# Posinormal Operators and Weyl's Theorem

**Buthainah A. H. Ahmed, Shaima Shawket Kadhim**
University of Baghdad - College of Science - Mathematics Department

## Abstract

An operator $A$ on a Hilbert space $H$ is a posinormal operator if there exists an interrupter $P \in B(H)$ such that $AA^* = A^*PA$.

Let $P(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is not posinormal$\}$, A is called totally posinormal if $P(A) \neq \phi$. In this paper we study some properties of posinormal operator and the set $P(A)$ and show that Weyl's theorem holds for a totally posinormal operator.

**Proposition 1.1**

If A and $A^*$ are unitarily equivalent on $B(H)$ then $P(A) = P(A^*)$.

**Proof:**

Let A and $A^*$ be unitarily equivalent then $A = U^*A^*U$ for a unitary operator U. Now if $A^*$ is posinormal, then A is posinormal [10]. If A is unitarily equivalent to $A^*$, then $A - \lambda I$ is unitarily equivalent to $(A - \lambda I)^*$ for all $\lambda \in \mathbb{C}$

$$A - \lambda I = U^*A^*U - \lambda U^*U = U^*(A^* - \lambda I)U$$

it follows that $\lambda \notin P(A^*)$ iff $A - \lambda I$ is a posinormal operator. Thus $P(A) = P(A^*)$.□

The following theorem describes some properties of the eigenspace of posinormal operators.

**Theorem 1.2**

Let $A \in B(H)$ where $\lambda$ and $\mu \notin P(A)$, then

1- $N_A(\lambda) \subseteq N_{A^*}(\bar{\lambda})$

2- $N_A(\lambda)$ reduces A.

3- $N_A(\lambda) \perp N_A(\mu)$ whenever $\lambda \neq \mu$.

**Proof:**

1- Let $x \in N_A(\lambda)$. Since $\lambda \notin P(A)$, then $(A - \lambda I)$ is posinormal and so $N(A - \lambda I) \subseteq N(A - \lambda I)^*$ [10], hence $x \in N_{A^*}(\bar{\lambda})$.

2- To prove that $N_A(\lambda)$ reduces A, it is enough to show that $(N_A(\lambda))^{\perp}$ is an invariant subspace of H under A. Let $x \in N_A(\lambda)^{\perp}$, it follows that $<x, y> = 0$ for all $y \in N_A(\lambda)$. But $N_A(\lambda) \subseteq N_{A^*}(\bar{\lambda})$ from [1], so $y \in N_{A^*}(\bar{\lambda})$. Now for all $y \in N_A(\lambda)$,

$<Ax, y> = <x, A^*y> = <x, \bar{\lambda}y> = \lambda <x, y> = 0$.

Thus, $Ax \in N_A(\lambda)^{\perp}$. Hence $N_A(\lambda)$ reduces A.

3- Let $x \in N_A(\lambda)$ and $y \in N_A(\mu)$, then $y \in N_{A^*}(\bar{\lambda})$. Thus, $\lambda <x, y> = <\lambda x, y> = <Ax, y> = <x, A^*y> = <x, \bar{\mu}y>$. Therefore $(\lambda - \mu) <x, y> = 0$. But $\lambda \neq \mu$, hence $<x, y> = 0$ and $N_A(\lambda) \perp N_A(\mu)$.□

**Theorem 1.3**

Let $A \in B(H)$ and $\lambda \notin P(A)$, then $A|N_A(\lambda)$ is normal.

**Proof:**

Let $x \in N_A(\lambda)$. From part (1) of theorem (1.2) we have

$A^*Ax = A^*(Ax) = A^*(\lambda x) = \lambda(A^*x) = |\lambda|^2 x$.

On the other hand,

$AA^*x = A(A^*x) = A(\bar{\lambda}x) = \bar{\lambda}(Ax) = |\lambda|^2 x$.

Therefore $AA^*x = A^*Ax$ for all $x \in N_A(\lambda)$, hence $A|N_A(\lambda)$ is normal.□

Recall that $\sigma_p(A) = \{\lambda \in \mathbb{C} : Ax = \lambda x\}$

**Theorem 1.4**

Let $A \in B(H)$ and $\lambda \notin P(A)$, then $\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ reduces A, and $A\Big|\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ is normal.

**Proof:**

We prove first that $\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ reduces A. let $x \in \overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$, then $x = \sum_i x_{\lambda_i}$ such that $\sum \|x_{\lambda_i}\|^2 < \infty$ where $x_{\lambda_i} \in N_A(\lambda_i)$, $i = 1,2,\ldots$

$$A(Ax) = A\left(\sum_i Ax_{\lambda_i}\right)$$
$$= A\left(\lambda_1 x_{\lambda_1} + \lambda_2 x_{\lambda_2} + \lambda_3 x_{\lambda_3} + \cdots\right)$$
$$= \lambda_1\left(Ax_{\lambda_1}\right) + \lambda_2\left(Ax_{\lambda_2}\right) + \lambda_3\left(Ax_{\lambda_3}\right) + \cdots$$

it follows that $Ax \in \overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ and $\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ is invariant under A.

Now

$$A(A^*x) = A\left(\sum_i A^*x_{\lambda_i}\right)$$
$$= A\left(A^*x_{\lambda_1} + A^*x_{\lambda_2} + A^*x_{\lambda_3} + \cdots\right)$$

since $\lambda \notin P(A)$, then $N_A(\lambda) \subseteq N_{A^*}(\bar{\lambda})$, so

$$A\left(A^*x\right) = A\left(\overline{\lambda}_1 x_{\lambda_1} + \overline{\lambda}_2 x_{\lambda_2} + \cdots\right)$$
$$= \lambda_1(A^*x_{\lambda_1}) + \lambda_2(A^*x_{\lambda_2}) + \cdots$$

therefore $\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ is invariant under $A^*$.

Now to prove $A\Big|\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ is normal. let $x \in \overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$, then $x = \sum_{i=1}^{\infty} x_{\lambda_i}$. By part (1) of theorem (1.2) we have

$$A^*Ax = A^*\left(\sum Ax_{\lambda_i}\right)$$
$$= A^*\left(Ax_{\lambda_1} + Ax_{\lambda_2} + \cdots\right)$$
$$= A^*\left(\lambda_1 x_{\lambda_1} + \lambda_2 x_{\lambda_2} + \cdots\right)$$
$$= \lambda_1\overline{\lambda}_1 x_{\lambda_1} + \lambda_2\overline{\lambda}_2 x_{\lambda_2} + \cdots$$
$$= |\lambda_1|^2 x_{\lambda_1} + |\lambda_2|^2 x_{\lambda_2} + \cdots$$
$$= \sum |\lambda_i|^2 x_{\lambda_i}$$

Using the same argument we can show that $AA^*x = \sum|\lambda_i|^2 x_{\lambda_i}$. Thus, $AA^*x = A^*Ax$ for all $x \in \overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$. Hence, $A\Big|\overline{\sum_{\lambda\in\sigma_p(A)} \oplus N_A(\lambda)}$ is normal.□

Recall that a family of closed subspaces of H is said to be total in case the only vector x in H, which is orthogonal to every subspace of H belonging to the family is $x = 0$ [5, P.167].

**Theorem 1.5**

Let A be a posinormal operator in $B(H)$. If the eigensubspaces of A form a total family, then A is normal.

**Proof:**

Let $H_0$ be the null space of $AA^* - A^*A$, the problem is to show that $H_0 = H$, i.e. $H_0^{\perp} = \{0\}$. Let $x \in N_A(\lambda)$, since A is posinormal, then $A^*x \in N_A(\lambda)$. Therefore,

$AA^*x = \lambda(A^*x) = A^*(\lambda x) = A^*Ax$. Thus, $\left(AA^* - A^*A\right)x = 0$, and therefore $x \in H_0$ and hence $N_A(\lambda) \subset H_0$ for all $\lambda$.□

§ 2 Weyl's Theorem

We show that Weyl's theorem holds for the class of totally posinormals. We define $\tilde{A} = |A|^{1/2} U |A|^{1/2}$, where $A = U|A|$ and $|A| = \left(AA^*\right)^{1/2}$, this definition appeared first in [1].

The following lemma which we needed is appeared in [8].

**Lemma 2.1:**

If $A \geq B \geq 0$ then for each $r \geq 0$

(i) $\left(B^{r/2}A^{p}B^{r/2}\right)^{1/q} \geq \left(B^{r/2}B^{p}B^{r/2}\right)^{1/q}$

and

(ii) $\left(A^{r/2}A^{p}A^{r/2}\right)^{1/q} \geq \left(A^{r/2}B^{p}A^{r/2}\right)^{1/q}$

holds for $P \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$

**Theorem 2.2**

Let A be posinormal. Then $\tilde{A} = |A|^{1/2}U|A|^{1/2}$ is hyponormal.

**Proof:**

From the posinormality of A we have $\lambda^2 A^*A \geq AA^*$ for some $\lambda > 0$ [10] and since the operator inequality $\lambda^2 U^*|A|^2 U \geq \lambda^2 |A|^2$ is satisfied [7]. Then $\lambda^2 U^*|A|^2 U \geq \lambda^2 |A|^2 \geq U|A|^2 U^*$. Let $S = \lambda^2 U^*|A|^2 U$, $T = \lambda^2 |A|^2$, and $V = U|A|^2 U^*$

Now,

$$(\tilde{A}^*\tilde{A}) = \left(|A|^{1/2}U^*|A|U|A|^{1/2}\right)$$

$$= \left(\left(\frac{1}{\lambda^2}T\right)^{1/4}\left(\frac{1}{\lambda^2}S\right)^{1/2}\left(\frac{1}{\lambda^2}T\right)^{1/4}\right)$$

$$\geq \left(\left(\frac{1}{\lambda^2}T\right)^{1/4}\left(\frac{1}{\lambda^2}T\right)^{1/2}\left(\frac{1}{\lambda^2}T\right)^{1/4}\right) \text{ [lemma 2.1.(a)]}$$

$$= \frac{1}{\lambda^2}T$$

$$(\tilde{A}\tilde{A}^*) = \left(|A|^{1/2}U^*|A|U|A|^{1/2}\right)$$

$$= \left(\frac{1}{\lambda^2}T\right)^{1/4}V^{1/2}\left(\frac{1}{\lambda^2}T\right)^{1/4}$$

$$\leq \left(\frac{1}{\lambda^2}T\right)^{1/4}\left(\frac{1}{\lambda^2}T\right)^{1/2}\left(\frac{1}{\lambda^2}T\right)^{1/4}$$

$$\leq \left(\frac{1}{\lambda^2}T\right)$$

Thus, $\tilde{A}^*\tilde{A} \geq \tilde{A}\tilde{A}^*$, it follows that $\tilde{A}$ is hyponormal.□

**Theorem 2.3**

Let A be totally posinormal. If $\lambda$ is an isolated point of $\sigma(A)$ then $\lambda \in \sigma_p(A)$ where $\sigma_p(A)$ denotes the point spectrum of A.

Before proving this theorem we need the following lemmas which appeared in [2].

**Lemma 2.4**

The spectrum of A, $\tilde{A}$ are identical, i.e. $\sigma(A) = \sigma(\tilde{A})$.

**Proof of theorem (2.3):**

Since $A + \lambda I$ is posinormal for all $\lambda \in \mathbb{C}$, we need only prove the case $\lambda = 0$. Let $0 \in \sigma(A)$, since $\sigma(A) = \sigma(\tilde{A})$ [Lemma 2.4] then $0 \in \sigma(\tilde{A})$ and 0 is an isolated point of $\sigma(\tilde{A})$ but $\tilde{A}$ is hyponormal then by [23, theorem 2] it follows that $x_0 \in H$ such that $\tilde{A}x_0 = 0$, $|A|^{1/2}U|A|^{1/2}x_0 = 0$, we have $U|A|^{1/2}x_0 \in N\left(|A|^{1/2}\right)$ but $N(A) \subset N(A^*)$ [20], it follows that $A^*(U|A|^{1/2}x_0) = |A|U^*U|A|^{1/2}x_0 = |A||A|^{1/2}x_0 = 0$. Hence $|A|^{1/2}x_0 = 0$ therefore we have $0 \in \sigma_p(A)$.□

**Proposition 2.5**

Let A be posinormal with $Ax = \lambda x$, $Ay = \mu y$, $\lambda \neq \mu$ and each of $\lambda$ and $\mu \notin P(A)$. Then $< x, y > = 0$.

**Proof:**

Since $\mu \notin P(A)$, then $A^*y = \bar{\mu}y$ [9], thus $\mu < x, y > = < x, A^*y > = < Ax, y > = \lambda < x, y >$. Since $\lambda \neq \mu$ then $< x, y > = 0$.□

Now before we give our main theorem we need the following lemma

**Lemma 2.6 [3, Lemma 3]:**

Let $A \in B(H)$, suppose A satisfied the following condition.

C 1: If $\{\lambda_n\}$ is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of A and $\{x_n\}$ is any sequence of the corresponding normalized eigenvalues then the sequence $\{x_n\}$ does not converge. Thus $\sigma(A) - \sigma_{00}(A) \subset \sigma_w(A)$.

**Theorem 2.7**

Let A be a totally posinormal operator. Then Weyl's theorem holds for A.

**Proof:**

By proposition (2.4), if A is a totally posinormal operator, then A satisfied Lemma (2.6)and hence $\sigma(A)-\sigma_{00}(A)\subseteq\sigma_w(A)$.

For the inclusion relation, $\sigma_w(A)\subseteq\sigma(A)-\sigma_{00}(A)$. Let $\lambda\in\sigma_{00}(A)$, since $A-\lambda I$ is posinormal, then we have $N(A-\lambda I)\subseteq N(A-\lambda I)^*=Ran(A-\lambda I)^{\perp}$, hence we have the following decomposition of $A-\lambda I$:

$$A-\lambda I=\begin{bmatrix}0 & 0\\ 0 & S\end{bmatrix}\quad\text{on}$$

$N(A-\lambda I)\oplus\overline{Ran(A-\lambda I)^*}$ . since $A=\begin{bmatrix}\lambda & 0\\ 0 & S+\lambda\end{bmatrix}$, where $S+\lambda$ is one - to - one and is totally posinormal operator on $\overline{Ran(A-\lambda I)^*}$. If $\lambda\in\sigma(S+\lambda)$, by theorem (2.2) we have $\lambda\in\sigma_p(S+\lambda)$ because $\lambda$ is an isolated point of $\sigma(S-\lambda)$. This is a contradiction. Hence $\lambda\notin\sigma(S+\lambda)$, therefore $0\neq\sigma(S)$.

Let $K=\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}$, then K is a compact operator and $A+K-\lambda I=\begin{bmatrix}I & 0\\ 0 & S\end{bmatrix}$ is an invertible operator. Therefore $\lambda\notin\sigma_w(A)$. Hence we have $\sigma_w(A)\subseteq\sigma(A)-\sigma_{00}(A)$ and the proof of the theorem is complete.□

**Proposition 2.8**

Let A be totally posinormal. Then there exists orthogonal reducing subspaces M and N for A such that $H=M\oplus N$ , $A\setminus M$ is normal on M and $\sigma_w(A\setminus N)=\sigma(A\setminus N)$.

Before we give the proof, we need to prove the following lemma.

**Lemma 2.9**

If A is totally posinormal and maps a subspace M into itself, then $A\big|_M$ is totally posinormal.

**Proof:**

Since A is totally posinormal, then $A-\lambda I$ is posinormal for all $\lambda\in\mathbb{C}$. Let $A_1-\lambda I$ be the restriction of $A-\lambda I$ to M for all $\lambda\in\mathbb{C}$. Then for $x,y\in M$

$$\begin{aligned}<x,(A-\lambda I)^*y> &=<(H-\lambda I)x,y>\\ &=<(A_1-\lambda I)x,y>\\ &=<x,(A-\lambda I)^*y>\quad\text{for all }\lambda\in\mathbb{C}\end{aligned}$$

In particular

$$\begin{aligned}\left|(A_1-\lambda I)^*x\right|^2 &=<(A_1-\lambda I)^*x,(A_1-\lambda I)^*x>\\ &=<(A_1-\lambda I)^*x,(A-\lambda I)^*x>\end{aligned}$$

Hence

$$\left\|(A_1-\lambda I)^*x\right\|^2\leq\left|(A_1-\lambda I)^*x\right|\left\|(A-\lambda I)^*x\right\|$$

, it follows that

$$\left|(A_1-\lambda I)^*x\right|^2\leq\left|(A-\lambda I)^*x\right|\leq\left\|\sqrt{P}\,(A-\lambda I)x\right\|=\left|\sqrt{P}\right|\left|(A_1-\lambda I)x\right|$$

for all $\lambda\in\mathbb{C}$ and $x\in M$ . Thus, $(A_1-\lambda I)$ is posinormal for all $\lambda\in\mathbb{C}$.

Therefore $A_1$ is totally posinormal.□

**Proof of Proposition (2.8):**

For $\lambda\in\sigma_p(A)$, let $M_\lambda=\{x:Ax=\lambda x\}$. Then $M_\lambda$ is a reducing subspace for A (theorem 1.2). Let $M=\bigoplus_{\lambda\in\sigma_p(A)}M_\lambda$ and $N=M^{\perp}$ Then M reduces A and $A\big|M$ is normal (theorem 1.3). Let $S=A\big|N$ then S is a totally posinormal operator on N by lemma (2.9). By theorem (2.7), Weyl's theorem holds for S. since $\sigma_{00}(S)=\phi$, it follows that $\sigma_w(S)=\sigma(S)$.□

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## الخلاصة


يقال للمؤثر A المعرف على فضاء هلبرت H بأنه موجب السوية إذا وجد مؤثر P في $B(H)$ بحيث أن $AA^{*} = A^{*}PA$ ليكن مجموعة كل القيم $\lambda \in \mathbb{C}$ بحيث أن $A - \lambda I$ ليست موجبة السوية نقول أن A موجب السوية كلياً إذا كان $A - \lambda I$ موجب السوية لكل $\lambda \in \mathbb{C}$. درسنا في هذا البحث بعض خواص المؤثرات موجبة السوية وخواص المجموعة $P(A)$ وبرهنا انه إذا كان A مؤثر موجب السوية كلياً فانه يحقق نظرية ويل.