Compactness of Fuzzy Cone Metric Spaces

Fadhel S. Fadhel[1] and Amer A. Abdullah[2]
Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad-Iraq.
[1]E-mail: dr.fsf@sc.nahrainuniv.edu.iq
[2]E-mail: amernet91@gmail.com

Abstract

Cone metric spaces may be considered as a generalization of determinist metric spaces or fuzzy metric spaces. In this paper, we will construct fuzzy cone metric space using different approaches, which is based on the definition of fuzzy points, and then study different types of compact subsets of such spaces, as well as, the relationship among them.

Keywords: Cone Metric Spaces, Fuzzy Metric Spaces, Cone Fuzzy Metric Spaces, Compactness of Fuzzy Cone Metric Space.

1. Introduction

The concept of fuzzy set was introduced earliarly by L.A. Zadeh in 1965 as a tool to model problems with uncertain phenomenon's, [9]. After that, this concept has been used in topology and analysis by different authors and then expansively developed the theory of fuzzy sets and its application in different directions of pure and applied mathematics, such as dynamic system, linear algebra, differential equations, etc. On the other hand, there have been a number of generalizations of metric spaces and among such generalizations is the cone metric spaces which is introduced by H. Long-Guang in 2007, [4], as a generalization of the metric space.

In cone metric spaces, the authors replaced the set of real numbers by ordering Banach space. By using their concept, different authors established many results of cone metric spaces and fixed point theorems in such spaces. In an earlier article given by T. Bag in 2013, [7]. In which he introduced an idea of fuzzy cone metric spaces and established some basic results and fixed point theorems in such spaces.

In this paper, we will introduce the basic concepts of fuzzy points, and introduce the definition of fuzzy cone metric spaces with respect to fuzzy points as a modified direction. Finally, as the main results of this work study the compactness of fuzzy cone metric spaces, the study of the relationship between different types of compactness.

2. Preliminaries:

In this section, some basic concepts and results related to this work will be included.

Definition 2.1. :

[4]**.** Let E be alway a real Banach space and p a subset of E. P is called a cone if and only if:

1. P is closed, nonempty, and $p \neq \{0\}$;
2. $a, b \in R$, $a.b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
3. If $x \in P$ and $-x \in P \Rightarrow x = 0$.
   Given a cone.

$P \subseteq E$, we define a partial ordering with respect to p by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ that with $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}(P)$, $\text{Int}(P)$ denotes the interior of P, [4].

Definition 2.2. :

[4]**.** The cone P is called normal if there is a number $k > 0$ such that for $x, y \in P$, $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$.The least positive number satisfying above is called the normal constant of P.

Definition 2.3. :

[4]**.** The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$, [4].

In the next definition, the concept of cone metric spaces will be illustrated.

Definition 2.4. :

[4]. Let X be a non-empty set. Suppose the mapping $d: X \times X \to E$ satisfies:

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 2.5.:

[8]. Let $E = \mathbb{R}^2$, $P = \{(x.y) \in E: x.y \geq 0\}$, $X = \mathbb{R}$, where $\mathbb{R}$ is the field of real number, and $d: X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a constant.

Definition 2.6. :

[6]. Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$, $n \in \mathbb{N}$ a sequence in X. Then:

1. $\{x_n\}$, $n \in \mathbb{N}$ converges to x whenever for every $c \in E$ with $c \gg 0$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.We denoted this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$, $n \in \mathbb{N}$ is a Cauchy sequence whenever for every $c \in E$ with $c \gg 0$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
3. (X, d) is complete cone metric space if every Cauchy sequence is convergent.

Next, some fundamental concepts in fuzzy set theory are given for completeness purpose.

Definition 2.7. :

[9]. Let X be the universal set and $\widetilde{A}$ be any subset of X, then $\widetilde{A}$ is called fuzzy subset of X, which is characterized by a membership function $\mu_{\widetilde{A}} : X \longrightarrow [0, 1]$, i.e.,

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) \mid x \in X, 0 \leq \mu_{\widetilde{A}}(x) \leq 1\}.$$

Definition 2.8. :

[9]. Let X be the universal set, and $\widetilde{A}$ be fuzzy subset of X with membership function $\mu_{\widetilde{A}}$. A fuzzy point $\tilde{p}_x^{\lambda}$ (or fuzzy singleton) of a fuzzy set $\widetilde{A}$ is also a fuzzy subset of X, where $x \in X$ is the support of the fuzzy point, and $\lambda \in (0, 1]$ is the grade of this fuzzy point, with membership function:

$$\mu_{\tilde{p}_x^{\lambda}}(y) = \begin{cases} \lambda, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

and $\tilde{p}_x^{1-\lambda}$ is the complement fuzzy point of $\tilde{p}_x^{\lambda}$, which is also denoted by ${}^{c}\tilde{p}_x^{\lambda}$.

Definition 2.9. :

[10]. Let X* be the set of all fuzzy points in X, i.e., $X^* = \{\tilde{q}_x^{\lambda} : x \in X, \lambda \in (0, 1]\}$. A function $d^*: X^* \times X^* \longrightarrow [0, \infty)$ is called fuzzy distance function if d* satisfies the following conditions:

1. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0$ if and only if $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.
2. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = d^*({}^{c}\tilde{q}_{x_2}^{\lambda_2}, {}^{c}\tilde{q}_{x_1}^{\lambda_1})$.
3. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \leq d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$.
4. If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) < r$, where $r > 0$, then there exist $\lambda' > \lambda_1 > \lambda_2$, such that $d^*(\tilde{q}_x^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) < r$.

Then (X*, d*) is called fuzzy metric space.

Example 2.10. :

[10]. Let (X, d) be the universal metric space, and let $\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2} \in X^*$; and suppose d* be defined as follows:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + d(x_1, x_2)$$

where $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in (0, 1]$, then (X*, d*) is a fuzzy metric space.

Definition 2.11. :

[10]. Let (X*, d*) be fuzzy metric space, $\widetilde{A}$ be s subset of X*, and $\{\tilde{q}_{x_n}^{\lambda_n}\}$ be a sequence of fuzzy points in $\widetilde{A}$. Then

1. $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is said to be converge to $\tilde{q}_{x}^{\lambda}$ if for all $\alpha > 0$, there exist $N \in \mathbb{N}$, such that:
   $d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x}^{\lambda}) < \alpha$, for all $n \geq N$.
2. $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is said to be Cauchy sequence if for all $\alpha > 0$, there exist $N \in \mathbb{N}$, such that:
   $d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) < \alpha$, for all $n, m \geq N$.
3. $\tilde{A}$ is said to be complete if every Cauchy sequence is convergent.

3. Contraction of Fuzzy Cone Metric Spaces:

The definition with examples and some basic results of fuzzy cone metric space will be given in this section.

Definition 3.1.:

Let X* be the set of all fuzzy points in X, and let E be a Banach space and $P \subseteq E$ is cone. Then a function $d^*: X^* \times X^* \longrightarrow E$ is called fuzzy cone distance function if d* satisfies the following conditions:

1. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0$ if and only if $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.
2. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1})$.
3. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \leq d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$.
4. If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \ll r$, where $r \gg 0$, then there exist $\lambda' > \lambda_1 > \lambda_2$, such that $d^*(\tilde{q}_{x}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) \ll r$.

Also, (X*, d*) is called fuzzy cone metric space.

Example 3.2.:

Let (X, d) be the universal metric space, let $\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2} \in X^*$, and let $E = \mathbb{R}^2$, $p = \{(x, y) \in E : x, y \geq 0\}$, and suppose d* be defined as follows:

$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2))$.

where $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in (0, 1]$, then (X*, d*) is a fuzzy cone metric space.

Solution:

1. If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0 \Leftrightarrow (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2)) = (0,0) \Leftrightarrow \max\{\lambda_1 - \lambda_2, 0\} = 0$ and $d(x_1, x_2) = 0 \Leftrightarrow \lambda_1 \leq \lambda_2$, and $x_1 = x_2$.
2. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2))$, which implies to
   $(\max\{(1-\lambda_2) - (1-\lambda_1,) 0\}$,
   $d(x_2, x_1)) = d^*(\tilde{q}_{x_2}^{1-\lambda_2}, \tilde{q}_{x_1}^{1-\lambda_1}) =$
   $d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1})$.
3. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) = (\max\{\lambda_1 - \lambda_3, 0\}, d(x_1, x_3))$
   $= (\max\{(\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3), 0\}, d(x_1, x_3))$
   $\leq (\max\{(\lambda_1 - \lambda_2), 0\} + \max\{(\lambda_2 - \lambda_3), 0\}, d(x_1, x_2) + d(x_2, x_3))$
   $= (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2)) + (\max\{\lambda_2 - \lambda_3, 0\}, d(x_2, x_3))$
   $= d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$.
4. If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \ll r$, where $r \gg 0$ and $r = (r_1, r_2)$.

   Since $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \geq 0$, then $\lambda_1 - \lambda_2 \geq 0$, i.e., $\max\{\lambda_1 - \lambda_2, 0\} = \lambda_1 - \lambda_2 > 0$.

   Which implies to
   $(\lambda_1 - \lambda_2, 0) < d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) =$
   $(\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2))$
   $= (\lambda_1 - \lambda_2, d(x_1, x_2)) \ll r = (r_1, r_2)$.
   Hence, $(\lambda_1 - \lambda_2, 0) \ll (r_1, r_2)$, i.e.,
   $(r_1, r_2) - (\lambda_1 - \lambda_2, 0) \in \mathrm{Int}(P)$, then
   $(r_1 - (\lambda_1 - \lambda_2)) > 0$, and $r_2 > 0$.
   Hence, $\lambda_1 - \lambda_2 < r_1$, i.e., $\lambda_1 < r_1 + \lambda_2$.
   Now let $\lambda' \in (0, 1]$ be chosen such that
   $\lambda_2 < \lambda_1 < \lambda' < \min\{1, r_1 + \lambda_2\}$.

   Which implies:
   $\lambda' \leq r_1 + \lambda_2$, and so $0 < \lambda' - \lambda_2 < r_1$. i.e.
   $r_1 - (\lambda' - \lambda_2) > 0$.
   Then, $(r_1 - (\lambda' - \lambda_2), r_2) \in \mathrm{Int}(P)$.

Or equivalently, $(r_1,r_2) - (\lambda' - \lambda_2, 0) \in \text{Int}(P)$. i.e. $(\lambda' - \lambda_2, 0) \ll (r_1, r_2)$.

Now:

$d^*(\tilde{q}_{x_2}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) = (\max\{\lambda' - \lambda_2, 0\}, d(x_2, x_2))$

$= (\max\{\lambda' - \lambda_2, 0\}, 0)$

$= (\lambda' - \lambda_2, 0) \ll (r_1, r_2) = r$.

Therefore, $d^*(\tilde{q}_{x_2}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) \ll r$.

Then (X^*, d^*) is a fuzzy cone metric space.

Definition 3.3.:

Let (X^*, d^*) be fuzzy cone metric space, $\tilde{A}$ be s subset of X^*, and $\{\tilde{q}_{x_n}^{\lambda_n}\}$ be a sequence of fuzzy points in $\tilde{A}$. Then

1. $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is said to be converge to $\tilde{q}_x^{\lambda}$ if for all $\varepsilon \gg 0$, there exist $N \in \mathbb{N}$, such that: $d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x_n}^{\lambda_n}) \ll \varepsilon$, for all $n \geq N$.
2. $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is said to be Cauchy sequence if for all $\varepsilon \gg 0$, there exist $N \in \mathbb{N}$, such that: $d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) \ll \varepsilon$, for all $n, m \geq N$.
3. $\tilde{A}$ is said to be complete if every Cauchy sequence is convergent.

Definition 3.4. :

Let (X^*, d^*) be a fuzzy cone metric space, and let $\tilde{q}_x^{\lambda}$ be a fuzzy point in X^*, then the fuzzy neighborhood of a point $\tilde{q}_x^{\lambda}$ is the fuzzy set $U_\varepsilon(\tilde{q}_x^{\lambda})$ consisting of all points $\tilde{q}_{x'}^{\lambda'}$, such that $d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x'}^{\lambda'}) \ll \varepsilon$. The number ε is called the radius of $U_\varepsilon(\tilde{q}_x^{\lambda})$, and $\tilde{q}_x^{\lambda}$ is the center of the neighborhood, i.e.,

$$\tilde{U}_\varepsilon(\tilde{q}_x^{\lambda}) = \{\tilde{q}_{x'}^{\lambda'} \tilde{\in} X^* \mid d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x'}^{\lambda'}) \ll \varepsilon, \text{ where } \varepsilon > 0\}.$$

Main Result

In this section, some types of compactness of fuzzy cone metric space are introduced and some theoretical results are proved.

Definition 4.1. :

Let (X^*, d^*) be a fuzzy cone metric space, a fuzzy set $\tilde{A}$ in X^* is said to be fuzzy compact if for all $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$, there exists a finite subcover $\tilde{U}_i$, $i = 1, 2, \ldots, n$; of $\tilde{A}$, such that $\tilde{A} \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, where $\tilde{U}_i \tilde{\in} U_{\tilde{q}_x^{\lambda}}$, $\forall$ i; and $U_{\tilde{q}_x^{\lambda}}$ is the fuzzy neighborhood system of $\tilde{q}_x^{\lambda}$

Definition 4.2.:

Let $\tilde{A}$ be a fuzzy subset of a fuzzy cone metric space (X^*, d^*) and let $\varepsilon \gg 0$. A finite fuzzy set $\tilde{W}$ of fuzzy points:

$\tilde{W} = \{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$, $x_1, x_2, \ldots, x_n \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0, 1]$

is called an ε-fuzzy net for $\tilde{A}$ if for every fuzzy point $\tilde{p}_x^{\lambda} \tilde{\in} \tilde{A}$, there exists $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{W}$, for some $i \in \{1, 2, \ldots, n\}$; such that $d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x_i}^{\lambda_i}) \ll \varepsilon$.

Definition 4.3. :

A fuzzy set $\tilde{A}$ of a fuzzy cone metric space (X^*, d^*) is said to be fuzzy pre-compact (or fuzzy totally bounded) if $\tilde{A}$ possess an ε-fuzzy net, for every $\varepsilon \gg 0$.

The next theorem gives the first direction of the relationship between compact and pre-compact fuzzy cone metric spaces.

Theorem 4.4. :

If (X^*, d^*) is compact fuzzy cone metric space, then (X^*, d^*) is pre-compact fuzzy cone metric space and complete.

***Proof*:**

Let (X^*, d^*) be a compact fuzzy cone metric space (X^*, d^*). Let $\tilde{U}_i$ be an open cover with center $\tilde{q}_{x_i}^{\lambda_i}$ with radius $\varepsilon \gg 0$. Since X^* is compact, then there exist a finite subcover of X^*, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in} X^*$ implies $\tilde{q}_x^{\lambda} \tilde{\in}$

$\bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in}$ X* there exist $i_0 \in \{1, 2, \ldots, n\}$; such that $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{U}_\varepsilon(\tilde{q}_{x_{i_0}}^{\lambda_{i_0}})$.

Then the set $\{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ is form an ε-fuzzy net for X*, i.e., (X*, d*) is pre-compact fuzzy cone metric space.

Now to prove (X*, d*) is complete. Assume to contrary that there exists a Cauchy sequence $\{\tilde{q}_{x_n}^{\lambda_n}\}$ of fuzzy point is not convergent, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in}$ X* there exist ε ≫ 0, such that $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\in} \tilde{U}_\varepsilon(\tilde{q}_x^{\lambda})$, for finitely many n. From above assumption we get $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., there exist $i_0 \in \{1, 2, \ldots, n\}$; such that $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\in} \tilde{U}_\varepsilon(\tilde{q}_{x_{i_0}}^{\lambda_{i_0}})$, for infinitely many n. Which is contradiction. Then $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is convergent, i.e., (X*, d*) is complete. ■

The next definition is introducing the concept of sequentially compact.

Definition 4.5.:

A fuzzy subset $\tilde{A}$ of a fuzzy cone metric space (X*, d*) is said to be fuzzy sequentially compact if every sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\in} \tilde{A}$ has a convergent subsequence.

Theorem 4.6.

Let (X*, d*) be a fuzzy cone metric space, then (X*,d*) is a sequentially compact fuzzy cone metric space if and only if (X*, d*) is a complete and pre-compact fuzzy cone metric space.

***Proof*:**

If (X*,d*) is a sequentially compact fuzzy cone metric space. Assume to contrary that X* is not fuzzy pre-compact. Then, there exists ε ≫ 0, such that X* possess no finite ε-fuzzy net.

Take $\tilde{p}_x^{\lambda} \tilde{\in}$ X*, hence there exists $\tilde{q}_{x_{1_i}}^{\lambda_{1_i}} \tilde{\in}$ X*, such that $d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x_{1_i}}^{\lambda_{1_i}}) \gg \varepsilon$, otherwise $\{\tilde{q}_{x_{1_i}}^{\lambda_{1_i}}\}$ is an ε-fuzzy net. Also, there exists $\tilde{q}_{x_{2_i}}^{\lambda_{2_i}} \tilde{\in}$ X*, such that:

$d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x_{2_i}}^{\lambda_{2_i}}) \gg \varepsilon$ and $d^*(\tilde{q}_{x_{1_i}}^{\lambda_{1_i}}, \tilde{q}_{x_{2_i}}^{\lambda_{2_i}}) \gg \varepsilon$, otherwise $\{\tilde{q}_{x_{1_i}}^{\lambda_{1_i}}, \tilde{q}_{x_{2_i}}^{\lambda_{2_i}}\}$ is an ε-fuzzy net. So on, we get a sequence of fuzzy points $\{\tilde{q}_{x_{1_i}}^{\lambda_{1_i}}, \tilde{q}_{x_{2_i}}^{\lambda_{2_i}}, \ldots\}$, such that $d^*(\tilde{q}_{x_{k_i}}^{\lambda_{k_i}}, \tilde{q}_{x_{m_i}}^{\lambda_{m_i}}) \gg \varepsilon$, ∀ k ≠ m. Therefore, $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ does not a convergent fuzzy sequence of (X*, d*), i.e., (X*,d*) is not fuzzy sequentially cone compact metric space, which is a contradiction., i.e., (X*, d*) is fuzzy sequentially cone compact metric space, which implies that (X*, d*) is pre-compact fuzzy cone metric space.

Now to prove X* is complete. Let $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ be a Cauchy sequence in X*. Since X* is sequentially compact, then $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ has convergent subsequence $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$, with a limit point $\tilde{q}_x^{\lambda}$. Then $\tilde{q}_x^{\lambda}$ is also limit point for $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$, i.e., $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ is convergent, then X* is complete.

If (X*, d*) is a pre-compact fuzzy cone metric space, then X* possess an ε-fuzzy net, for every ε ≫ 0. Let $\tilde{W} = \{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ be an ε-fuzzy net for X*. Let $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ be a subsequence of $\{\tilde{q}_{x_n}^{\lambda_n}\}$, then there exist $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{W}$, such that $d^*(\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}, \tilde{q}_{x_i}^{\lambda_i}) \ll \varepsilon$, i.e.,

$\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ is form a Cauchy sequence, and since X* is complete, then $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ is convergent. Therefore, the sequence $\{\tilde{q}_{x_n}^{\lambda_n}\}$ has a convergent subsequence. Hence (X*, d*) is sequentially compact fuzzy cone metric space. ■

Lemma 4.7.:

If (X*, d*) is sequentially compact fuzzy cone metric space, and $\tilde{G}_\alpha, \forall \alpha \in A$ is infinitely open cover for X*. Then every ball of radius $\varepsilon \gg 0$ is contained in one of the open sets $\tilde{G}_\alpha$

Proof:

Assume for contrary that for any $n \in \mathbb{N}$ there is an open ball $\tilde{B}_n$ with center $\tilde{q}_x^\lambda$ and radius 1/n which is not contained in $\tilde{G}_\alpha, \forall \alpha \in A$. Since X* is sequentially compact, then every sequence in X* has a convergent subsequence. Therefore, the subsequence $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ of $\{\tilde{q}_{x_n}^{\lambda_n}\}$ in $\tilde{B}_n$ is convergent to $\tilde{q}_x^\lambda \tilde{\in}$ X*. Since $\tilde{G}_\alpha$ is an open cover for X* there exist an index $\alpha_0 \in A$, and an open set $\tilde{G}_{\alpha_0}$ such that $\tilde{q}_x^\lambda \tilde{\in} \tilde{G}_{\alpha_0}$. Since $\tilde{G}_{\alpha_0}$ is open and $\tilde{q}_x^\lambda \tilde{\in} \tilde{G}_{\alpha_0}$, then $\tilde{U}_\varepsilon(\tilde{q}_x^\lambda) \tilde{\subseteq} \tilde{G}_{\alpha_0}$, and since $\tilde{q}_x^\lambda$ is a limit point for a subsequence of the sequence $\{\tilde{q}_{x_n}^{\lambda_n}\}$, then $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\subseteq} \tilde{U}_\varepsilon(\tilde{q}_x^\lambda)$ for finitely many n. Which implies $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\subseteq} \tilde{G}_{\alpha_0}$, which is a contradiction, i.e., every ball is contained in one of the open sets $\tilde{G}_\alpha$, for some α. ■

Theorem 4.8:

Let (X*, d*) be a sequentially compact fuzzy cone metric space, then (X*, d*) is a compact fuzzy cone metric space.

***Proof*:**

Let $\tilde{G}_\alpha$ be infinitely open cover for X* and by using the above lemma, we have every open ball is contained in one of the open set $\tilde{G}_\alpha$. Since X* is sequentially compact then X* is a pre compact by using theorem (4.6). Then X* has an ε-fuzzy net, for each $\varepsilon \gg 0$. Let $\tilde{W} = \{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ be an ε-fuzzy net for X*, then each $\tilde{q}_x^\lambda \tilde{\in}$ X* belong to the union of ball $\tilde{U}_\varepsilon(\tilde{q}_{x_i}^{\lambda_i})$, i=1,2,…,n. Now each $\tilde{U}_\varepsilon(\tilde{q}_{x_i}^{\lambda_i})$ is contained in one of $\tilde{G}_\alpha$, say $\tilde{G}_{\alpha_i}$, i=1,2,…,n. Then the collection of $\tilde{G}_{\alpha_i}$, i=1,2,…,n is a finite sub cover for X*. Then X* is compact fuzzy. ■

The following theorem show that every pre-compact is compact.

Theorem 4.9:

Let (X*, d*) be a pre-compact fuzzy cone metric space and complete, then (X*, d*) is a compact fuzzy cone metric space.

***Proof*:**

If (X*, d*) be a pre-compact fuzzy cone metric space and complete, then (X*, d*) is a sequentially compact fuzzy cone metric space by using Theorem (4.6) and Theorem (4.8). which implies (X*, d*) is a compact fuzzy cone metric space. ■

Definition 4.10:

A fuzzy cone metric space (X*, d*) is said to be fuzzy countably compact if every open countably cover has finite subcover.

Theorem 4.11:

Let (X*, d*) be a compact fuzzy cone metric space, then (X*, d*) is countably compact fuzzy cone metric space.

***Proof*:**

Let $\tilde{G}_\alpha, \forall \alpha \in A$ be countably open cover for X*, and since X*, is compact, then $\tilde{G}_\alpha, \forall \alpha \in A$ has finite subcover which covering X*. Hence (X*, d*) is also fuzzy countably compact cone metric space. ■

Theorem 4.12 :

Let (X*, d*) be a fuzzy countably compact fuzzy cone metric space, then (X*, d*) is also fuzzy pre-compact cone space.

***Proof*:**

Let $\tilde{U}_i$ be a countable open cover for X*, with center $\tilde{q}_{x_i}^{\lambda_i}$ with radius $\varepsilon \gg 0$. Since X* is countably compact fuzzy, then there exist a finite subcover of X*, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in}$ X* implies $\tilde{q}_x^{\lambda} \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in}$ X* there exist $i_0 \in \{1, 2, \ldots, n\}$; such that $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{U}_\varepsilon(\tilde{q}_{x_{i_0}}^{\lambda_{i_0}})$. Then the set $\{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ is form an ε-fuzzy net for X*, i.e., (X*, d*) is pre-compact fuzzy cone metric space. ■

Theorem 4.13 :

If (X*, d*) is complete and pre-compact fuzzy countably compact fuzzy cone metric space, then (X*, d*) is also countably compact fuzzy cone space.

***Proof*:**

If (X*, d*) is complete and pre-compact fuzzy countably compact fuzzy cone metric space, then by using Theorem (4.9) and Theorem (4.11), which implies (X*, d*) is countably compact fuzzy cone space. ■

Theorem 4.14 :

Let (X*, d*) be a sequentially compact fuzzy cone metric space, then (X*, d*) is a countably compact fuzzy cone metric space.

***Proof*:**

Since (X*, d*) is sequentially compact fuzzy cone metric space, and by using Theorem (4.8) and Theorem (4.11), then (X*, d*) is a countably compact fuzzy cone metric space. ■

Definition 4.15. :

A fuzzy set $\tilde{A}$ of X* is said to be fuzzy locally cone compact if for all $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$, $x \in X$, $\lambda \in (0, 1]$ there exists a fuzzy neighborhood $\tilde{U}_\varepsilon(\tilde{q}_x^{\lambda})$ of $\tilde{q}_x^{\lambda}$, such that $\tilde{U}_\varepsilon(\tilde{q}_x^{\lambda})$ is fuzzy compact set.

Theorem 4.16 :

Every fuzzy compact cone metric space (X*, d*) is fuzzy locally compact cone metric space.

***Proof*:**

Since (X*, d*) is compact fuzzy cone metric space, then every cover $\tilde{U}_i$ has finite subcover, i.e., X* $\tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$. Now for each $\tilde{q}_x^{\lambda} \tilde{\in}$ X* implies $\tilde{U}_\varepsilon(\tilde{q}_x^{\lambda}) \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in}$ X* has compact $\tilde{U}_\varepsilon(\tilde{q}_x^{\lambda})$. (X*, d*) is locally compact fuzzy cone metric space. ■

Theorem 4.17 :

If (X*, d*) is pre-compact fuzzy cone metric space and complete, then (X*, d*) is locally compact fuzzy cone metric space.

***Proof*:**

If (X*, d*) is pre-compact fuzzy cone metric space, then by using theorem (4.9) and theorem (4.16), we have (X*, d*) is locally compact fuzzy cone metric space. ■

Theorem 4.18 :

If (X*, d*) is sequentially compact fuzzy cone metric space, then (X*, d*) is locally compact fuzzy cone metric space.

***Proof*:**

If (X*, d*) is sequentially compact fuzzy cone metric space, then by using Theorem (4.8) and Theorem (4.16), we have (X*, d*) is locally compact fuzzy cone metric space. ■

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الخلاصة

النظام المتري الكون يعتبر تعميم للنظام المتري العادي وكذلك للنظام المتري الضبابي. في هذا البحث سوف نقدم النظام المتري الضبابي المخروطي بواسطة تعريف النقاط الضبابية، وكذالك سندرس عدة انواع من التراص لهذا الفضاء، وكذالك دراسة العلاقات بينهم.