Efficiency of Monte Carlo Methods

By

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Keywords: - Hit or Miss method, Estimation, Confidence Interval, Sample Mean method, Efficiency

Abstract

In this paper, the Hit or Miss Monte Carlo method is reviewed. the number of repeated trials for evaluating an ordinary integral according to Hit or Miss Monte Carlo method is estimated. For sufficiently large number of trials, the 100(1- α)% confidence interval estimation for Hit or Miss method is approximated. Efficiency comparison is made between the Hit or Miss method with that of the Sample Mean Monte Carlo method.

Introduction

Throughout the literature, there are many deterministic quadrature formulas for computation of ordinary integrals with well behaved integrands such as Gauss quadrature, Trapezoidal, and Simpson Rules [1]. These rules become less attractive especially in the case of multidimensional integrals where the application of such rules lead to sever difficulties [2].

is often more convenient to compute such integrals by Monte Carlo method, which, although less accurate than conventional quadrature formulas, but it is mush simpler to use [1].

For computing one-dimensional integrals by a good numerical integration scheme is evidence, the Monte Carlo method is not competitive in this case, because each integral can be represented as an expected value (parameter) and estimating an integral by the Monte Carlo method is equivalent to estimating an unknown parameter.

The Hit or Miss Monte Carlo Method [3]

The method is based on the geometrical interpretation of an integral as an area. The technique is simple for computing one dimensional integrals.

$$\lambda = \int_a^b f(x)dx \quad \text{.................................(1)}$$

Assume that the integrand f(x) is bounded

$$0 \le f(x) \le c \qquad for \qquad a \le x \le b$$

Let X and Y be two independent random variables (r.v) uniformly distributed

U(a, b) and U(0, c) respectively with probability density functions (p.d.f.'s)

$$f_1(x) = \begin{cases} \dfrac{1}{b-a} & ,a \le x \le b \\ 0 & ,elsewhere \end{cases}$$

$$and \qquad f_2(y) = \begin{cases} \dfrac{1}{c} & ,0 \le y \le c \\ 0 & ,elsewhere \end{cases}$$

and cumulative distribution functions (c.d.f.'s)

$$F_1(x) = \frac{x-a}{b-a} \qquad ,a \le x \le b$$

$$and \qquad F_2(y) = \frac{y}{c} \qquad ,0 \le y \le c$$

Then the random vector (X,Y) define on the rectangle region

A ={(x,y) : a ≤ x ≤ b , 0 ≤ y ≤ c } (fig 1) with joint p.d.f.

$$f(x,y) = \frac{1}{(b-a)c} \qquad ,(x,y) \in A \quad \text{..........(2)}$$
$$= 0 \qquad ,elsewhere$$

Let p be the probability that the random vector (X, Y) falls within the area under the curve f(x). Define the subset S of **A** as S= {(x, y) : y ≤ f (x)}

ISSN:1814 5922

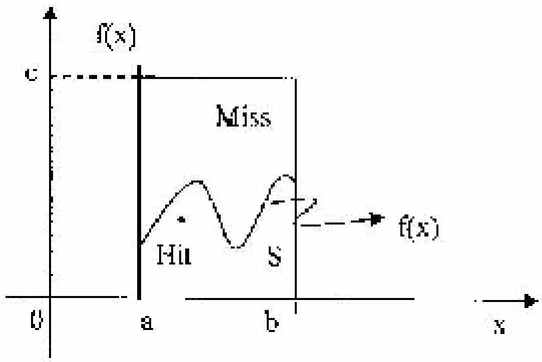

Fig (1): Graphical representation of the Hit or Miss Monte Carlo method

The area under the curve f(x)

Area $S = \int_a^b f(x)\,dx$, then

$$p = \frac{Area\,S}{Area\ \ A} = \frac{\int_a^b f(x)\,dx}{c(b-a)} = \frac{\lambda}{c(b-a)} \quad \ldots\ldots(3)$$

If N independent random vectors (X_1, Y_1), (X_2, Y_2)... (X_N, Y_N) are generated from the p.d.f. of equation (2), then the parameter þ can be estimated by

$$\hat{þ} = \frac{N_H}{N} \quad \ldots\ldots(4)$$

where N_H is the number of "hits" on which $f(X_i) \geq Y_i$, i = 1,2,3, ..., N

From equations (3) and (4), the integral λ can be estimated by

$$\hat{\lambda}_1 = c(b-a)\frac{N_H}{N} \quad \ldots\ldots(5)$$

That is, we estimate the integral λ of equation (1) by taking a sample of size N from a p.d.f. of equation (2) and counting the number N_H of hits (below the curve f(x)) and then applying equation (5).

The HM-Algorithm describes the necessary steps for estimating the integral λ of equation (1).

HM-Algorithm

1- Read a, b, c.

2- Generate a sequence $\{U_i\}$ of 2N uniform random numbers.

3- Arrange the uniform random numbers into N pairs $(U_1,U_1'),(U_2,U_2'),\ldots,(U_N,U_N')$ in any fashion such that each random numbers U_i is used exactly once.

4- Compute $X_i = a + (b-a)\,U_i$ and $f(X_i)$, i = 1,2, ..., N

5- Count the number of occasions N_H for which $f(X_i) \geq C\,U_i'$

6- Estimate the integral λ by $\hat{\lambda} = c\,(b-a)\,\frac{N_H}{N}$

Estimating the number of Trials

Since each Monte Carlo trial is classified into two categories "Hit" or "Miss" with probability þ of Hit, then the N trials constitutes a Bernoulli trials. Accordingly N_H can be considered as a r.v. represent the number Hits in the N repeated independent Bernoulli trials, then

$N_H \sim b(N, þ)$ with $E(N_H) = Nþ$ and $var(N_H) = Np(1-p)$

$$E(\hat{\lambda}_1) = \frac{c(b-a)}{N}E(N_H) = \frac{c(b-a)}{N}Nþ = pc(b-a) = \lambda \quad \ldots\ldots(6)$$

That is $\hat{\lambda}_1$ is an unbiased estimator of the integral λ

$$\delta_1^2 = var(\hat{\lambda}_1) = \left[\frac{c(b-a)}{N}\right]^2 Var(N_H) = \left[\frac{c(b-a)}{N}\right]^2 Nþ(1-p)$$

$$= \frac{1}{N}[c(b-a)]^2\,p\,(1-p) = \frac{\lambda}{N}[c(b-a)-\lambda] \quad \ldots\ldots(7)$$

and the standard deviation $\delta_1 = N^{-1/2}\,[c(b-a)\lambda]^{1/2}$

To estimate the number of needed trials to perform according to Hit or Miss Monte Carlo method, we utilize the Chebyshev's inequality

$$Pr(|\hat{\lambda}_1 - \lambda| < \epsilon) = Pr\left(|\hat{\lambda}_1 - \lambda| < \frac{\epsilon}{\delta_1}\cdot\delta_1\right) \geq 1 - \frac{\delta_1^2}{\epsilon^2} \geq \alpha \quad \text{Say} \ldots\ldots(8)$$

That implies $\epsilon^2(1-\alpha) \geq \delta_1^2$ ....(9)

From inequality (9) and equation (7) we have

$$\epsilon^2(1-\alpha) \geq \frac{1}{N}\,p\,(1-p)\,[c(b-a)]^2 \quad \ldots\ldots(10)$$

Solving inequality (10) with repeat to N, we have

$$N \geq \frac{p(1-p)[c(b-a)]^2}{(1-\alpha)\varepsilon^2} \quad \text{.....................(11)}$$

which is the required number of trials for inequality (8) to hold.

Confidence Interval Estimation

For sufficiently large number of trails N, then according to the central limit theorem, the r. v.

$\frac{\hat{\lambda}_1 - \lambda}{\sigma_1}$ is distributed approximately N(0,1)

so, we can find form N(0,1) table two number $\pm Z_{1-\frac{\alpha}{2}}$ such that

$$\Pr(-Z_{1-\frac{\alpha}{2}} < \frac{\hat{\lambda}_1 - \lambda}{\delta_1} < Z_{1-\frac{\alpha}{2}}) = 1-\alpha \quad \text{......(12)}$$

The event

$-Z_{1-\frac{\alpha}{2}} < \frac{\hat{\lambda}_1 - \lambda}{\delta_1} < Z_{1-\frac{\alpha}{2}}$ is equivalent to

$$\hat{\lambda}_1 - Z_{1-\frac{\alpha}{2}}\hat{\delta}_1 < \lambda < \hat{\lambda}_1 - Z_{1-\frac{\alpha}{2}}\hat{\delta}_1 \quad \text{.............(13)}$$

From (13) and (7), we have the 100(1-α)% confidence interval for λ is

$$\hat{\lambda}_1 - c(b-a)\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}Z_{1-\frac{\alpha}{2}} \quad ,$$

$$\hat{\lambda}_1 - c(b-a)Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}Z_{1-\frac{\alpha}{2}} \quad \text{...............(14)}$$

The sample means Monte Carlo method [4]

This method is based on the representation of integral as a mean value. To carry out the method for computing the integral of equation (1) is by rewriting the integral as an expected value of certain r.v. as

$$\lambda = \int_a^b \frac{f(x)}{g(x)} g(x)dx \quad \text{..............................(15)}$$

where g(x) is any p.d.f. defined at sample space Ω satisfying the well known conditions

(i) $g(x)>0$, $\forall x \in \Omega$

(ii) $\int_{x \in \Omega} g(x)dx = 1$

Then $\lambda = E\left[\frac{f(X)}{g(X)}\right]$ ....(16)

For simplicity, let the r.v X~U(a,b) where p.d.f.

$$g(x) = \begin{cases} \frac{1}{b-a} & , a \leq x \leq b \\ 0 & , elsewhere \end{cases}$$ and c.d.f

$$G(x) = \Pr(X \leq x) = \frac{x-a}{b-a} \quad \text{..............................(17)}$$

In this case

$$E[f(X)] = \int_a^b f(x)g(x)dx = \int_a^b f(x)\frac{1}{b-a}dx$$

$$= \frac{1}{b-a}\int_a^b f(x)dx = \frac{\lambda}{b-a}$$

that implies

$$\lambda = (b-a)E[f(X)] \quad \text{.......................(18)}$$

If N independent r.v $X_1, X_2, ..., X_N$ are generated from the p.d.f in the equation (17) then λ can be estimated by its sample mean

$$\hat{\lambda}_2 = (b-a)\frac{1}{N}\sum_{i=1}^{N} f(x_i) \quad \text{.....................(19)}$$

where $\hat{\lambda}_2$ is unbiased estimator of the integral λ, since

$$E(\hat{\lambda}_2) = (b-a)\frac{1}{N}\sum_{i=1}^{N} E[f(X_i)] =$$

$$(b-a)\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda}{b-a} = \frac{b-a}{N}\cdot\frac{N\lambda}{b-a} = \lambda$$

With variance of $\hat{\lambda}_2$

$$\delta_2^2 = \text{var}(\hat{\lambda}_2) = E(\hat{\lambda}_2^2) - \left[E(\hat{\lambda}_2)\right]^2 = E(\hat{\lambda}_2^2) - \lambda^2 \qquad (20)$$

where

$$E(\hat{\lambda}_2^2) = \left(\frac{b-a}{N}\right)^2 E\left[\left\{\sum_{i=1}^{N} f(X_i)\right\}^2\right] = \left(\frac{b-a}{N}\right)^2 E\left[\sum_{i=1}^{N} f^2(X_i) + \sum_{i \neq j}^{N-1}\sum^{N} f(X_i) f(X_j)\right]$$

$$= \left(\frac{b-a}{N}\right)^2 \left[\sum_{i=1}^{N} E[f^2(X_i)] + \sum\sum_{i \neq j} E[f(X_i)] E[f(X_j)]\right]$$

$$= \left(\frac{b-a}{N}\right)^2 \left[\frac{N}{b-a}\int_a^b f^2(x)dx + N(N-1)\frac{\lambda^2}{(b-a)^2}\right]$$

$$= \frac{b-a}{N}\left[\int_a^b f^2(x)dx + \frac{N-1}{b-a}\lambda^2\right] \qquad (21)$$

The equation (20) becom

$$\delta_2^2 = \text{var}(\hat{\lambda}_2) = \frac{b-a}{N}\int_a^b f^2(x)dx + \frac{N-1}{N}\lambda^2 - \lambda^2$$

$$= \frac{b-a}{N}\int_a^b f^2(x)dx - \frac{\lambda^2}{N}$$

$$= \frac{1}{N}\left[(b-a)\int_a^b f^2(x)dx - \lambda^2\right] \qquad (22)$$

The SM-Algorithm describes the necessary steps for estimating the integral λ of equation (1)

SM-Algorithm

1- Read a and b
2- Generate $U_1, U_2, U_3, \ldots, U_N$ from U (0,1)
3- Compute $X_i = a + (b-a)U_i$ and $g(X_i)$,i=1,2,…,N
4- Estimate the integral λ by $\hat{\lambda}_2 = \frac{b-a}{N}\sum_{i=1}^{N} g(X_i)$

Efficiency of the two Monte Carlo Methods

Assume that the two Monte Carlo methods (Hit or Miss and sample mean) for estimatery the integer λ are exist.

As shown earlier that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unbiased estimated estimators for integral λ of equation (1)

Then we say that the first method is more efficient than the second method if $\varepsilon = \frac{t_1 \text{var}(\hat{\lambda}_1)}{t_2 \text{var}(\hat{\lambda}_2)} < 1$, where t_1 and t_2 are respectively the units of computing time for evaluation $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

To compare the efficiency of the hit miss Monte Carlo methods with that of the sample mean Monte Carlo method we have to show

$$\text{var}(\hat{\lambda}_2) < \text{var}(\hat{\lambda}_1)$$

we subtract equation (7) from equation (20). we have

$$\text{var}(\hat{\lambda}_2) - \text{var}(\hat{\lambda}_1) = \frac{\lambda}{N}\left[c(b-a) - \lambda\right] - \frac{1}{N}\left[(b-a)\int_a^b f^2(x)dx - \lambda^2\right]$$

$$= \frac{b-a}{N}\left[\lambda c - \frac{\lambda^2}{b-a} - \int_a^b f^2(x)dx + \frac{\lambda^2}{b-a}\right]$$

$$= \frac{b-a}{N}\left[\lambda c - \int_a^b f^2(x)dx\right]$$

since $f(x) \le c$ then $\lambda c - \int_a^b g^2(x)dx \ge 0$

(23)

and that implies $\text{var}(\hat{\lambda}_1) - \text{var}(\hat{\lambda}_2) \ge 0$

if we assume that the computing time for $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are approximately equal , we conclude that the sample mean method is more efficient than the hit or miss method

Discussion

If we assume that f(x) is measured with some error, then we can observe instead of the f(x)

$f^*(x_i) = f(x_i) + e_i$ i=1, 2,…,N ,where the error $\{e_i\}$ are independent identically distributed r.v^s with zero mean and the same variance , with |e| <k<∞

Let (X, Y) be a random vector having the p.d.f.

$$f(x,y) = \begin{cases} \dfrac{1}{c_1(b-a)} & , a \le x \le b, 0 \le y \le c_1 \\ 0 & , elsewhere \end{cases}$$

where $c_1 \ge f(x) + k$

Again, we can show that an unbiased estimator for the hit or miss and the sample mean Monte Carlo methods are respectively

$$\lambda_1^* = c_1(b-a)\frac{N_H}{N} \qquad (24)$$

and

$$\lambda_2^* = \frac{1}{N}(b-a)\sum_{i=1}^{N} f^*(x_i) \qquad (25)$$

Where these estimators are converge almost surely in mean square to λ and that the sample mean method is again more efficient than the hit or miss method.

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المستخلص

في هذا البحث استعرض لطريقة مونت كارلو "الرفض أو القبول"، تخمين عدد المحاولات لحساب التكامل الاعتيادي حسب طريقة الرفض أو القبول، تقريب فترة الثقة التخمينية لطريقة الرفض والقبول عند مستوى أهمية (1-α)، مقارنة كفاءة طريقة الرفض والقبول مع طريقة معدل العينة.