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Numerical Solution of Variable Delay Differential Equations Using Runge-Kutta Methods

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Abstract

In this paper, we study one of the most important types of functional differential equations, namely, Variable Delay Differential Equations in which the delay term appears as a function of time t, and we shall also use important methods in numerical analysis, namely, Runge-Kutta methods to approximate the variable delay differential equation. Finally, a comparison is made between the results obtained by Runge-Kutta method and the analytic method.

Introduction

Among the most important branches of ordinary differential equations is the so called delay differential equations, which are ordinary differential equations that consist of one or some of its arguments evaluated at a time which is differ by any fixed number or variable that is called the time delay or the time lag.

Delay differential equations are used to describe many phenomenon of physical interest. Delay differential equations contain in addition derivatives which depend on the solution at previous times, such type of equations arise in models throughout the sciences, such as mathematical modeling of population growth and in many problems of control theory, [Driver, 1977].

Delay differential equations are of sufficient importance in modeling real-life problems to merit the attention of numerical analysis, in which delay differential equations arise in many areas of mathematical modeling, for example, population dynamics (taking into account the gestation times), infections diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling, for example, the body's reaction to CO_2, etc. in circulating blood) and chemical kinetics (such as mixing reactants), the navigational control of ships and air crafts (with respectively large and short lags) and more general control problems, [Reinhold K., 2002].

In recent years, the theory of this class of equations had become an independent trend and the literatures on this subject comprise over 1000 titles.Also, theory of delay differential equations can be considered as a general-ization to the theory of ordinary differential equations and hence numerical methods are necessary in some cases in which the theoretical solution is not available, [Stark, 1970].

Exhibition to Delay Differential Equations:

In physics and other engineering subjects, the rate of change of a process y(t) often does not depend only on the value of y(t) at time t, but also on the values of the process in the past [Reinhold, 2002].We can define delay differential equation "DDE" to be an equation in an unknown function y(t) and some of its derivatives, evaluated at arguments that differ by any variable function with respect to time t such as $\tau_1(t), \tau_2(t), \ldots, \tau_n(t)$. The general form of the n^{th} order DDE is given by:

$$F(t; y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_k(t)), y'(t-\tau_1(t)), \ldots, y'(t-\tau_1(t)), \ldots, y^{(n)}(t-\tau_k(t))) = g(t) \quad \ldots\ldots (1)$$

where f is a given function and $\tau_1(t)$, $\tau_2(t)$, …, $\tau_k(t)$ are given functions called the "time delay", [Bellman, 1963].

In some literatures, equation (1) is called differential equation with deviating arguments, or an equation with time lag or a differential difference equation, or functional differential equation. Also, equation (1) is called homogenous DDE in case of g(t) = 0, otherwise this equation is called non-homogenous DDE, [Driver, 1977], [El'sgolte, 1964], [Namik, 1966].

A remarkable phenomena of great importance is that, the ODE could be considered as

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a special case of DDE with $\tau_1(t), \tau_2(t), \ldots, \tau_n(t) = 0$. The emphasis will be, in general, on the linear equations with constant coefficient of the first order and with one delay (because as it is in ordinary differential equation "ODE's" any higher order differential equation could be transformed easily to a linear system of the first order). Such form of a first order DDE with one delay is given as:

$$a_0 y'(t) + a_1 y'(t - \tau(t)) + b_0 y(t) - b_1 y(t - \tau(t)) = g(t) \qquad (2)$$

where $g(t)$ is a given continuous function, $\tau(t)$ is a function with respect to time t and a_0, a_1, b_0 and b_1 are constants.

Three types of delay differential equations arise, namely retarded, neutral and mixed (sometimes called of advanced type). The first type means an equation where the rate of change of state is determined by the present and past states of the equations, for example equation (2) where $a_0 \neq 0$ and $a_1 = 0$, if the rate of change of state depends on its own past values as well, the equation is then of neutral type for example equation (2) where $a_0 \neq 0$ and $a_1 \neq 0$, $b_1 = 0$, the third type is a combination of the two type, [Al-Saddy, 2000].

It is important to remark that, the theory of neutral and mixed differential equations is more advanced than this of retarded type, and in most real life applications, the differential equation with retarded arguments is the most encountered one. The main difficulty between delay differential equations is the kind of initial conditions that should be used, which are differ from ordinary differential equations, so that one should specify in delay differential equations the initial condition on some interval of length $\tau^* = \max_{t \in D} \tau(t)$, where D is the domain of definite of time t, say $[t_0 - \tau(t), t_0]$ and then try to find the solution of equation (2) for all $t \geq t_0$. Thus, we set $y(t) = \phi_0$ for all $t_0 - \tau^* \leq t \leq t_0$, where $\phi_0(t)$ is some given continuous extension of $\phi_0(t)$ into a function $y(t)$ which satisfy equation (2) for all $t \geq t_0$, [Halanay, 1966].

Solution of Delay Differential Equations

The best well known theoretical method for solving delay differential equations is the method of steps or the method of successive integrations which is given for certain time step with step size (say $\tau^* = \max_{t \in D} \tau(t)$), we must find this solution for all $t > t_0$ divided into these steps with length $\tau(t)$ also. [Nadia, 2001], [Al Saady, 2000].

For simplicity and illustration purpose, consider DDE's of the form:

$$y'(t) = f(t, y(t), y(t-\tau(t)), y'(t-\tau(t))), \; t \in D$$

with initial condition $y(t) = \phi_0(t)$, $t \in [t_0 - \tau^*, t_0]$, and $\tau(t)$, $\phi_0(t)$ are given condition. For such equation, the solution is constructed step by step as follows:

One can form the solution in the next time step interval $[t_0, t_0 + \tau^*]$ by solving the following equation:

$$y'(t) = f(t, y(t), \phi(t - \tau(t)), \phi'(t - \tau(t))),$$
$$\text{for } t_0 \leq t \leq t_0 + \tau$$

with initial condition $y(t_0) = \phi(t_0)$. Suppose that $\phi_1(t)$ is the desired first step solution, which exists by virtue of continuity hypotheses.

Similarly, if $\phi_1(t)$ is defined on the whole segment $[t_0, t_0 + \tau^*]$, hence by forming the new equation,

$$y'(t) = f(t, y(t), \phi_1(t - \tau(t)), \phi'_1(t - \tau(t))),$$
$$\text{for } t_0 + \tau(t) \leq t \leq t_0 + 2\tau^*$$

One can find the solution $\phi_2(t)$ with the initial condition $y(t_0 + \tau^*) = \phi(t + \tau^*)$. In general, by assuming that $\phi_{k-1}(t)$ is defined on the interval $[t_0 + (k-2)\tau^*, t_0 + (k-1)\tau^*]$, the following equation constructed:

$$y'(t) = f(t, y(t), \phi_{k-1}(t-\tau(t)), \phi'_{k-1}(t-\tau(t)))$$

for $t_0 + (k-1)\tau^* \leq t \leq t_0 k\tau^*$, and considering the solution of this equation with the initial condition

$$y(t_0 + (k-1)\tau^*) = \phi_{k-1}(t_0 + (k-1)\tau^*)$$

which is denoted by $\phi_k(t)$, [Driver, 1977], [Al-Saady, 2000].

Numerical Solution of Delay Differential Equations

When the analytical solution goes for each further step, the procedure of solution will be more complicated till a very complex situation is encountered when t is much greater than t_0. Thus, analytical solution in this cases irreliable and one should consult certain numerical technique to solve such delay differential equations.

In many real life problems, DDE's of the form.

$$y'(t) = f(t, y(t), y(t - \tau(t)), y'(t - \tau(t)))$$

is too complicated to be solved, or when $\phi_k(t)$ for the successive steps becomes more and more complicated such that the next step solution $\phi_{k+1}(t)$ could not be evaluated.

Therefore, the need for numerical methods is necessary. Due to these problems and because of the importance of the delay differential equations in which there is a few methods for solving them numerically; a modified approach for solving DDE's is required. This approach is the Runge-Kutta methods.

Formulation of Runge-Kutta methods for DDE's

For simplicity purpose, consider the delay differential equation with variable delay τ(t), given by:

$$y'(t) = f(t, y(t), y(t-\tau(t))), \ t \in [t_0, T]$$

where, $y(t) = \varphi_0(t)$, $x_0 - \tau(t) \le t \le t_0$, $T \subset R^+$.

By replacing the general form r-stages Runge-Kutta methods for solving delay differential equations which is given by:

$$y_{n+1} = y_n + h\sum_{i=1}^{r} c_i k_i$$

Where:

$$k_i = f(t_n + ha_i, \ y_n + h\sum_{j=1}^{r} b_{ij}k_j, y_{n-\tau(t_n)} + h\sum_{j=1}^{r} b_{ij}k_j)$$

and

$$a_i = \sum_{j=1}^{r} b_{ij}$$

where c_i, a_i and b_{ij}, for all $i, j = 1, 2, \ldots, r$; are constants to be determined.

For convenience, we design the process by an array of constants, and classify Runge-Kutta methods as follows.

$$\begin{array}{cccc|c} b_{11} & b_{12} & \cdots & b_{1j} & a_1 \\ b_{21} & b_{22} & \cdots & b_{2j} & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & a_i \\ \hline c_1 & c_2 & \cdots & c_j & \end{array}$$

and it is easy to classify Runge-Kutta methods, as follows:

- If $b_{ij} = 0, \forall i < j$, then the method is called semi – explicit.
- If $b_{ij} = 0, \forall i \le j$, then the method is called explicit.
- Otherwise it is called implicit.

Interpolation and Extrapolation Methods, [Stark, 1970]

Suppose we are given a table of values such that:

x	f(x)
x_0	$f(x_0)$
x_1	$f(x_1)$
⋮	⋮
x_n	$f(x_n)$

Thus, for n + 1 different values of x, not necessary equally spaced, the corresponding values of f(x) are given. We assume here that both the x_i and the corresponding $f(x_i)$ are given either exactly, or within some specified accuracy. Figure (1) shows the function $f(x)$ and the corresponding tabulated values, which are shown as a heavy black point on the curve.

Now, let us suppose that we wish to know $f(\overline{x})$, where $\overline{x}$ is, for the sake of discussion, somewhere between x_3 and x_4, as shown in figure (1). To use linear interpolation, we draw a straight line between two tabulated points, one on each side (with some loss of accuracy, we can use two points both of which were at one side of the desired points; in this case, the process would be called extrapolation). Of the unknown point $\overline{x}$, just in this case, draw a straight line AD between the tabulated points x_3 and x_4.

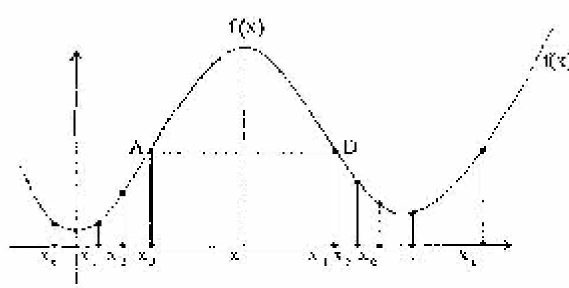

Figure (1).

Having drawn this line, as in figure (1), we now approximate the curve in the region between, in this case, x_3 and x_4 by the straight line, which is shown magnified in figure (2) using similar approach, we form the proportion:

$$\frac{BC}{AC} = \frac{DE}{AE}$$

which we can solve for BC, to give:

$$BC = \frac{AC}{AE} DE$$
$$= \frac{\bar{x} - x_3}{x_4 - x_3}\left[f(x_4) - f(x_3)\right]$$

our resulting interpolated value for $f(\bar{x})$ is then

$$p(\bar{x}) = f(\bar{x})_{int}$$
$$= f(x_3) + \frac{\bar{x} - x_3}{x_4 - x_3}\left[f(x_4) - f(x_3)\right],$$

where $p(\bar{x})$ is the interpolation approximation to $f(\bar{x})$.

In general, suppose we wish to find the value of $f(x)$ for some x located between x_i and x_{i+1} in the table of given values; then the interpolated value $p(x)$, which is only an approximation for $f(x)$, is given by:

$$p(x) = f(x_i) + \frac{x - x_i}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right]$$

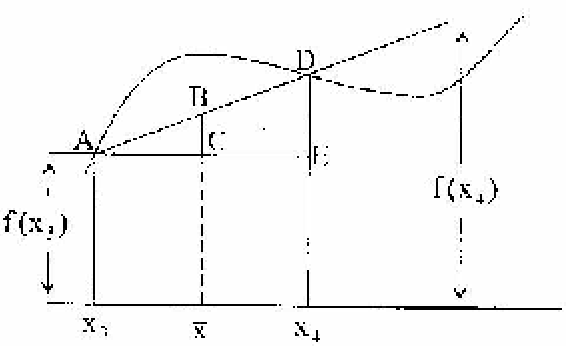

Figure (2).

The next example illustrate the procedure used in solving delay differential equations with variable delay using Runge-Kutta methods.

Illustrative Example

Suppose as an example, we want to solve the following differential equation with small delay

$$y'(t) = 1 - y(e^{1-\frac{1}{t}}), \quad t \geq 0$$

where

$$y(t) = \ln(t), \quad 0 < t \leq 0.1$$

hence using the 4^{th} order explicit Runge-Kutta method defined by:

$$y_{n+1} = y_n + h\sum_{i=1}^{4} c_i k_i, \quad n = 0,1,\ldots,N$$

where:

$$k_i = f\left(t_n + h a_i,\ y_n + h\sum_{j=1}^{r} b_{ij} k_j,\ y_{n-\tau(t_n)} + h\sum_{j=1}^{r} b_{ij} k_j\right)$$

and

$$a_i = \sum_{j=1}^{4} b_{ij}$$

where c_i, a_i, b_{ij}, for all i, j = 1,2,3,4 and N is the number of subdivisions of each subinterval.

Therefore, to the above considered problem, we have:

$$k_1 = f(t_n, y_n, y_{n-\tau})$$
$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1, y_{n-\tau} + \frac{h}{2}k_1\right)$$
$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2, y_{n-\tau} + \frac{h}{2}k_2\right)$$
$$k_4 = f(t_n + h, y_n + hk_3, y_{n-\tau} + hk_3)$$

with step length h = 0.1 and initial condition $y(0.1) = \ln(0.1) = 2.3.25$ and upon using the interpolation and extrapolation methods discussed in section (6) depending on the restarded value of y obtained from Runge-Kutta method to be either interpolation if the retardation lies between two known solutions, or extrapolation if retardation lies outside an interval of known solution, see Figure (3).

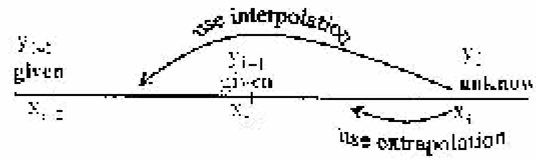

Figure (3).

The obtained results and its comparison with the exact solution are given in Table (1) and Figure (4).

Table (1).

t_i	***Exact Solution***	***Numerical solution***
0.2	- 1.6094	- 1.6103
0.3	- 1.20397	- 1.206342
0.4	- 0.91629	- 0.878359
0.5	- 0.693147	- 0.656251
0.6	- 0.510825	- 0.475311
0.7	- 0.356674	- 0.324967
0.8	- 0.223143	- 0.192333
0.9	- 0.105360	- 0.075469
1.0	0	- 0.029277

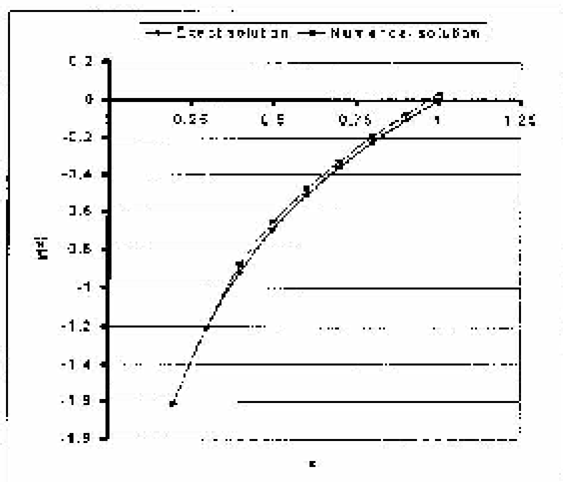

Figure (4)

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المستخلص

في هذا البحث قمنا بدراسة واحدة من أنواع المعادلات التفاضلية، ألا وهي المعادلات التفاضلية التباطؤية ذات التباطؤ المتغير في الزمن t. حيث قمنا بحل هذا النوع من المعادلات عددياً باستخدام طرائق رانك كوتا لحل المعادلات التفاضلية، حيث تم مقارنة النتائج من خلال المقارنة بين الحل العددي والحل التحليلي للمسألة.