k\}, \{b_k\} and a constant d such that

\[ 2d\|A^{-1}\| < 1 \]. Thereafter

\[ \lim_{k} e_k \leq 3k \cdot \frac{\lim_{k} b_k + \lim_{k} a_k}{1 - 2d\|A\|} \]

**Theorm 2.3.4:** [2] if \( \beta = 2T_k(C_1 + 2C_2 + 8) < 1 \), then algorithm computes the seq\{x_k\} such that

\[ \lim_{k} \|A^{1/2}(x_k - \alpha)\| \leq \frac{3}{T_k} \frac{3(Tk C_1 + 1)}{1 - \beta} \|A^{1/2}\|\|x_k\| \]

**Corollary 2.3.5:** [2] brief the numerical estate of the “steepest-descent algorithm. It shows that the algorithm may be neither well behaved nor numerically stable”.

From the above lemmas we at once seal the asymptotic attitude of the series \{x_n\} calculated by algorithm (2.3.1).

3. Samples analysis

“Numerical tests confirm that the residual vectors sometimes depend” on \( T_k \). This means that algorithms are not well behaved. However, if \( \|A x_k - b\| \equiv \|A^{1/2}(x_k - \alpha)\|/\|A^{-1/2}\| \), then the residual vectors \( r^*_k \) rely at worst on \( T_k^{1/2} \).

“Numerical stability and/or the well-behaved property may be achieved by the use of iterative refinement” even if the residual are calculate in single accuracy. From theorems [7] it follows up that algorithm with iterative refinement in single accuracy is numerically stable whenever \( 1 > T k^{3/2} C \), and it is well behaved where \( Tk^2 \) is at maximum of rank unity.

4. Result and Discussion

The researchers progressing a new group of conjugate gradient process for unaffected optimization problems. A new non monotonous line search style is suggested to warranty the global concourse of those conjugate gradient methods down about reasonable case in specific, Lu-Storey & Polak–Ribiere conjugate gradient process are specific situation of the new class of conjugate gradient methods.
By considering the local lipschitz constant \[24\] of the derivative of the objective functions, we can discover and appropriate step size and basically reduces the function valuation at every iteration.

“Numerical results show that these new conjugate gradient methods are” active in reducing large –scale non – convex non – quadratic functions.

4.1 Numerical Examples

Example 4.1.1 [10]

Let \(x_k\) and \(x_{k-1}\) be the calculate vectors and \(r_k, r_{k-1}\) the corresponding residual vectors.

Let \(f \ell(Ar_k) = v_k\) be the calculate vector which is used for the calculation of \(c_k\).

We suggest the following algorithm for the calculation of \(\mu_k\). Let

\[ f \ell((y_k, r_k - c_k v_k)) = w_1, \]
\[ f \ell((y_k, r_{k-1} - r_k + c_k v_k)) = w_2. \]

So the counting of \(w_1\) and \(w_2\) does not require moreover matrix–vector multiplication. cyclic a portion of the analysis before it is likely to show that

\[ w_1 = (y_k, A(z_k - \alpha) + \delta w_1), \]
\[ |\delta w_1| \leq \delta \|A\| \|y_k\| \|x_k\| C_3, \]
\[ w_2 = (y_k, A y_k + \delta w_2), |\delta w_2| \leq \delta \|A\| \|y_k\| \|x_k\| C_4, \]

Where \(C_3 \equiv C_1 + 1\) and \(C_4 \equiv 2C_1 + 1\). From this we get

\[ \frac{w_1}{w_2} = \mu_k (1 + \delta \mu_k) \]
\[ |\delta \mu_k| \leq T \|A\| \|y_k\| \|x_k\| \left(\frac{C_3}{w_1} + \frac{C_4}{w_2}\right). \]

This suggest the following algorithm for the calculation of \(\bar{\mu}_k\),

\[ \bar{\mu}_k = \begin{cases} \frac{w_1}{w_2} & \text{if } T \|A\| \|y_k\| \|x_k\| \left(\frac{C_3}{w_1} + \frac{C_4}{w_2}\right) < 1 \\ 0 & \text{otherwise} \end{cases} \]

Hence, \(\bar{\mu}_k = \mu_k (1 + \delta \mu_k), |\delta \mu_k| \leq 1\) is content. Reminder that \(\bar{\mu}_k = 0\) means in order to \(z_k = x_{k+1} = f \ell(x_k - c_k r_k)\) is gained by one step of the steepest-descent algorithm.

“This can be interpreted as the initialization of the cg algorithm from the” vector \(x_k\).

It may also be observed while vector \(z_k\) and \(y_k\) need not be stocked. One step of the algorithm can be perfect having five vectors \(x_k, x_{k-1}, r_k, r_{k-1}\) and \(v_k = A r_k\) in storage and use two matrix–vector multiplications.

We have complete much numerical workout employ this algorithm. In almost condition the algorithm was well behaved in the spectral norm. Much, in a few situation (about 5 present) numerical tests experimentally certain the sharpness of the error bounds in Corollory (2.3.5).

Example 4.1.2: (Encyclopedia) gaze the linear system \(Ax=b\) given by

\[
Ax = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

We will execute the two steps of the conjugate gradient method beginning with the initial rating \(x_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\) “in order to find approximate solution to the system”. resolution to assurance the accurate solution is:

\[
X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 7 \\ 11 \\ 11 \end{bmatrix} \approx \begin{bmatrix} 0.0909 \\ 0.6364 \end{bmatrix}
\]

Ours first stage is to enumerate the residue vector \(r_0\) related with \(x_0\). This residue is calculate for the form: \(b - Ax_0 = r_0\) is equal to:

\[
r_0 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ -3 \end{bmatrix}
\]

Since that is the first iteration we would used the residue vector \(r_0\) as our initial search direction \(P_0\): the style of chosen \(p_k\) would alteration in otherwise iterations.

We now count the scalar \(a_0\) employ the related

\[
a_0 = \frac{r_0^T r_0}{p_0^T A p_0} = \frac{[-8 -3][-8 -3]}{[-8 -3][1 3][-8 -3]} = \frac{73}{331}
\]

We can now calculate \(x_1\) using the form:

\[
x_1 = x_0 + a_0 p_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{73}{331} \left[ \begin{bmatrix} -8 \\ -3 \end{bmatrix} \right] = \begin{bmatrix} 0.2356 \\ 0.3384 \end{bmatrix}
\]

That outcome complete the first iteration, the outcome existence an "amended" convergent solution to the system, \(x_1\). We may now proceed on and calculate the following residue vector \(r_1\) using the formula.
Our next step in the method is to calculate \( \beta_0 \) that will eventually be used to decide the next discussion direction \( p_1 \):

\[
\beta = \frac{r_1^T r_1}{r_0^T r_0} = \frac{[-0.2810, 0.7492]^T [ -0.2810, 0.7492 ]}{[-3, -3]^T [-3, -3]} = 0.0088
\]

Now using \( \beta_0 \). We can enumerate the next discussion direction \( p_1 \) using the form:

\[
p_1 = r_1 - \beta_0 p_0 = [-0.2810, 0.7492] + 0.0088 [-8, -3] = [-0.3511, 0.7229]
\]

We now enumerate the scalar \( \alpha_1 \) using our new gained \( p_1 \)

Using the same procedure as that used for \( \alpha_0 \)

\[
\alpha_1 = \frac{r_1^T r_1}{p_1^T A p_1} = \frac{[-0.2810, 0.7492]^T [-0.2810, 0.7492]}{[-0.2810, 0.7492]^T [-3, -3] [-3, -3]} = 0.4122
\]

Lastly, we discovery that \( x_2 \) utilizes the himself process as that applied to discovery \( x_1 \)

\[
x_2 = x_1 + \alpha_2 p_1 = [0.2356, 0.3384] + 0.4122 [-0.3511, 0.7229] = [0.0909, 0.6364]
\]

The outcome, \( x_2 \) is a “better” approach to the system solution than \( x_1 \) and \( x_0 \) If accurate arithmetical were to be applied in this example instead of limited-accuracy, then the precise solution would in theory be to attend next \( n=2 \) repetition (n being the order of the system).

**Example 4.1.3, [18]**

As a numerical example we gaze the well-known five-point difference approach of the Poisson equation with homogeneous “Dirichlet boundary conditions using an equidistant mesh of mesh width 1 in a rectangle having side lengths 12 and 6”. The system of difference equations” depends of \( n = 55 \) linear equations, the associated coefficient matrix \( A \) is a band matrix of band width 11. In addition”, \( A \) is an M-matrix so that \( A^{-1}, U^{-1} \) are nonnegative. Hence the condition numbers \( \sigma_D, \sigma_R \) of the unknown’s \( x_i \) can be specific simply as solutions of the linear systems

\[
A \sigma_D = \tau_D, \quad A \sigma_0 = \tau_0, \quad U \sigma_1 = X, \quad \sigma_R = \sigma_0 + \sigma_1,
\]

Where

\[
x_j = \sum_{k=1}^{n} |z_k^j| e_{jk} + \sum_{k=1}^{n} |\bar{u}_{jk} x_k| (j = 1, ..., n).
\]

“The matrix A and the right-hand side \( y=(1, ..., 1) \) of the system of difference equations were first multiplied by 0.9973 and then rounded symmetrically (B15) or truncated (C15), respectively, to 15 binary digits. Next, the solution vector of the linear system was computed. The arithmetic operations of the floating-point arithmetic were carried out in the form \( \text{fl}_N(a \circ b) \)” wherever \( \text{fl}_N \) denotes symmetrical round (BN) “or chopping (CN) to N binary digits and. a \circ b \) for \( \circ = +, - , \times , / \) is the result of about 9 decimal digits floating-point arithmetic of the desk top computer, the condition numbers were computed in the built-in floating-point arithmetic of the computer using the weights” \( \varepsilon_{\text{r}} \) specified by

\[
\bar{m}_{it} = 0: \varepsilon_t \bar{m}_{it} = 0 \quad \bar{m}_{it} = 0 \text{ or } u_{tk} = 0: \varepsilon_t \bar{m}_{it} = 0 \quad a_{ik}^{-t} = 0 \text{ or } \bar{m}_{it} = 0 \text{ or } \bar{u}_{ik} = 0: \varepsilon_t \bar{m}_{it} = 0.
\]

And in back substitution

\[
\bar{u}_{ik} = 0 \quad or \bar{x}_{k} = 0: \varepsilon_t \bar{m}_{it} = 0 \quad or \quad f(\bar{u}_{ik}, \bar{X}_{k}) = 0: \varepsilon_t \bar{m}_{it} = 0 \quad and \quad \varepsilon_t \bar{m}_{it} = 1 \text{ else.}
\]

Table (1) display a chain of relative data and rounding condition numbers the solutions \( x_i \) and the connected residual condition numbers together with the error and residual percentages

\[
P_i \% = 100 \left| \frac{p_i x_i}{\rho_i R_i} \right|, \quad Q_i \% = \left| \frac{(A \bar{E} - y_i)}{\tau_i} \right| \eta_R
\]

The “floating-point accuracy constant has the value”

\[
\eta_R = 2^{-N}, (CN): \eta_R = 2^{-N+1}
\]

It is clear from Table (1) that the error proportion of the floating-point arithmetic (C15) are safely greater than those of the arithmetic (B 15) for A is an M-matrix and the right-hand side \( y \geq 0 \) This reality has been explained before. The maximal relative error of the computed solutions is overestimated by a factor of about 11 and 8, respectively.
Five-point difference approximation of the Poisson equation, condition numbers of the solutions $x_i$, error and residual percentages.

<table>
<thead>
<tr>
<th>Float.pt</th>
<th>Arithmetic</th>
<th>$\rho_i^D$</th>
<th>$\rho_i^R$</th>
<th>$P_i^R$</th>
<th>$Q_i^R$</th>
<th>$\tau_i^D$</th>
<th>$\tau_i^R$</th>
<th>$P_i^%$</th>
<th>$Q_i^%$</th>
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<td>70</td>
<td>1</td>
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Furthermore see Ex [2,427-428].

5. Conclusion

We have shown in this paper that the relative error of computed vector $x_k$ by cg. Algorithm at worst on $T_k^{3/2}$ for the relative computer precision $T$. Moreover, for numerous functional situation the desired precision is great than $T_k^{3/2}$. This is a fully favorable outcome.

As we mentioned previously, our process is not achieved in analyzing traditional cg. Algorithms, but we proved that at fully several of them has identical numerical ownership.

We watch experimentally that the calculated series first approximate the exact solution of the system $\alpha$ at least as quick as the Chebyshev repetition.

Moreover, in many condition the error $\|x_k-\alpha\|_d$ is safely minimal than the over restricted.

Recommendation

A modern group of adaptive algorithms is suggested to be found “on a uniformly” spread strings arrangement by affect its gravity vector to affixed combine symmetric form. The process is used to the well-known reference signal based (RSB) and it will be former and the linearly forced lower variance (LCMV) ray previous as two application models. The action of the extra bonds is equal to gather a second step in the derived adaptive algorithm. Only, a difference grow for the RSB situation ago no “direction-of-arrival (“DOA) notification of the required indicative is obtainable, who leads to a two –stage frame for combine the enjoined bands. Match to the classic algorithms. The suggest one’s ability obtain a faster gathering speed and higher stable state produce signal – to- chaos-plus-noise ratio, give the himself step size.

Reference
