Minimizing Error Bounds in Lacunary Interpolation by Spline Function, (0,1,4) Case

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Abstract

In this paper, we changed the boundary conditions which are given by [6, chapter two] from second derivative to third derivative. Accordingly, it was observed that and we show that the change of the boundary conditions affect in minimizing the error bounds for lacunary interpolation by spline function.

Introduction

Recently Jwamer, K. H. [6] obtained the error bounds for (0,1,4) lacunary interpolation of certain function by deficient sextic spline. In this paper we study same lacunary interpolation, but the essential difference here being in the boundary condition, in order to improve the state in which the change of the boundary condition affect on minimizing the error bounds and this fact is the main object of this work. Similar idea used in [4] for (0,2,3) and (0,2,4) cases.

In another communication we shall give the applications of spline functions in obtaining the approximation solution of boundary value problems. For more about applications of spline functions, see [2, 3 and 5]

For description our problem, let $\Delta: 0 = X_0 < X_2 < \dots < X_{2m} = 1$ be a uniform partition of the interval [0,1] with $X_i = \frac{i}{2m}$, i=0, 2, ..., $2m$ and $n=2m-1$. We define the class of spline function $S_p(6,4,n)$ as follow :

Any element $S_\Delta(X) \in Sp(6,4,n)$ if the following two conditions are satisfied [6]:

$$(1.0)\begin{cases} (i) \quad S_\Delta(X) \in C^4[0,1] \\ (ii) \quad S_\Delta(X) \text{ is a polynomial of degree six in each } [X_{2i}, X_{2i+2}],\ i = 0,1,\dots, m-1. \end{cases}$$

In the subsequent section, we prove the following:

Theorem 1.1:

Given arbitrary numbers $f(X_{2i})$, $f^{(r)}(t_{2i})$, i=0,1,...,m-1; r=0, 1, 4 and $f'''(X_0), f'''(X_{2m})$, there exits a unique spline $S_\Delta(X) \in S_p(6,4,n)$ such that

$$(1.1)\begin{cases} S_n(X_{2i}) = f(X_{2i}),\ i = 0,1,\dots,m \\ S_n^{(r)}(t_{2i}) = f^{(r)}(t_{2i}),\ i = 0,1,\dots,m-1; \\ \qquad r = 0,\ 1,\ 4, \\ S_n'''(X_0) = f'''(X_0), \\ S_n'''(X_{2m}) = f'''(X_{2m}) \end{cases}$$

Theorem 1.2:

Let $f \in C^5[0,1]$ and $S_n(X) \in S_p(6,4,n)$ be a unique spline satisfying the conditions of Theorem 1.1, then

$$\left|S_n^{(r)}(X) - f^{(r)}(X)\right| \le 23m^{r-5}\omega(f^5;\frac{1}{m}) + 3.5m^{r-5} + \|f^{(5)}\|$$

r=0,1,2,3,4.

Where $\omega(f^{(5)};\frac{1}{m})$ denotes the modulus of continuity of $f^{(5)}$ and $\|f^{(5)}\| = \max\{|f^{(5)}(X)|; 0 \le X \le 1\}$.

Theorem 1.3:

Let $f \in C^6[0,1]$ and $S_n(X) \in S_p(6,4,n)$ be a unique spline satisfying the conditions of Theorem 1.1, then

$$\left|S_n^{(r)}(X) - f^{(r)}(X)\right| \le 335m^{r-6}\omega(f^6;\frac{1}{m}) + 45m^{r-6} + \|f^{(6)}\|,\quad r = 0,1,2,3,4,5,$$

Where $\omega(f^{(6)};\frac{1}{m})$ denotes the modulus of continuity of $f^{(6)}$ and $\|f^{(6)}\| = \max\{|f^{(6)}(X)|; 0 \le x \le 1\}$.

2. Technical Preliminaries:

If $P(X)$ is a polynomial of degree six on [0,1] (because we want to construct a spline function of degree six), then we have

$$(2.0)\quad P(X)=P(0)A_0(X)+P(\tfrac{1}{3})A_1(X)-P(1)A_2(X)+ P'(\tfrac{1}{3})A_3(X)+P'''(0)A_4(X)+P'''(1)A_5(X)- P^{(4)}(\tfrac{1}{3})A_6(X)$$

Where

$$(2.1)\quad \begin{cases} A_0(X)=\frac{1}{67}(-243X^6+972X^5-1215X^4+ 879X^2-460X+67), \\ A_1(X)=\frac{1}{268}(729X^6-2916X^5+364X^4- 3240X^2+1782X), \\ A_2(X)=\frac{1}{268}(243X^6-972X^5+1215X^4- 276X^2+58X), \\ A_3(X)=\frac{1}{134}(-243X^6+972X^5-1215X^4+ 678X^2-192X), \\ A_4(X)=\frac{1}{12060}(999X^6+3393X^5- 3950X^4+2010X^3-451X^2+37X), \\ A_5(X)=\frac{1}{12060}(189X^6-153X^5-60X^4- 51X^2+7X), \\ A_6(X)=\frac{1}{72360}(-3159X^6+9018X^5- 6750X^4+1109X^2-218X). \end{cases}$$

In the subsequent section we need the following values:

For $f\in C^6[0,1]$ we have the following expansions[4]

$$f(X_{2i+2})=f(X_{2i})+2hf'(X_{2i})+ 2h^2f''(X_{2i})+\frac{4}{3}h^3f'''(X_{2i})+ \frac{2}{3}h^4f^{(4)}(X_{2i})+\frac{4}{15}h^5f^{(5)}(\lambda_{1,2i}), \quad X_{2i}<\lambda_{1,2i}<X_{2i+2}$$

$$(2.2)\quad f(X_{2i-2})=f(X_{2i})-2hf'(X_{2i})+ 2h^2f''(X_{2i})-\frac{4}{3}h^3f'''(X_{2i})- \frac{2}{3}h^4f^{(4)}(X_{2i})-\frac{4}{15}h^5f^{(5)}(\lambda_{2,2i}) \quad X_{2i-2}<\lambda_{2,2i}<X_{2i}$$

$$f(t_{2i})=f(X_{2i})+\frac{2}{3}hf'(X_{2i})+ \frac{2}{9}h^2f''(X_{2i})+\frac{4}{81}h^3f'''(X_{2i})+ \frac{2}{243}h^4f^{(4)}(X_{2i})+\frac{4}{3645}h^5f^{(5)}(\lambda_{3,2i}), \quad X_{2i}<\lambda_{3,2i}<t_{2i}$$

$$f(t_{2i-2})=f(X_{2i})-\frac{4}{3}hf'(X_{2i})+ \frac{8}{9}h^2f''(X_{2i})-\frac{32}{81}h^3f'''(X_{2i})+ \frac{32}{243}h^4f^{(4)}(X_{2i})-\frac{128}{3645}h^5f^{(5)}(\lambda_{4,2i}) \quad t_{2i-2}<\lambda_{4,2i}<X_{2i}$$

$$f'(t_{2i})=f'(X_{2i})+\frac{2}{3}hf''(X_{2i})- \frac{2}{9}h^2f'''(X_{2i})+\frac{4}{81}h^3f^{(4)}(X_{2i})+ \frac{2}{243}h^4f^{(5)}(\lambda_{5,2i}),\ X_{2i}<\lambda_{5,2i}<t_{2i}$$

$$f'(t_{2i-2})=f'(X_{2i})-\frac{4}{3}hf''(X_{2i})- \frac{8}{9}h^2f'''(X_{2i})-\frac{8}{81}h^3f^{(4)}(X_{2i})+ \frac{32}{81}h^4f^{(5)}(\lambda_{6,2i}),\ t_{2i-2}<\lambda_{6,2i}<X_{2i}$$

$$f'''(X_{2i+2})=f'''(X_{2i})+2hf^{(4)}(X_{2i})+ 2h^2f^{(5)}(\lambda_{7,2i}),\ X_{2i}<\lambda_{7,2i}<X_{2i+2}$$

$$f'''(X_{2i-2})=f'''-2hf^{(4)}(X_{2i})+ 2hf^{(4)}(X_{2i})+2h^2f^{(5)}(\lambda_{8,2i}), \quad X_{2i-2}<\lambda_{8,2i}<X_{2i}$$

$$f''(t_{2i})=f''(X_{2i})+\frac{2}{3}hf^{(4)}(X_{2i})+ \frac{2}{9}h^2f^{(5)}(\lambda_{9,2i}),\ X_{2i}<\lambda_{9,2i}<t_{2i}$$

$$f^{(4)}(t_{2i})=f^{(4)}(X_{2i})+\frac{2}{3}hf^{(5)}(\lambda_{10,2i}), \quad X_{2i}<\lambda_{10,2i}<t_{2i}$$

$$f^{(4)}(t_{2i-2})=f^{(4)}(X_{2i})-\frac{4}{3}hf^{(5)}(\lambda_{11,2i}), \quad t_{2i-2}<\lambda_{11,2i}<X_{2i}$$

Moreover for $f\in C^6[0,1]$ we have

$$f(X_{2i+2})=f(X_{2i})+2hf'(X_{2i})+ 2h^2f''(X_{2i})+\frac{4}{3}h^3f'''(X_{2i})+ +\frac{2}{3}h^4f^{(4)}(X_{2i})+\frac{4}{15}h^5f^{(5)}(X_{2i})+ \frac{4}{45}h^6f^{(6)}(\lambda_{1,2i}),\ X_{2i}<\lambda_{1,2i}<X_{2i+2}$$

$$f(X_{2i-2})=f(X_{2i})-2hf'(X_{2i})+$$
$$2h^2 f''(X_{2i})\ \frac{4}{3}h^3 f'''(X_{2i})+$$
$$+\frac{2}{3}h^4 f^{(4)}(X_{2i})-\frac{4}{15}h^5 f^{(5)}(X_{2i})+$$
$$\frac{4}{15}h^6 f^{(6)}(\lambda_{2,2i}),\ X_{2i-2}<\lambda_{2,2i}<X_{2i}$$
$$f(t_{2i})=f(X_{2i})+\frac{2}{3}hf'(X_{2i})+$$
$$\frac{2}{9}h^2 f''(X_{2i})-\frac{4}{81}h^3 f'''(X_{2i})-$$
$$+\frac{2}{243}h^4 f^{(4)}(X_{2i})+\frac{4}{3645}h^5 f^{(5)}(X_{2i})+$$
$$\frac{4}{32805}h^6 f^{(6)}(\lambda_{3,i}),\ X_{2i}<\lambda_{3,2i}<t_{2i}$$
$$f(t_{2i-2})=f(X_{2i})-\frac{4}{3}hf'(X_{2i})+$$
$$\frac{8}{9}h^2 f''(X_{2i})-\frac{32}{81}h^3 f'''(X_{2i})+$$
$$+\frac{32}{243}h^4 f^{(4)}(X_{2i})-\frac{128}{3645}h^5 f^{(5)}(X_{2i})+$$
$$\frac{256}{32805}h^6 f^{(6)}(\lambda_{4,2i}),\ t_{2i-2}<\lambda_{4,2i}<X_{2i}$$
$$f'(t_{2i})=f'(X_{2i})\frac{2}{3}hf''(X_{2i})$$
$$\frac{2}{9}h^2 f'''(X_{2i})+\frac{4}{81}h^3 f^{(4)}(X_{2i})+$$
$$+\frac{2}{243}h^4 f^{(5)}(X_{2i})-\frac{4}{3645}h^5 f^{(6)}(\lambda_{5,2i}),$$
$$X_{2i}<\lambda_{5,2i}<t_{2i},$$
$$f'(t_{2i-2})=f'(X_{2i})-\frac{4}{3}hf''(X_{2i})+$$
(2.3)
$$\frac{8}{9}h^2 f'''(X_{2i})-\frac{32}{81}h^3 f^{(4)}(X_{2i})+$$
$$+\frac{32}{243}h^4 f^{(5)}(X_{2i})-\frac{128}{3645}h^5 f^{(6)}(\lambda_{6,2i}),$$
$$t_{2i-2}<\lambda_{6,2i}<X_{2i}$$
$$f''(X_{2i+2})=f''(X_{2i})+2hf'''(X_{2i})+$$
$$2h^2 f^{(4)}(X_{2i})+\frac{4}{3}h^3 f^{(5)}(\lambda_{7,2i}),$$
$$X_{2i}<\lambda_{7,2i},X_{2i+2}$$
$$f''(X_{2i-2})=f''(X_{2i})\ 2hf'''(X_{2i})+$$
$$2h^2 f^{(4)}(X_{2i})\ \frac{4}{3}h^3 f^{(5)}(\lambda_{8,2i}),$$
$$X_{2i-2}<\lambda_{8,2i}<X_{2i}$$
$$f''(t_{2i})=f''(X_{2i})+\frac{2}{3}hf'''(X_{2i})+$$
$$\frac{2}{9}h^2 f^{(4)}(X_{2i})+\frac{4}{81}h^3 f^{(5)}(\lambda_{9,2i}),$$
$$X_{2i}<\lambda_{9,2i}<t_{2i}$$
$$f^{(4)}(t_{2i})=f^{(4)}(X_{2i})-\frac{2}{3}hf^{(5)}(X_{2i})+$$
$$\frac{2}{9}h^2 f^{(6)}(\lambda_{10,2i}),\ X_{2i}<\lambda_{10,2i}<t_{2i}$$
$$f^{(4)}(t_{2i-2})=f^{(4)}(X_{2i})-\frac{4}{3}hf^{(5)}(X_{2i})+$$
$$\frac{8}{9}h^2 f^{(6)}(\lambda_{11,2i}),\ t_{2i-2}<\lambda_{11,2i}<X_{2i}$$
$$f^{(5)}(t_{2i})=f^{(5)}(X_{2i})+\frac{2}{3}hf^{(6)}(\lambda_{12,2i})$$
$$X_{2i}<\lambda_{12,2i}<t_{2i}$$

3. Proof of Theorem 1.1

The proof depends on the following representation of $S_n(X)$ for $2ih<X<(2i+2)h$, $i=0,1,\ldots,m-1$, we have

$$(3.0)\quad S_n(X)=f(X_{2i})A_1\left(\frac{X-2ih}{2h}\right)+f(t_{2i})A_2\left(\frac{X-2ih}{2h}\right)+f(X_{2i+2})A_3\left(\frac{X-2ih}{2h}\right)-$$
$$-2hf'(t_{2i})A_4\left(\frac{X-2ih}{2h}\right)+8h^3 S'''_n(X_{2i})A_5\left(\frac{X-2ih}{2h}\right)+8h^3 S'''_n(X_{2i+2})A_6\left(\frac{X-2ih}{2h}\right)+$$
$$+16h^4 f^{(4)}(t_{2i})A_7\left(\frac{X-2ih}{2h}\right).$$

On using (3.0) and conditions

(3.1) $\quad S'''_n(0)=f'''(0),\ S'''_n(1)=f'''(1)$

We see that $S_n(X)$ as given by (3.0) satisfies (1.0) and is sextic in $[X_{2i}, X_{2i+2}]$, $i=0,1,\ldots,m-1$. We also need to show that whether it is possible to determine $S'''_n(X_{2i})$, $i=1,2,\ldots,m-1$ uniquely. For this purpose we use the fact that $S_n^{(4)}(X_{2i-})=S_n^{(4)}(X_{2i+})$, $i=1,2,\ldots,m-1$; with the help of (3.0) and (3.1) reduce to

$$(3.2)\ 2h^3 S'''_n(X_{2i+2})-\frac{400}{67}h^3 S'''_n(X_{2i})-$$
$$\frac{4}{67}h^3 S'''_n(X_{2i-2})\ \frac{3645}{134}f(X_{2i})-\frac{10932}{536}f(t_{2i})-$$
$$\frac{3645}{536}f(X_{2i-2})+\frac{3645}{134}hf'(t_{2i})-\frac{150}{67}h^4 f^{(4)}(t_{2i})-$$
$$3h^4 f^{(4)}(t_{2i-2})\ ,\ i=1,2,\ldots,m-1.$$

But (3.2) is a strictly tri diagonal dominant system which has a unique solution [?]. Thus $S'''_n(X_{2i})$, $i=1,2,\ldots,m-1$ can be obtained uniquely by the system (3.2) which establishes Theorem 1.1.

4. Estimates:

In order to prove the Theorem 1.2 and 1.3 we needs the following

Lemma 4.0:

Let us write $E_{2i}=\left|S_n'''(X_{2i})-f'''(X_{2i})\right|$, then for $f\in C^5[0,1]$, we have

(4.0) $\max E_{2i}\le \frac{391}{270}h^2 w(f^{(5)};\frac{1}{m})$

and for $f\in C^6[0,1]$ we have

(4.1) $\max E_{2i} \le \frac{329}{405}h^3 w(f^{(6)};\frac{1}{m})$

Proof:

Form (3.2) we have

$$2h^3(S_n'''(X_{2i+2})-f'''(X_{2i+2}))-\frac{400}{67}h^3(S_n'''(X_{2i})-f'''(X_{2i}))-\frac{4}{67}h^3(S_n'''(X_{2i-2})-f'''(X_{2i-2}))-$$

$$-2h^3 f'''(X_{2i+2})+\frac{400}{67}h^3 f'''(X_{2i})+\frac{4}{67}h^3 f'''(X_{2i-2})+\frac{3645}{134}f(X_{2i})-\frac{10935}{536}f(t_{2i})-$$

$$\frac{3645}{536}f(X_{2i+2})+\frac{3645}{134}hf'(t_{2i})+\frac{150}{67}h^4 f^{(4)}(t_{2i})$$

$$-3h^4 f^{(4)}(t_{2i+2})=-\frac{3}{134}h^5 f^{(5)}(\lambda_{1,2i})-\frac{234}{134}h^5 f^{(5)}(\lambda_{1,2i})+\frac{15}{67}h^5 f^{(5)}(\lambda_{2,2i})+$$

$$\frac{100}{67}h^5 f^{(5)}(\lambda_{16,2i})-\frac{224}{134}h^5 f^{(5)}(\lambda_{4,2i})+\frac{8}{67}h^5 f^{(5)}(\lambda_{7,2i})-4h^5 f^{(5)}(\lambda_{3,2i})=$$

$$\frac{391}{67}h^5\theta_6 w(f^{(5)};\frac{1}{m}),\ |\theta_6|\le 1$$

The result (4.0) follows on using the property of diagonal dominant [2]. The proof of (4.1) is similar.

Lemma 4.1 :

Let $f\in C^5[0,1]$ then

(i) $\left|S_n^{(4)}(t_{2i})-f^{(4)}(t_{2i})\right|\le\frac{14923}{18090}h^2 w(f^{(5)};\frac{1}{m})$,

(ii) $\left|S_n^{(4)}(X_{2i-})-f^{(4)}(X_{2i})\right|\le\frac{514}{67}hw(f^{(5)};\frac{1}{m})$,

(iii) $\left|S_n^{(4)}(X_{2i}+)-f^{(4)}(X_{2i})\right|\le\frac{1322}{135}hw(f^{(5)};\frac{1}{m})$,

(iv) $\left|S_n^{(5)}(t_{2i})\right|\le\frac{905}{1809670}w(f^{(5)};\frac{1}{m})+\left\|f^{(5)}\right\|$.

Proof:

Form (3.0) we have

$$h^4S_n^{(4)}(t_{2i})-\frac{540}{67}f(X_{2i})+\frac{405}{67}f(t_{2i})+\frac{135}{67}f(X_{2i+2})-\frac{540}{67}hf'(t_{2i})-$$

$$\frac{84}{606}h^3S_n'''(X_{2i})-\frac{11}{201}h^3S_n'''(X_{2i+2})-\frac{44}{201}h^4 f^{(4)}(t_{2i}).$$

Hence

$$h^4(S_n^{(4)}(t_{2i})-f^{(4)}(t_{2i}))=-\frac{2}{9}h^5 f^{(5)}(\lambda_{9,2i})+$$

$$\frac{4}{603}h^5 f^{(5)}(\lambda_{1,2i})-\frac{40}{603}h^5 f^{(5)}(\lambda_{2,2i})-$$

$$\frac{22}{201}h^5 f^{(5)}(\lambda_{7,2i})-\frac{88}{603}h^5 f^{(5)}(\lambda_{13,2i})-$$

$$\frac{11}{201}h^3(S_n'''(X_{2i+2})-f'''(X_{2i+2}))-$$

$$\frac{84}{603}h^3(S_n'''(X_{2i})-f'''(X_{2i}))-$$

$$\frac{328}{603}h^5\theta_1 w(f^{(5)};\frac{1}{m})-\frac{11}{201}h^3(S_n'''(X_{2i+2})-$$

$$f'''(X_{2i+2}))-\frac{84}{603}h^3(S_n'''(X_{2i})-f'''(X_{2i}));$$

$$|\theta_1|\le 1.$$

By using (4.0), the Lemma4.1(i) follows, the proof of the Lemma4.1 (ii-iv) are similar, we have only mention that

$$h^4S_n^{(4)}(X_{2i-})=-\frac{3645}{134}f(X_{2i})+\frac{10935}{536}f(t_{2i})+$$

$$\frac{3645}{536}f(X_{2i+2})-\frac{3645}{134}hf'(t_{2i})-$$

$$\frac{266}{67}h^3S_n'''(X_{2i})-\frac{4}{67}h^3S_n'''(X_{2i+2})$$

$$\frac{130}{67}h^4 f^{(4)}(t_{2i}).$$

$$h^4S_n^{(4)}(X_{2i-})=2h^3S_n'''(X_{2i+2})+$$

$$2h^3S_n'''(X_{2i})-4h^4 f^{(4)}(t_{2i+2}),$$

and

$$h^4S_n^{(4)}(t_{2i})=\frac{3645}{134}f(X_{2i})-\frac{10935}{536}f(t_{2i})-$$

$$\frac{3645}{536}f(X_{2i+2})+\frac{3645}{134}hf'(t_{2i})+$$

$$\frac{465}{134}h^3S_n'''(X_{2i})+\frac{75}{134}h^3S_n'''(X_{2i-2})-$$

$$\frac{150}{67}h^4 f^{(4)}(t_{2i}).$$

Lemma 4.2: Let $f\in C^6[0,1]$, then

(i) $\left|S_n^{(4)}(t_{2i})-f^{(4)}(t_{2i})\right|\le\frac{9157}{27135}h^3 w(f^{(6)};\frac{1}{m})$,

(ii) $\left|S_n^{(4)}(X_{2i-})-f^{(4)}(X_{2i})\right|\le\frac{80383}{9045}hw(f^{(6)};\frac{1}{m})$,

(iii) $\left|S_n^{(4)}(X_{2i+})-f^{(4)}(X_{2i})\right|\le\frac{153281}{9045}hw(f^{(6)};\frac{1}{m})$,

(iv) $\left|S_n^{(5)}(t_{2i})\right|\le\frac{181342}{9045}w(f^{(6)};\frac{1}{m})+\left\|f^{(6)}\right\|$.

Proof :

Form (2.2) and (3.0) we have

$$h^2S_n''(t_2)=-\frac{540}{67}f(X_2)+\frac{405}{67}f(t_2)+$$

$$\frac{135}{67}f(X_{2i+2})-\frac{540}{67}hf'(t_2)-\frac{84}{603}h S_n'''(X_{2i})-$$

$$\frac{11}{201}h^3S_n'''(X_{2i+2})-\frac{44}{201}h^4f^{(4)}(t_{2i}).$$

Hence

$$h^2(S_n''(t_2)-f''(t_2))=-\frac{4}{81}h^6f^{(6)}(\lambda_{32i})+$$

$$\frac{4}{5427}h^6f^{(6)}(\lambda_{32i})+\frac{12}{67}h^6f^{(6)}(\lambda_{12i})$$

$$\frac{16}{1809}h^6f^{(6)}(\lambda_{52i})-\frac{44}{603}h^6f^{(6)}(\lambda_{52i})$$

$$\frac{88}{1809}h^6f^{(6)}(\lambda_{102i})-\frac{84}{603}(h^3(S_n'''(X_{2i})-$$

$$f'''(X_{2i}))-\frac{11}{201}h^3(S_n'''(X_{2i})-f'''(X_{2i+2}))$$

$$\frac{976}{1809}h^5\theta_2 w(f^{(5)};\frac{1}{m})-\frac{84}{603}h^3(S_n'''(X_{2i})-$$

$$f'''(X_{2i}))-\frac{11}{201}h^3(S_n'''(X_{2i+2})-f'''(X_{2i+2})),$$

$$|\theta_2|\le 1.$$

On using (4.1), the Lemma 4.2(i) follows. The Proof of the Lemma 1.2(ii-iv) are similar, we only mention that

$$h^3S_n'''(X_{2i})=\frac{3645}{67}f(X_{2i})-\frac{10935}{268}f(t_{2i})-$$

$$\frac{3645}{268}f(X_{2i+2})+\frac{501}{67}h^4f^{(4)}(t_{2i})+$$

$$\frac{3645}{67}hf'(t_{2i})-\frac{113}{134}h^3S_n'''(X_{2i})-$$

$$\frac{51}{134}h^3S_n'''(X_{2i+2}).$$

$$h^3S_n^{(3)}(X_{2i})=-\frac{3645}{134}f(X_{2i})+\frac{10935}{536}f(t_{2i})+$$

$$\frac{3645}{536}f(X_{2i})-\frac{3645}{134}hf'(t_{2i})-$$

$$\frac{897}{134}h^3S_n'''(X_{2i})+\frac{327}{134}h^3S_n'''(X_{2i})$$

$$\frac{552}{67}h^4f^{(4)}(t_{2i}),$$

and

$$h^3S_n^{(3)}(t_{2i})=\frac{10935}{268}f(X_{2i})-\frac{32805}{1072}f(t_{2i})+$$

$$\frac{10935}{1072}f(X_{2i+2})-\frac{10935}{268}hf'(t_{2i})$$

$$\frac{999}{134}h^3S_n'''(X_{2i})+\frac{189}{134}h^3S_n'''(X_{2i+2})$$

$$\frac{1053}{134}h^4f^{(4)}(t_{2i}).$$

5. Proof of Theorem 1.2.

For $0\le Z\le 1$, we have

(5.0) $\quad A_0(Z)+A_1(Z)+A_2(Z)=1.$

Let $X_{2i}\le X\le X_{2i+2}$. On using (5.0) and (3.1), we have obtain

$$(5.1)\ S_m^{(4)}(X)-f^{(4)}(X)=(S_m^{(4)}(X_{2i})-$$

$$f^{(4)}(X))A_0(\frac{X-2ih}{2h})+(S_m^{(4)}(X_{2i+2})$$

$$f^{(4)}(X))A_2(\frac{X-2ih}{2h})+(S_n^{(4)}(t_{2i})-f^{(4)}(X))$$

$$A_1(\frac{X-2ih}{2h})+2hS_n^{(5)}(t_{2i})A_1(\frac{X-2ih}{2h})=$$

$$L_1+L_2+L_3+L_4$$

From (2.1) it follows that

$|A_0(X)|\le 1$, $|A_1(X)|\le 1$, $|A_2(X)|<1$

and $|A_3(X)|<1$

Since $f^{(4)}(X)=f^{(4)}(X_{2i})+(X-X_{2i})f^{(5)}(\lambda)$,

$X_{2i}<\lambda<X$,

therefore

$$L_1=(S_n^{(4)}(X_{2i})-f^{(4)}(X))A_0(\frac{X-2ih}{2h})$$

$$=(S_n^{(4)}(X_{2i})-f^{(4)}(X_{2i})-(X-X_{2i})f^{(5)}(\lambda))$$

$$A_0(\frac{X-2ih}{2h}).$$

On using (4.1.2) and $|X-X_{2i}|\le 2h$, we obtain

$$(5.2)\qquad L_1<\frac{514}{67}hw(f^{(5)};\frac{1}{m})+2h\|f^{(5)}\|$$

Similarly,

$$(5.3)\quad L_2\le\frac{1322}{135}hw(f^{(5)};\frac{1}{m})+2h\|f^{(5)}\|,$$

and

$$(5.3)\ L_3=(S_n^{(4)}(t_{2i})-f^{(4)}(X))A_1(\frac{X-2ih}{2h})$$

$$=(S_n^{(4)}(t_{2i})-f^{(4)}(t_{2i})+f^{(4)}(t_{2i})-$$

$$f^{(4)}(X_{2i})-(X-X_{2i})f^{(5)}(\lambda)).$$

Form (2.2) we obtain

$$f^{(4)}(t_{2i})-f^{(4)}(X_{2i})=\frac{2}{3}hf^{(5)}(\lambda_{112i})$$

and

$$S_n^{(4)}(t_{2i})-f^{(4)}(t_{2i}).$$

Put this result in (5.3) we obtain

$$|L_3|\le\frac{8}{3}h\|f^{(5)}\|.$$

Therefore,

$$L_4=2hS_n^{(5)}(t_{2i})A_1(\frac{X-2ih}{2h}).$$

On using (4.9) we obtain

$$(5.5)\ |L_4|<\frac{1810}{67}hw(f^{(5)};\frac{1}{m})+2h\|f^{(5)}\|.$$

Putting (5.2) – (5.5) in (5.1) we obtain

$$(5.6)\ S_n^{(4)}(X)-f^{(4)}(X)<45hw(f^{(5)};\frac{1}{m})-\frac{2}{3}\|f^{(5)}\|$$

This proves (1.2) for $P=4$. To prove (1.2) for $P=3$, since

$$S_n^{(3)}(X)-f^{(3)}(X)=\int_{t_{2i}}^{X}(S_n^{(4)}(t)-f^{(4)}(t))dt+ S_n^{(3)}(t_{2i})-f^{(3)}(t_{2i}).$$

On using Lemma4.1 (i) and (5.6) we obtain

$$\left|S_n^{(3)}(X)-f^{(3)}(X)\right|\le 23m^{-2}w(f^{(3)};\frac{1}{m})+m^{-4}\left\|f^{(4)}\right\|.$$

Which prove (1.2) for $P=3$, The proof of (1.2) for P=0, 1, 2 follows immediately on the lines of Jwamer.H [6]. This completes the proof of Theorem1.2 the proof of Theorem1.3 is similar to that of Theorem1.2. Therefore we omit it .

Conclusion

In this paper we conclude that some times charge of the boundary conditions affect on minimizing error bounds in the subject of lacunary interpolation by spline functions.

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الخلاصة

في هذا البحث، تم تغيير شروط الحدود لـ [6؛ فصل الثاني] من الإشتقاق الثاني إلى الإشتقاق الثالث. وبناء على ذلك، ظهر بأن تغييرات شروط الحدود تؤثر في تقليل الخطأ للاندراج الفراغي بواسطة دالة سبلاين.