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On Lie's Reduction Theorem with an Application to Isentropic Fluid Spheres

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Abstract

Assuming that, the interior of the star is filled up with perfect fluid the corresponding Einstein's equations one represented ordinary differential equation of 2nd order involving a parameter K. Different value of K are responsible for different models of the fluid sphere.

There exist many papers [2], [3], [4], [5] and [6] study this problem by seek a series solution $\left(y = \sum A_k z^k\right)$ and found special cases for parameter K include particular solutions with physical analysis.

In the present paper, the general solution (optimal solution) have been found by use similarity method of differential equations and specially Lie's reduction theorem. the solution that we obtained was in terms of special functions, namely the confluent hypergeometric functions.

1. Introduction

Vaidya P.C and Tikekar have utilized the space-time with hypersurfaces (t=const.) as spheroids to describe the gravitational field inside the superdense stars [2]. Physically the interior of stars is a perfect fluid and mathematically the Einstein's equation reduces to an ordinary differential equation of 2nd order which involves a parameter K. Vaidya analysed the model for K = -2.

Ramesh Tikekar (1990) discussed the model corresponding to K = -7 [3].

In 1996 Maharaj and P.G.L. Leach presented a new class of algebric solution for all the negative integral value of K [4].

Gupta and Jasim (2000, 2003) have obtained the most exact general solution with some restricted conditions [6], [7].

In the present article the authors have obtained a new general solution (optimal solution) by using the very powerful technique; similarity transformation with help of Matlab V.6 .

2. Basic equations

The Vaidya – Tikeker space – time have hypersurfaces t= constant as 3 – spheroid and can be

expressed by the metric

$$ds^2 = -\left[1 - \frac{kr^2}{R^2}\right]\left[1 - \frac{r^2}{R^2}\right]^{-1} dr^2 - \qquad \ldots\ldots\ldots\ldots\ldots(2.1)$$

$$r^2\left(d\theta^2 + \sin^2\theta d\phi^2 - y^2(r)dt^2\right)$$

where $k = 1 - \frac{b^2}{R^2}$, the metric (2.1) is regular and positive definite at all points $r^2 < R^2$ and its describes the fluid filled star the Einstein's field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi\left[(\rho + P)v_i v_j - Pg_{ij}\right] \ldots(2.2)$$

i = 1,2,3

j = 1,2,3

Supply the following expression for every density ρ, Pressure P, flow vector V^i with isotropy conditions $\left(T_1^1 = T_2^2 = T_3^3\right)$

$$8\pi\rho = \frac{3(1-k)}{R^2}\left(1 - \frac{kr^2}{3R^2}\right)\left(1 - \frac{kr^2}{R^2}\right)^{-2} \ldots(2.3)$$

$$8\pi P = \left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{kr^2}{R^2}\right)^{-1}\left(\frac{2y'}{ry} + \frac{1}{r^2}\right) \ldots(2.4)$$

$$v^i = \left(0, 0, 0, y^{-1}\right) \ldots\ldots\ldots\ldots\ldots\ldots(2.5)$$

and the isotropy condition

$$y''\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{kr^2}{R^2}\right) - \frac{1}{r}y'$$
$$\left[1 - k + k(1-k) - \frac{(1-k)k}{R^2}r^2 y\right] = 0 \qquad \ldots(2.6)$$

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with $c = \frac{k}{R^2}$ and the equation (2.6)

Assume the form

$$\left(1 - x^2\right)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + (1-k)y = 0 \quad (2.7)$$

for $k < 0$ or $k > 1$ with

$$x = \sqrt{\frac{k}{k-1}}\sqrt{1 - \frac{r^2}{R^2}}$$

Once we solve the linear equation (2.7) the expression for pressure and flow vector can be obtained easily .

Vaidya–Tikekar have solved the equation (2.7) for k = – 2 and . Tikekar discussed the problem with k = – 7 .

In the next section the authors obtained a new general solution (optimal solution) of equation (2.7) and construct a most general model. The method start with the usual symmetry solution but a useful manipulation sends the whole solution into closed form with help of Matlab V.6.

3. Mathematical Formulation

We consider a one – parameter Lie group of transformation

$$\left.\begin{aligned} x^* &= x + \varepsilon X(x,y) + O(\varepsilon^2) \\ y^* &= y + \varepsilon Y(x,y) + O(\varepsilon^2) \end{aligned}\right\} \quad (3.1)$$

Then the extended transformation , will be

$$\left.\begin{aligned} P^* &= P + \zeta(x,y;p) + O(\varepsilon^2) \\ q^* &= q + \xi(x,y;p,q) + O(\varepsilon^2) \end{aligned}\right\} \quad (3.2)$$

Where

$$P = \frac{dy}{dx}, q = \frac{d^2y}{dx^2}, P^* = \frac{dy^*}{dx^*}, q^* = \frac{d^2y^*}{dx^{*2}} \quad (3.3)$$

We derive ζ and ξ as follows ;

$$\left.\begin{aligned} \zeta &= Y_x + PY_y - PX_x - P^2X_y \\ \xi &= Y_{xx} + 2PY_{xy} + P^2Y_{yy} - PX_{xx} \\ &\quad -2P^2X_{xy} - P^3X_{yy} - qX_x - qPX_y \end{aligned}\right\} \quad (3.4)$$

The infinitesimal generators $D^{(1)}$ and $D^{(2)}$ are :

$$D^{(1)} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p} \quad (3.5)$$

$$D^{(2)} = D^{(1)} + \xi\frac{\partial}{\partial q} \quad (3.6)$$

4. Group – Invariant solution

If we write equation (2.7) in the primed form and substituted from equations (3.1) and (3.2) for the primed variables and simplification, we find equating coefficients of $p^n q^m$ give the infinitesimal elements (X,Y) leaving invariant the equation (2.7) .

Now, we find the determining equations for X and Y as follows:

$$q + \begin{bmatrix} Y_{xx} - 2pY_{xy} + p^2Y_{yy} - pX_{xx} \\ -2p^2Y_{xy} - p^3X_{yy} - qX_x - qpX_y \end{bmatrix}\varepsilon - O(\varepsilon^2)$$
$$-qx^2 - x^2Y_{xx}\varepsilon - 2px^2Y_{xy}\varepsilon - p^2x^2Y_{yy}\varepsilon +$$
$$x^2pX_{xx}\varepsilon + x^2p^2X_{xy}\varepsilon + 2x^2p^3X_{yy}\varepsilon$$
$$+ x^2qpX_y\varepsilon + x^2qX_x\varepsilon + O(\varepsilon^2) - 2xqX\varepsilon +$$
$$O(\varepsilon^2) + xp + xY_x\varepsilon - xpX_x\varepsilon - xp^2X_y\varepsilon$$
$$+ O(\varepsilon^2) + pX\varepsilon + O(\varepsilon^2) + y + Y\varepsilon - ky - kY\varepsilon -$$
$$O(\varepsilon^2) = 0 \quad (4.1)$$

we find a primary set of determining equarions is the following :

Monomails	Coefficient
p	$2Y_{yx} - X_{xx} - 2x^2Y_{yx} + x^2X_{xx} - xX_x - X = 0$ ...... (4.2)
p^2	$Y_{yy} - 2X_{xy} - x^2Y_{yy} - 2x^2X_{xy} - xX_y = 0$ ...... (4.3)
p^3	$-X_{yy} - x^2X_{yy} = 0$ ...... (4.4)
q	$-X_x + x^2X_x - 2xX = 0$ ...... (4.5)
qp	$-X_y + x^2X_y = 0$ ...... (4.6)

and we found terms for ε empty from the derivatives in extended equation (4.1)

$$Y_{xx} - x^2Y_{xx} + xY_x + Y - kY = 0$$

This implies

$$\left(1 - x^2\right)Y_{xx} + xY_x + (1-k)Y = 0 \quad (4.7)$$

Now, from equation (4.6)

$$\Rightarrow \left(x^2 - 1\right)X_y = 0 \Rightarrow X_y = 0$$

$$\therefore X(x,y) = \alpha + H(x)$$

$$X_x = H'(x) \quad (4.8)$$

substitute (4.8) in equation (4.5), we have

$$H(x) = \frac{x^2 - 1}{x^2 - 1} = 1$$

we know $X(x) = \frac{x^2 - 1}{2x} H'(x)$ in equation (4.5), this implies $X(x,y) = zero$

Now, rewrite equation (4.1) using the result obtained above, the second set of the determining equations is the following:

Monomails	Coefficient
p	$2Y_{yx} - 2x^2 Y_{yx} = 0$ .... (4.9)
p^2	$Y_{yy} - x^2 Y_{yy}$ .... (4.10)

From equation (4.10), we have $(1 - x^2)Y_{yy} = 0 \Rightarrow Y_{yy} = 0$ ....(4.11)

This implies to $Y(x,y) = c_1 y + R(x)$

But we find $c_1 = 0$ from equation (4.11)

$\therefore\ Y(x,y) = R(x)$

i.e. $Y(x,y)$ is a function of x only we find the second order differential equation (2.7) is invariant to the twice – extended group

$$x^* = x$$

$$y^* = y + Y(x)\varepsilon + O(\varepsilon^2)$$

Where

$$Y(x) = c_1(x^2-1)^{\gamma-1} P_1\left(\sqrt{2-k} - \frac{1}{2}, \frac{3}{2}; x\right) + c_2(x^2-1)^{\gamma+1} P_1\left(\sqrt{2-k} - \frac{1}{2}, \frac{3}{2}; x\right)$$

by Matlabe program.

In [5], [8] the series of confluent hypergeometric

$${}_1P_1(a, c; x) = \lim_{b\to\infty} {}_2P_1\left(a, b, c, \frac{x}{b}\right) = 1 + \frac{a}{1!c}x + \frac{a(a+1)}{c(c+1)2!}x^2 + \cdots$$

is well defined provided c is not negative integer, and convergefor |x|<1 and any solutions in other regions are obtainable by analytic continuation of these solution

therefore we find

$P\left(\frac{3}{2}, \frac{3}{2}; x\right)$ when $k = -2$

$P\left(\frac{5}{2}, \frac{3}{2}; x\right)$ when $k = -7$

$P\left(\frac{7}{2}, \frac{3}{2}; x\right)$ when $k = -14$

$\vdots \qquad \vdots$

Now, there exist a simple relations between the confluent hypergeometric with different parameters by ± 1 [8].

In functions (which obtained it ,we find $\frac{3}{2} = c$ is constant value, but a variable value dependent of value k such that

$$(c-a)\,{}_1P_1(a, c; x) - (c-a)\,{}_1P_1(a-1, c; x) - a(1-x)\,{}_1P_1(a-1, c; x)$$

This implies

$$-\,{}_1P_1\left(\frac{5}{2}, \frac{3}{2}; x\right) - {}_1P_1\left(\frac{3}{2}, \frac{3}{2}; x\right) - \frac{5}{2}(1-x)\,{}_1P_1\left(\frac{7}{2}, \frac{3}{2}; x\right)$$

when $a = 5/2$; $c = 3/2$, $k = -7$

$${}_1P_1\left(\frac{3}{2}, \frac{3}{2}; x\right) = {}_1P_1\left(\frac{5}{2}, \frac{3}{2}; x\right) - \frac{5}{2}(1-x)\,{}_1P_1\left(\frac{7}{2}, \frac{3}{2}; x\right)$$

as the same as when $a = 7/2$ and $c = 3/2$, $k = -14$

therefore, we consequently

${}_1P_1\left(\frac{3}{2}, \frac{3}{2}; x\right) = e^x$ by using Matlabe program V.6.

5. Lie's Reduction Theorem [4]

Let the general form of Ordinary Differential Equation of the second order is

$$w(x, y, \dot{y}, \ddot{y}) = 0 \quad \text{..................} (5.1)$$

We always be written as a pair coupled DE of the first order as follow

$$\left.\begin{aligned} \dot{y} &= u \\ w(x, y, u, \dot{u}) &= 0 \end{aligned}\right\} \text{.........................(5.2)}$$

Equation (5.2) determine a two – parameter family of curves in 3–dimensional space. Equation (5.2) are invariant to the once–extended group (X, Y, ζ) the transformations of the group carry each of these curves into other curves of the family.

Each one – parameter family of curves defines a surface in (x,y,u) – space and denoted by equation $\phi(x, y, u, c) = 0$ ---- (5.3)

Equation (5.3) is invariant

i.e. $0 = \phi(x', y', u', c) = \phi(X, Y, U; c)$.......(5.4)

and satisfy $X\phi_x + Y\phi_y + \zeta\phi_u = 0$ ....(5.5)

The characteristics equations of which are

$$\frac{dx}{X(x,y)}=\frac{dy}{Y(x,y)}=\frac{du}{\zeta(x,y,\dot{y})}\quad \ldots\ldots\ldots(5.6)$$

If $p(x,y)$ and $q(x,y)$ are two integrals of equation (5.4) the general solution for ϕ is an arbitrary function G of P and q , the function $P(x,y)$ being an integral of the first pair of equation (5.4) is a group invariant , the function $q(x,y,u)=q(x,y,\dot{y})$ which is an invariant of the once extended group called a first differential invariant .

((if we adopt the invariant P and first differential invariant q as new variables , the second – order differential equation $w(x,y,\dot{y},\ddot{y})=0$ will reduce to a first order differential equation in P and q)) .

6. Application of Lie's Reduction Theorem

If we write a differential equation (2.7) in the form $u(x,y,\dot{y},\ddot{y})=0$ where

$$u(x,y,\dot{y},\ddot{y})=\ddot{y}+\frac{x}{1-x^2}\dot{y}+\frac{1-k}{1-x^2}y=0$$

then u satisfy the condition $Xu_x+Yu_y+\zeta u_{\dot{y}}+\xi u_{\ddot{y}}=0$

The finitesimal coefficient of the group are :

$X(x,y)=0$, $Y(x,y)=Y(x)$, $\zeta=\dot{y}(x)$ and $\xi=\ddot{y}(x)$

direct substitution now shows that

$$Xu_x+Yu_y+\zeta u_{\dot{y}}+\xi u_{\ddot{y}}=\ddot{Y}+P(x)\dot{Y}+Q(x)Y=0 \quad \text{as required}$$

Now, by using the characteristic equation of which are

$$\frac{dx}{X(x,y)}=\frac{dy}{y(x)}=\frac{d\dot{y}}{\dot{Y}(x)}\Rightarrow\frac{dx}{0}=\frac{dy}{Y(x)}=\frac{d\dot{y}}{\dot{Y}(x)}$$

from the first equality that

$\frac{2}{5}xe^{x}\left(x^2-1\right)^{3/4}+\frac{3}{5}e^{x}(-1)^{3/4}=a$ where a constant , x is hypergeometric $\left\{\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2}\right],x^2\right\}$ and

$Y(x)=\left(x^2-1\right)^{3/4}e^{x}$

The greater generality of solution in terms of hypergeometric functions outweights tractability of the solution presented [4] .

Substituting this value of x in the second and third term tells us that we can treat Y and $\dot{Y}$ as a constant when integrating the second equality

$$\frac{dy}{Y(x)}=\frac{d\dot{y}}{\dot{Y}(x)}\Rightarrow\frac{dy}{c-\frac{3}{2x}e^{x}(-1)^{3/4}}=\frac{d\dot{y}}{c+\frac{3}{2}xe^{x}\left(x^2-1\right)^{-1/4}}$$

$$\int\left[c-\frac{3}{2x}e^{x}(-1)^{3/4}\right]d\dot{y}=\int\left[c+\frac{3}{2}xe^{x}\left(x^2-1\right)^{-1/4}\right]dy$$

$$\Rightarrow\left[c-\frac{3}{2x}e^{x}(-1)^{3/4}\right]\dot{y}-\left[c+\frac{3}{2}xe^{x}\left(x^2-1\right)^{-1/4}\right]y$$

This implies $Y(x)\dot{y}-y\dot{Y}=b$ where b is a second constant

Thus $\frac{2}{5}xe^{x}\left(x^2-1\right)^{3/4}+\frac{3}{5}e^{x}(-1)^{3/4}=p$ is an invariant ------(6.1)

and $\left(c-\frac{3}{2x}e^{x}(-1)^{3/4}\right)\dot{y}-\left(c-\frac{3}{2}xe^{x}\left(x^2-1\right)^{-1/4}\right)y=q$ is a first differential equation ------- (6.2)

Differentiating we find

$$\frac{dq}{dx}=\ddot{y}Y-y\ddot{Y}=R(x)Y(x)-P(x)q$$

$$\therefore\frac{dq}{dx}=R(x)Y(x)-P(x)q\quad\ldots\ldots\ldots\text{(Linear equation)}$$

$$\Rightarrow q=D\sqrt{x^2-1}\quad\text{-------}(6.3)$$

by substitute (6.2) in (6.3) we have

$$\dot{y}-\frac{\dot{Y}}{Y}y=\frac{D\sqrt{x^2-1}}{Y}\quad\ldots\ldots\ldots\text{(Linear equation)}$$

This implies

$$y=D\begin{bmatrix}\frac{1}{2}e^{x/2}\left(x^2-1\right)^{3/4}E_i(1,2x+2)\\ \frac{1}{2}e^{-x/2}\left(x^2-1\right)^{3/4}E_i(1,2x-2)\end{bmatrix}$$

Such that $E_i(x)$ is called (Cauchy Newton function) and defined by

$$E_i(x)=\int_{\infty}^{x}\frac{e^t}{t}dt$$

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المستخلص

افترضنا أن كرات المائع والممثلة بمائع تام قد مثلت بمعادلة تفاضلية من الرتبة الثانية والتي يلعب بها الوسيط K دور كبير في تحديد النموذج .

لقد سبقنا الكثيرون [7] ، [6] ، [4] ، [3] ، [2] في دراسة هذا النموذج باستخدام الحل المتسلسل الذي انتج حالات خاصة للوسيط (k=-14 , k=-7, k=-2) k...) ضمن حلول خاصة للمسألة مع تحليل فيزياوي لتلك الحالات .

في بحثنا هذا تناولنا هذا النموذج مستخدمين طريقة تماثل المعادلات التفاضلية وعلى وجه الخصوص نظرية لي للاختزال وأثبتنا كفاءة هذه الطريقة بحصولنا على الحل الأمثل والأعم بالاعتماد على الدوال فوق الهندسية الاندماجية التي بدورها أعطتنا جميع الحالات الخاصة وبينت أن قيمة الوسيط k = -2 تولد كافة الحالات الأخرى (k=-14 , k=-7,...) وتعطي الحل العام (الأمثل) للمسألة .